Clustering graphs using the spectrum of the nonlinear p-Laplacian

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(joint work with Matthias Hein)

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- The set of edges joining two subsets A, B is denoted by $E(A, B)$.

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Many applications:

Image analysis, Social networks, Bioinformatics, IT security, ...

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$$h_G(k) = \min_{\substack{A_1, \dots, A_k \subseteq V \\ \textit{disjoint}}} \max_{i=1,\dots,k} \frac{|E(A_i, \overline{A_i})|}{|A_i|}$$

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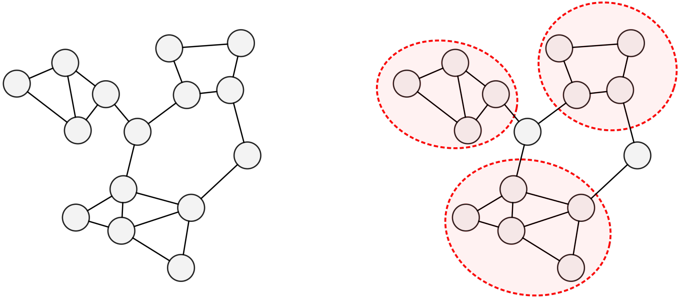

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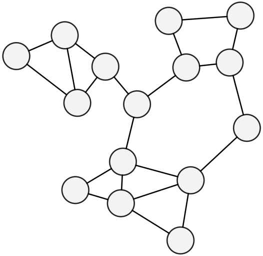

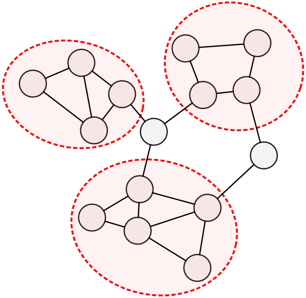

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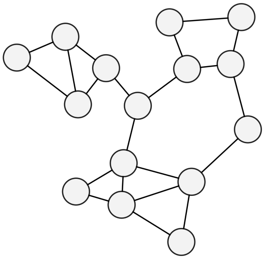

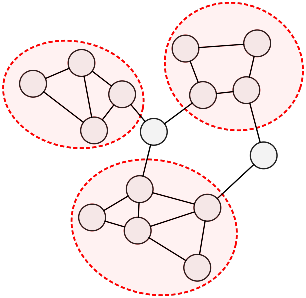

Spectral-based approach

Introduce the Laplacian matrix of the graph

$$\begin{array}{rccl} L_2 : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & f & \mapsto & (L_2 f)_i = \sum_{j:ij\in E}(f_i - f_j) \end{array}$$

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Select the disjoint subsets $A_1, \dots, A_k$ inside V using the eigenvalues/vectors of L_2 and the associated **nodal domains**

Nodal domains

Number of Nodal Domains of a vector $g \in \mathbb{R}^n$:

$$\boxed{\begin{array}{c}\nu(g) = \text{overall number of maximal connected components in} \\ G(\{i : g_i > 0\}) \text{ and } G(\{i : g_i < 0\})\end{array}}$$

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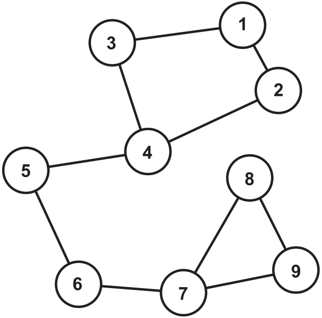

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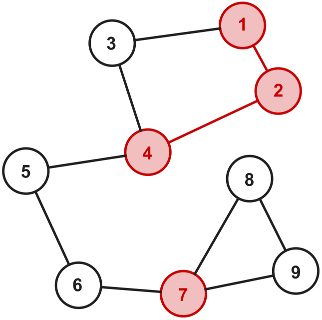

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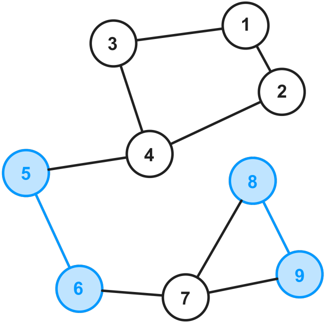

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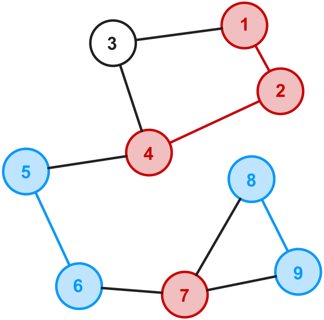

$\nu(g) = 4$

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Two main theoretical results in support of these choices:

Fiedler theorem: $\nu(f_2) = 2$
i.e. $G(A^+)$ and $G(A^-)$ are connected

Cheeger inequality:
$\lambda_2/2 \leq h_G(2) \leq \sqrt{2\, d_{\max}\, \lambda_2}$

Nodal domains and Cheeger inequality for $k > 2$

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k-way Cheeger inequalities [Lee, Gharan, Trevisan, Miclo, ..., 2010-13]

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extend nodal domain and Cheeger theorems to L_p, for a general $p \geq 1$
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$$\lambda_k^{(p)} = \min_{\substack{g_1,\dots,g_k \in \mathbb{R}^n \\ \textit{linearly independent}}} \ \max_{f\in \mathrm{span}\{g_1,\dots,g_k\}} R_p(f)$$

are eigenvalues of L_p such that $0 \leq \lambda_1^{(p)} \leq \lambda_2^{(p)} \leq \cdots \leq \lambda_n^{(p)}$ and the corresponding eigenvectors can be chosen to be pairwise independent.

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- $\lambda_k^{(p)} = 0$ if and only if G has exactly k connected components
- Each eigenvector f of L_p has zero p-mean: $\sum_{i \in V} |f_i|^{p-2} f_i = 0$, for any $p \geq 1$

A p-Laplacian nodal domain theorem

The nodal domain theorem for the linear case

f_k = eigenvector associated to the k-th eigenvalue of L_2. Then
$$\nu(f_k) \leq k$$

extends to L_p, for any $p \geq 1$, but restricted to variational spectrum.

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- If $G = \circ\!-\!\!\!-\!\circ\!-\!\!\!-\!\circ\!-\!\!\!-\!\circ \cdots\cdots\cdots \circ$

  then $\quad \nu(f_k) = k, \qquad k = 1, 2, 3, \dots$

> The bound can not be improved

High-order Cheeger inequalities based on L_p

The k-way Cheeger inequality for the linear Laplacian

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The k-way Cheeger inequality for the linear Laplacian

λ_k = k-th smallest eigenvalue of L_2. Then

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extends to L_p, for any $p \geq 1$ and we get a better bound (for small p)

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Let $\lambda_k^{(p)}$ be the k-th variational eigenvalue of L_p. Then

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- Propose clustering algorithms based on the eigenvalues/vectors of L_p

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Thank you!