On nonstationary preconditioned iterative regularization methods for image deblurring

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Outline

1. Image Deblur
2. Tikhonov Regularization
3. AIT, ARIT, APIT and APRIT
4. Conclusions and Future Work

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Image Deblur

The image deblur problem is a deconvolution problem.

$$g(x,y) = \int K(s-x, t-y) f(s,t) dsdt = K * f, \tag{1}$$

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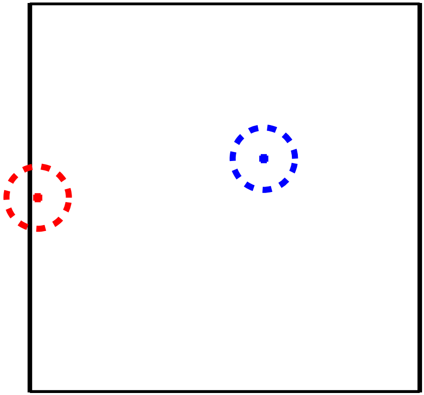
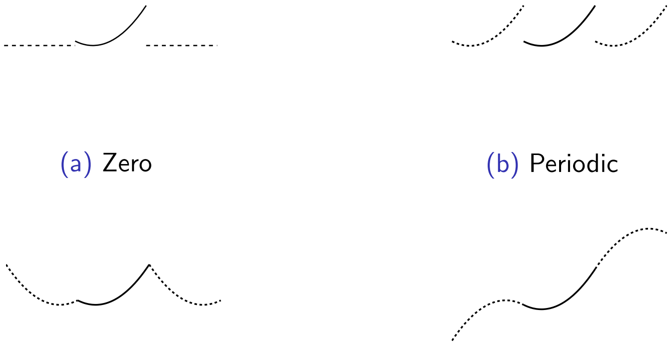
We can not directly solve this linear system for x because A is *severely ill-conditioned*, we have to use regularization methods.

In practical application we only have finite data, so a problem arises near the boundary. In particular we need to make assumptions on what is outside the field of view.





(a) Zero

(b) Periodic

(c) Reflexive

(d) Antireflexive

- Reflexive: [Ng et al., SISC1999];
- Antireflexive: [S. Serra-Capizzano, SISC2003].

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Tikhonov Regularization

The general Tikhonov regularization has the form

$$x_\alpha = \arg\min_x \|Ax - b\|_2^2 + \alpha \|Lx\|_2^2, \tag{3}$$

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By applying a refinement technique we obtain the *Iterated Tikhonov* method

$$x_{k+1} = x_k + (A^*A + \alpha L^*L)^{-1} A^*(b - Ax_k),$$

which can be seen as a preconditioned Landweber.

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Iterated Tikhonov works on the error equation

$$Ae_k \approx r_k, \tag{4}$$

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Let C be such that

$$\|(C - A)z\|_2 \leq \rho \|Az\|_2, \qquad \forall z, \tag{5}$$

for some $0 < \rho < \frac{1}{2}$.

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We approximate equation (4) with

$$Ce_k \approx r_k.$$

Algorithm (AIT, [M. Donatelli and M. Hanke, IP2013])

Let x_0 be fixed and set $r_0 = b - Ax_0$. Choose $\tau = \frac{1+2\rho}{1-2\rho}$ with ρ as in (5)*, and fix $q \in [2\rho, 1]$.*

While $\|r_k\|_2 > \tau \|\eta\|_2$, let $\tau_k = \|r_k\|_2 / \|\eta\|_2$ and $q_k = \max\left\{q, 2\rho + \frac{1+\rho}{\tau_k}\right\}$, compute

$$h_k = C^*(CC^* + \alpha_k I)^{-1} r_k,$$

where α_k is such that

$$\|r_k - Ch_k\|_2 = q_k \|r_k\|_2 ,$$

and update

$$x_{k+1} = x_k + h_k,$$
$$r_{k+1} = b - Ax_{k+1}.$$

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In AIT, $L = I$.

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Proposition

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Theorem

Let $\delta \mapsto b^\delta$ *a function such that* $\forall\delta \ \left\|b^\delta - b_{true}\right\|_2 < \delta$. *Let* C, τ, *and* q *as in AIT and denote with* x^δ *the approximation obtained with AIT with right hand side* b^δ. *For* $\delta \to 0$ *we have that* $x^\delta \to x_0^\dagger$, *which is the nearest solution to* x_0 *of the system.*

Algorithm (ARIT, [A. B. and M. Donatelli, submitted 2015])

Let x_0 *be fixed and set* $r_0 = b - Ax_0$. *Choose* $\tau = \frac{1+2\rho}{1-2\rho}$ *with* ρ *as in* (5), *and fix* $q \in [2\rho, 1]$.

While $\|r_k\|_2 > \tau \|\eta\|_2$, *let* $\tau_k = \|r_k\|_2 / \|\eta\|_2$ *and* $q_k = \max\left\{q, 2\rho + \frac{1+\rho}{\tau_k}\right\}$, *compute*

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$$\begin{aligned} x_{k+1} &= x_k + h_k, \\ r_{k+1} &= b - Ax_{k+1}. \end{aligned}$$

Remark

For ARIT to work we need to make the same assumptions of AIT on C and A and the following

- $\mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}$;
- $C\big|_{\mathcal{N}(L)} = A\big|_{\mathcal{N}(L)}$;
- *L and C are diagonalized by the same unitary transformation.*

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As $\delta \to 0$ *we have that* $x^\delta \to x_0^\dagger$, *with the same notation of AIT.*

Algorithm (APIT, [A. B. and M. Donatelli, submitted 2015])

Let x_0 *be fixed and set* $r_0 = b - Ax_0$. *Choose* $\tau = \frac{1+2\rho}{1-2\rho}$ *with* ρ *as in* (5), *and fix* $q \in [2\rho, 1]$.

Let Ω *be a closed and convex set such that* $x_{true} \in \Omega$ *and* P_Ω *the metric projection over it.*

While $\|r_k\|_2 > \tau \|\eta\|_2$, *let* $\tau_k = \|r_k\|_2 / \|\eta\|_2$ *and* $q_k = \max\left\{q, 2\rho + \frac{1+\rho}{\tau_k}\right\}$, *compute*

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Algorithm (APRIT, [A. B. and M. Donatelli, submitted 2015])

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- $L = L_1 \otimes I + I \otimes L_1$, where L_1 is defined as

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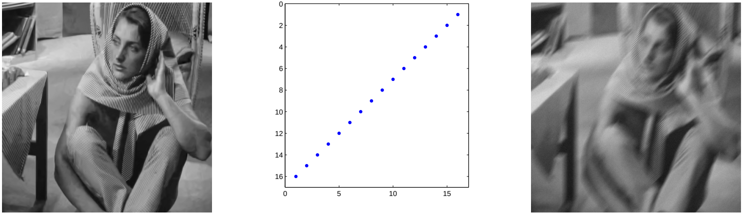


(a) (b) (c)

Figure: Barbara Test case: (a) Barbara Test image, (b) Diagonal motion PSF, (c) Blurred image, $\|\eta\|_2 = 0.03\,\|b_{true}\|_2$.

Since the image is generic we use the *antireflexive* boundary conditions.

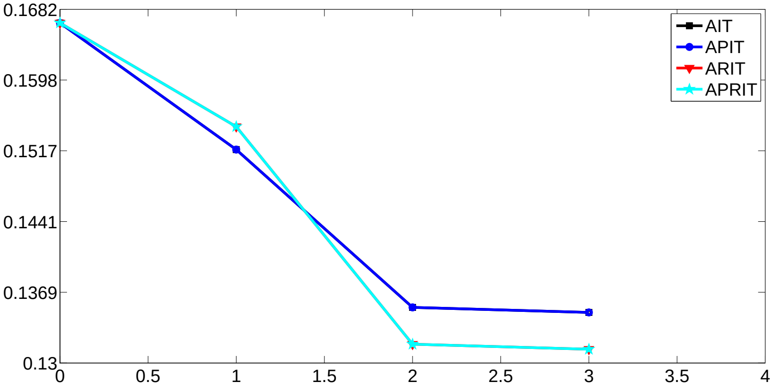
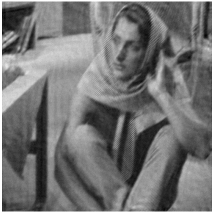


Figure: Barbara Test Case: RRE History

Method	RRE	Computational Time
AIT	0.13489	0.88789 sec.
ARIT	**0.13132**	1.2488 sec.
APIT	0.13489	0.88299 sec.
APRIT	0.13132	1.2694 sec.
Hybrid	0.15919 (Optimal: 0.13337)	19.3781 sec.
TwIST	0.14142	7.2969 sec.
FlexiAT	0.15443	10.6369 sec.
RRAT	0.15665	26.888 sec.
NN-ReStart-GAT	0.16502	162.0278 sec.

Table: Barbara: Relative errors comparisons

- Hybrid: [J. Chung, J. Nagy and, D. O'Leary, ETNA2008];
- TwIST: [J. Bioucas-Dias and M. Figueiredo, IEEE2007];
- FlexiAT: [S. Gazzola and J. Nagy, SISC2014];
- RRAT: [A. Neuman, L. Reichel and, H. Sadok, NA2012];
- NN-ReStart-GAT: [S. Gazzola and J. Nagy, SISC2014].



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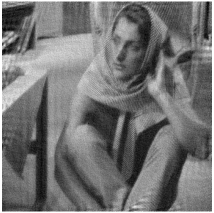

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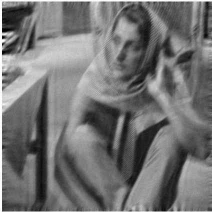

(d)

Figure: Barbara Reconstructions: (a) ARIT, (b) APIT, (c) Hybrid, (d) TwIST.

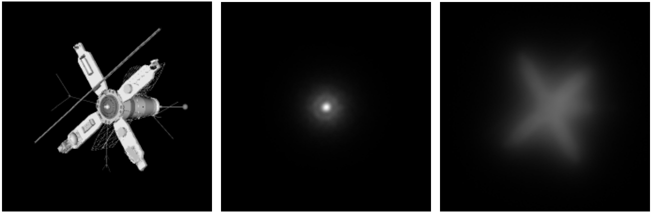

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Figure: Satellite Test case: (a) Test image, (b) Astronomic PSF, (c) Blurred image, $\|\eta\|_2 = 0.01 \|b_{true}\|_2$.

Since the image is all black around the boundaries we use the *zero* boundary conditions.

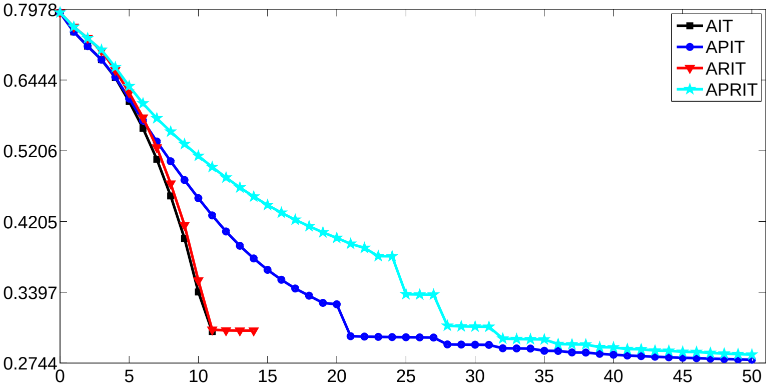
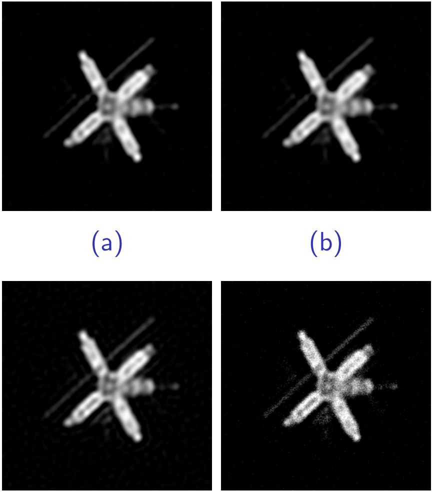


Figure: Satellite Test Case: RRE History

Method	RRE	Computational Time
AIT	0.30174	1.2155 sec.
ARIT	0.30262	1.8452 sec.
APIT	**0.27713**	7.3488 sec.
APRIT	0.2814	6.5561 sec.
Hybrid	0.40035	5.6961 sec.
TwIST	0.32599	7.0781 sec.
FlexiAT	0.31521	2.4719 sec.
RRAT	0.3087	1.5309 sec.
NN-ReStart-GAT	0.32599	20.1723 sec.

Table: Satellite: Relative errors comparisons



(a) (b)

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Figure: Satellite Reconstructions: (a) APIT, (b) APRIT, (c) RRAT, (d) NN-Restart-GAT.

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Possible future improvements

- Non spatially invariant blur;
- More advanced regularization term ($p-$norms, $TV, \ldots$);
- Blind deconvolution.

Thank you for your attention.