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Braunschweig

On Computing Matrix Polynomial Roots

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Outline

- **Nonlinear eigenvalue problem**
- **Hermite interpolation problem**
- **Linearization in Hermite basis**
- **A clarifying example**

Nonlinear eigenvalue problem

$$F(z)x = 0$$

- $F : \mathbb{C} \to \mathbb{C}^{n\times n}$: a given matrix-valued function
- $z \in \mathbb{C}$: *eigenvalue*
- $x \in \mathbb{C}^{n\times 1}$: *eigenvector*

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Application: different areas of mechanics, such as machine foundations, electronic model of metal strip, etc, see:
A comprehensive collection of the eigenvalue problems from models of real life as well as structured ones is studied in:

T. Betcke, N. J. Higham, V. Mehrmann, C.Schröder, F. Tisseur, *NLEVP: A Collection of Nonlinear Eigenvalue Problems*, ACM Trans. Math. Software 39(2) (2013) 1–28.

Hermite interpolation problem

$$F(z)x = 0$$

Assume F is not given explicitly:
For the interpolation nodes $z_0, \cdots, z_n \in \mathbb{C}$, let

$$F^{(j)}(z_i) = F_{i,j}, \qquad i = 0, \cdots, n, \ j = 0, \cdots, s_i - 1$$

where $s_i \geqslant 1$ denote the confluency of the ith interpolation node

$$s + 1 = \sum_{i=0}^{n} s_i.$$

The Hermite interpolation problem is to construct a matrix polynomial $P(z)$ of degree s such that

$$\left.\frac{d^j P(z)}{dz^j}\right|_{z=z_i} = F_{i,j}, \qquad i = 0, \cdots, n, \ j = 0, \cdots, s_i - 1.$$

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A solution to the interpolation problem

A solution to the interpolation problem is presented in:

B. Sadiq, D. Viswanath, *Barycentric Hermite interpolation*, SIAM J. Sci. Comput. 35 (2013) 1254–1270.

We begin with the definition of the polynomial $\omega(z)$ of degree $s+1$:

$$\omega(z) = \prod_{i=0}^{n} (z - z_i)^{s_i},$$

The Modified barycentric form of the Hermite interpolant

$$P(z) = \omega(z) \sum_{i=0}^{n} \sum_{j=0}^{s_i - 1} \frac{M_{i,j} w_i}{(z - z_i)^{s_i - j}},$$

where $M_{i,j} := \sum_{k=0}^{j} \frac{F_{i,k}}{k!} \mathfrak{I}_{j-k}$, and $w_{i,k}$ are the generalized barycentric weights.

Generalized barycentric weights

Define the sequences of $\mathcal{J}_r$ as follows:

$$\mathcal{J}_0 = 1, \qquad k\mathcal{J}_k = \mathcal{P}_k + \sum_{j=1}^{k-1} \mathcal{J}_j \mathcal{P}_{k-j}, \quad k \geqslant 1$$

where $\mathcal{P}_r := \sum_{j \neq k} \frac{s_j}{(z_j - z_k)^r}$, $r \geqslant 1$.

Now, we have:

$$w_{i,k} = w_i \mathcal{J}_k, \qquad w_i := \prod_{\substack{k=0, \\ k \neq i}}^{n} \frac{1}{(z_i - z_k)^{s_k}}.$$

Modified Hermite barycentric interpolant

Now, instead of the eigenproblem $F(z)x = 0$, we consider:

Polynomial eigenvalue problem

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Polynomial eigenvalue problem

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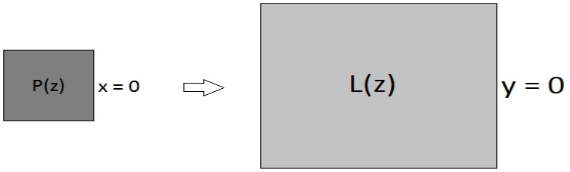
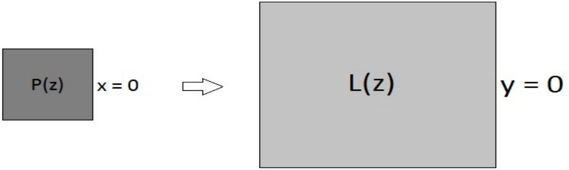
The classical approach to numerically solving polynomial eigenvalue problems is linearization.

Linearization



Linearization



A pencil $L(z)$ is a linearization of $P(z)$, if there exist unimodular matrix polynomials $E(z), F(z)$ such that

$$E(z)L(z)F(z) = \begin{bmatrix} P(z) & 0 \\ 0 & I \end{bmatrix}.$$

Linearization

Consider the interpolation nodes z_i, $i = 0, \cdots, n$, with the confluency s_i, and $N \times N$ matrix polynomial $P(z)$ of degree s in the Hermite form, satisfying interpolation conditions.

$$P(z) = \omega(z) \sum_{i=0}^{n} \sum_{j=0}^{s_i-1} \frac{M_{i,j} w_i}{(z - z_i)^{s_i-j}}.$$

The pencil $L(z)$ of size $(s+2)N$ is a linearization for $P(z)$:

$$L(z) = \begin{bmatrix} J_0 & & & & -M_0 \\ & J_1 & & & -M_1 \\ & & \ddots & & \vdots \\ & & & J_n & -M_n \\ W_0 & W_1 & \dots & W_n & 0 \end{bmatrix}.$$

Linearization

with the matrix

$$J_i = zI_{Ns_i} - \begin{bmatrix} z_i I_N & I_N & & \\ & z_i I_N & I_N & \\ & & \ddots & I_N \\ & & & z_i I_N \end{bmatrix} \in \mathbb{C}^{Ns_i \times Ns_i}, \qquad i = 0, \cdots, n.$$

The vectors

$$W_i = \begin{bmatrix} w_i I_N & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{N \times Ns_i} \qquad i = 0, \cdots, n,$$

and

$$M_i = \begin{bmatrix} M_{i,s_i-1} & \cdots & M_{i,1} & M_{i,0} \end{bmatrix}^T \in \mathbb{C}^{Ns_i \times N} \qquad i = 0, \cdots, n,$$

can be recovered directly from the interpolant.

Example

The eigenvalues of $F(z)$ are desired.

$F(z_0)$	$F'(z_0)$	$F''(z_0)$	$s_0 = 3$
$F(z_1)$	$F'(z_1)$		$s_1 = 2$
$F(z_2)$			$s_2 = 1$

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The polynomial interpolant exists and is given by:

$$P(z) = \omega(z)\left(\frac{M_{0,0}w_0}{(z-z_0)^3} + \frac{M_{0,1}w_0}{(z-z_0)^2} + \frac{M_{0,2}w_0}{(z-z_0)} + \frac{M_{1,0}w_1}{(z-z_1)^2} + \frac{M_{1,1}w_1}{(z-z_1)} + \frac{M_{2,0}w_2}{(z-z_2)}\right),$$

where

$$\omega(z) = (z-z_0)^3(z-z_1)^2(z-z_2),$$

$$w_i := \prod_{\substack{k=0,\\ k\neq i}}^{n} \frac{1}{(z_i - z_k)^{s_k}}.$$

The linearization is given by:

$$L(z) = \begin{pmatrix} (z-z_0)I_N & -I_N & & & & & -M_{0,2} \\ & (z-z_0)I_N & -I_N & & & & -M_{0,1} \\ & & (z-z_0)I_N & 0 & & & -M_{0,0} \\ & & & (z-z_1)I_N & -I_N & & -M_{1,1} \\ & & & & (z-z_1)I_N & 0 & -M_{1,0} \\ & & & & & (z-z_2)I_N & -M_{2,0} \\ w_0 I_N & 0 & 0 & w_1 I_N & 0 & w_2 I_N & 0 \end{pmatrix}.$$

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Eigenvalue problem

Nonlinear eigenproblem $F(z)x = 0 \Rightarrow$ Polynomial eigenproblem $P(z)x = 0$

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$$det\ P(z) = 0 \quad \Leftrightarrow \quad det\ L(z) = 0$$

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- Further work is needed on the computational complexity.