A strategy for updating an AINV preconditioner

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(joint work with D. Bertaccini[2])

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Outline

1. Motivation
   - The problem we're working on
2. AINV preconditioner
   - The symmetric case
   - The non symmetric case
3. The Proposed Approach
   - The Starting Point
   - Theoretical Results (The Symmetric Case)
   - Numerical Results

Defining the Problem
Succession of Sparse Linear System

We consider the efficient construction of **preconditioners** for sequences of linear systems of the form:

$$A_k \mathbf{x}_k = \mathbf{b}_k, \quad k \geq 0, \quad \{A_k\}_{k\geq 0} \subset \mathbb{R}^{n\times n},$$
$$\{\mathbf{x}_k\}_{k\geq 0}, \{\mathbf{b}_k\}_{k\geq 0} \subset \mathbb{R}^n,$$

where $\{A\}_k$ is a **sparse matrix** for each $k \geq 0$.
Possible sources for the problem are represented by:

- PDE discretization,
- Differential-Algebraic Equations,
- Eigenvalues and eigenvectors computation (*shift and invert* algorithms).

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Defining the Problem
Preconditioning Iterative Krylov Solvers

Our goal is to generate a **sequence of preconditioners** $\{M_k\}_k$, to improve the convergence properties, in terms of **execution time** and **number of iterations**, of the Krylov Solvers for the algebraically equivalent system $M_k^{-1}A_k\mathbf{x}_k = M_k^{-1}\mathbf{b}_k$ i.e., such that the transformed linear systems presents a **cluster of eigenvalues**,

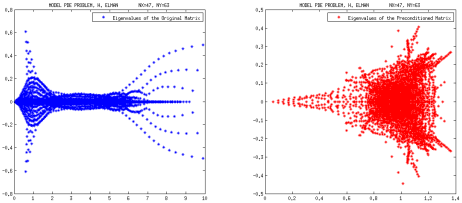

Figure: Cluster of Eigenvalues.

The AINV preconditioner
Incomplete Factorization Preconditioner

The **AINV preconditioner**[Benzi-Meyer-Tuma '96, Benzi-Cullum-Tuma 2000] is based on:

- an incomplete factorization of the matrix A^{-1},
- an A-conjugation process, starting from a set of conjugate directions $\{\mathbf{z}_i\}_{i=1,\dots,n}$ to write an SPD matrix A in the form: $Z^T A Z = D$.

$$Z = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{z}_1 & \mathbf{z}_2 & \dots & \mathbf{z}_n \\ \vdots & \vdots & & \vdots \end{pmatrix}, D = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix}, \tag{1}$$

And so we have that:

- $p_i = \mathbf{z}_i^T A \mathbf{z}_i$ and $A^{-1} = Z D^{-1} Z^T$.

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The AINV preconditioner
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To **obtain a preconditioner** $M^{-1} = ZD^{-1}Z^T$, a dropping strategy for going from Z to its sparse approximation $\tilde{Z}$ is needed.

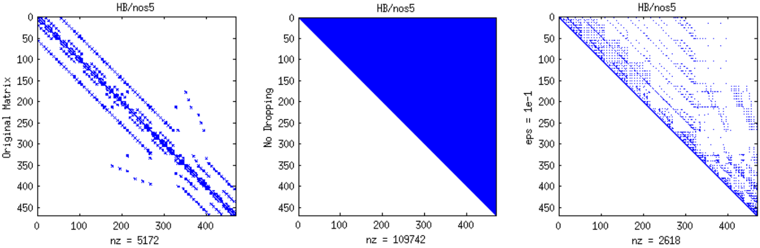

Figure: AINV dropping example.

The AINV preconditioner
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It's possible to go from a symmetric A-conjugation algorithm to a full A-biconjugation, [Benzi-Tuma 98, Benzi-Haws-Tuma 2000]. To obtain this we need:

- two family of A-orthogonal direction

and then the factorization is of the form:

$$W^T AZ = D = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix}, \qquad (2)$$

where $p_i = \mathbf{w}_i^t A\mathbf{z}_i \neq 0$, and W and Z are **non singular** and **orthogonal triangular** matrices.

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Construction of the Preconditioner Update

An AINV update strategy

The construction of the strategies proposed here is developed starting on the observations stated in the conclusion of [Benzi-Bertaccini 2003] for:

$$A_\alpha \mathbf{x} = \mathbf{b}, \quad A_\alpha = A + \alpha B, \quad \alpha \in [\alpha_1, \alpha_2], \tag{3}$$

where A, B are **symmetric**, **positive definite** and **sparse.**
As in [Benzi-Bertacini 2003, Bertaccini 2004] we are going to start with an AINV preconditioner for the A_{α_1} matrix and generate the order k-preconditioner for the sequence, setting $\tilde{Z}_k = \text{diag}(\tilde{Z}, k)$.

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Construction of the Preconditioner Update

The Preconditioner Definition

The **starting update strategy** is:

$$M_{\alpha,k}^{-1} = \tilde{Z}(\tilde{D} + \alpha E_k)^{-1}\tilde{Z}^T, \; E_k = \begin{cases} 0 & k = -1, \\ B & k = 0, \\ \tilde{Z}_k^T B \tilde{Z}_k & k \geq 1, \end{cases} . \quad (4)$$

Now what we want is to **modify also** the Z-factor of the factorization accordingly to further improve the convergence of the underlying Krylov method.

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Construction of the Preconditioner Update

The New Preconditioner Definition

We start computing **two reference preconditioner**:

$$\begin{aligned} M_{(1)}^{-1} =& \tilde{Z}_1 \tilde{D}_1^{-1} \tilde{Z}_1^T \approx A_{\alpha_1}^{-1} = (A + \alpha_1 B)^{-1}, \\ M_{(2)}^{-1} =& \tilde{Z}_2 \tilde{D}_2^{-1} \tilde{Z}_2^T \approx A_{\alpha_2}^{-1} = (A + \alpha_2 B)^{-1}. \end{aligned}$$

Then we are going to **define the updated matrix** $\tilde{Z}_\alpha$ as:

$$\tilde{Z}_\alpha = \begin{cases} \dfrac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} \tilde{Z}_1 + \dfrac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \tilde{Z}_2 & \alpha \in [\alpha_1, \alpha_2], \\ I & \alpha > \alpha_2. \end{cases}$$

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where $\tilde{Z}_{\alpha,k}$ is the matrix formed by the upper k diagonal of the Z_{α} matrix.

The preconditioner defined in this way is such that:

- the update $\tilde{Z}$-matrix of the factorization fixes the pattern of the updated matrix,
- the condition number of the $\tilde{Z}_{\alpha}$ factor is under control (corollary of the norm bound for triangular factors in [Lemeire '75]):

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Construction of the Preconditioner Update

Theoretical Results

Condition number bound

The condition number $\kappa(Z_\alpha)$ is bounded by the product of the constants:

$$\|Z_\alpha\| \leq \sqrt{n + \frac{n(n+1)}{2} M_Z^2},$$

$$\|Z_\alpha^{-1}\| \leq \frac{\sqrt{(M_Z+1)^{2n} + 2n(M_Z+2) - 1}}{M_Z+2},$$

where $M_Z = \max\left\{\max_{i<j} |\tilde{z}_{i,j}^{\alpha_1}|, \max_{i<j} |\tilde{z}_{i,j}^{\alpha_2}|\right\}$.

Construction of the Preconditioner Update

Theorem (Eigenvalues Clustering)

Theorem. Consider the sequence of linear systems in equation (3) and that there exists a $\delta \geq 0$ such that the eigenvalues of the matrix $Z_\alpha^{-T} E_k Z_\alpha^{-1} - B$ are:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_t \geq \delta > \sigma_{t+1} \geq \ldots \geq \sigma_n,$$

and $t \ll n$. Then if we have that:

$$\max_{\alpha \in (\alpha_1, \alpha_2)} |\alpha| \|D^{-1} E_k\| < \frac{1}{2}$$

we have that there exists matrices C_α, Δ and F and a constant c_α such that:

$$M_{\alpha,k}^{-1} A_\alpha = I + M_{\alpha,k}^{-1} C_\alpha + \Delta + F,$$

Construction of the Preconditioner Update

Theorem (Eigenvalues Clustering)

where $Rank(\Delta) = t \leq n$, independently from α, and:

$$\|F\|_2 \leq \frac{2|\alpha|c_\alpha \delta}{\lambda_{\min}(A_{\alpha_1})} \left(\frac{\|Z_{\alpha_2}\|_2}{\min_{r=1,2,\ldots,n} \mathbf{z}_{:,r}^{\alpha_1}} \right)^2,$$

where $Z_{\alpha_1} = (\mathbf{z}_{:,1}^{\alpha_1}, \ldots, \mathbf{z}_{:,n}^{\alpha_1})$. Besides we have that:

$$\|M_{\alpha,k}^{-1} C_\alpha\|_2 \leq \|M_{\alpha,k}^{-1} A_{\alpha_1}\| + \max_{r=1,2,\ldots,n} \frac{\lambda_{\max}(A_{\alpha_1})\|\mathbf{z}_{:,r}^{\alpha_1}\|_2^2}{\lambda_{\min}(A_{\alpha_2})\|\mathbf{z}_{:,r}^{\alpha_2}\|_2^2} \|M_{\alpha,k}^{-1} A_{\alpha_2}\|.$$

where $Z_{\alpha_2} = (\mathbf{z}_{:,1}^{\alpha_2}, \ldots, \mathbf{z}_{:,n}^{\alpha_2})$.

Construction of the Preconditioner Update

Theoretical Results

The **results of the theorem** is that,

Theorem (Eigenvalues Clustering):

We have written the product $M_{\alpha,k}^{-1}A_\alpha$ as the sum of some small norm matrices and a small rank matrix. $\Rightarrow$ the spectrum presents at least a **cluster** with **some outlier eigenvalues**.

Working with this framework we also have that if A_{α_1} and A_{α_2} are SPD with Z_α factors of bandwidth m_1 and m_2 we have

$$|z_{i,j}^{\alpha}| \leq \sqrt{d_j^{(\alpha_1)}}c_2^{(\alpha_1)}t_{(\alpha_1)}^{j-i} + \sqrt{d_j^{(\alpha_2)}}c_2^{(\alpha_2)}t_{(\alpha_2)}^{j-i},$$

$$t_{(\alpha)} = \left(\frac{\sqrt{\kappa(A_\alpha)}-1}{\sqrt{\kappa(A_\alpha)}+1}\right)^{\frac{1}{n}}.$$

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Towards the non-symmetrical case . . .
(A case that we are still analyzing)

Adapting the strategy to the non symmetric case,

$$M_{(1)}^{-1} = \tilde{Z}_1 \tilde{D}_1^{-1} \tilde{W}_1^T \approx A_{\alpha_1}^{-1} = (A + \alpha_1 B)^{-1},$$
$$M_{(2)}^{-1} = \tilde{Z}_2 \tilde{D}_2^{-1} \tilde{W}_2^T \approx A_{\alpha_2}^{-1} = (A + \alpha_2 B)^{-1}.$$

therefore,

$$\tilde{Z}_\alpha = \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} \tilde{Z}_1 + \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \tilde{Z}_2 \quad \alpha \in [\alpha_1, \alpha_2], \quad I, \alpha > \alpha_2$$
$$\tilde{W}_\alpha = \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} \tilde{W}_1 + \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \tilde{W}_2 \quad \alpha \in [\alpha_1, \alpha_2], \quad I, \alpha > \alpha_2$$

finally we have, $M_{\alpha,k}^{-1} = \tilde{Z}_\alpha (\tilde{D}_1 + \alpha E_k)^{-1} \tilde{W}_\alpha^T$, where the matrix $E_k = \tilde{Z}_{\alpha,k}^T B \tilde{W}_{\alpha,k}$ for $k \geq 1$.

Numerical Results
The Set of Matrices

To test the proposed strategy with symmetric matrices we consider the sequence:

$$(A + \alpha B)\mathbf{x}_\alpha = \mathbf{b}_\alpha, \quad \alpha \in (0, \alpha_2), \tag{5}$$

where B is the matrix representing the 5-point discretization of the laplacian operator with null Dirichlet boundary conditions.

Matrix name	$nnz(A)$	Size	κ	Results
msc01050	26198	1050	8.9970e+15	2
nasa1824	39208	1824	6.6264e+06	3

Table: Matrices from University of Florida Sparse Matrix Collection.

More experiments and details are in,

F. Durastante, *Interpolant Update of Preconditioners for Sequences of Large Linear Systems*, Mathematical Methods, Computational Techniques and Intelligent Systems (MAMECTIS '15). Vol 41, pp. 40-47, 2015, WSEAS Press.

Numerical Results

The symmetric case

SYMMETRIC TEST MATRIX FROM MSC/NASTRAN STARF8.OUT2								
		CG		*ILU*(0)		*AINV*		
α	$\kappa(A)$	T	IT	T	IT	T	IT	
0.00e+00	9.03e+15	0.41634	901†	0.50552	994†	0.04636	194	
1.49e-05	1.65e+06	0.28591	1050†	0.48536	1050†	0.25033	1050†	
2.38e-04	1.89e+05	0.13340	491	0.49070	1050†	0.24848	1050†	
1.50e-03	2.54e+04	0.08251	303	0.49052	1050†	0.25081	1050†	
3.10e-02	4.56e+03	0.05304	192	0.49249	1046†	0.25032	1050†	
2.40e-01	3.68e+03	0.06115	222	0.48309	1042	0.24669	1050	
		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		*AINV*$_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
0.00e+00	9.03e+15	**0.04602**	194	0.04616	194	53.58705	0.04560	194
1.49e-05	1.65e+06	**0.06667**	274	0.07622	274	52.54392	0.01359	59
2.38e-04	1.89e+05	**0.06082**	240	0.06610	239	51.45668	0.01009	45
1.50e-03	2.54e+04	**0.05809**	239	0.06611	235	51.49022	0.00889	40
3.10e-02	4.56e+03	0.08082	332	**0.06974**	236	50.76489	0.00784	36
2.40e-01	3.68e+03	0.13324	549	**0.00742**	32	50.42413	0.00789	32

Table: Matrix Boeing/msc01050 - Preconditioner Update.

Numerical Results

The symmetric case

STRUCTURE FROM NASA LANGLEY, 1824 DEGREES OF FREEDOM

		CG		$ILU(0)$		$AINV$		
α	$\kappa(A)$	T	IT	T	IT	T	IT	
0.00e+00	6.63e+06	0.61787	1822†	0.22174	267	0.12274	348	
1.49e-05	3.99e+05	0.61250	1818†	0.15446	250	0.06663	188	
2.38e-04	5.22e+04	0.25546	742	0.24582	404	0.08493	242	
1.50e-03	1.70e+04	0.15175	438	0.42829	703	0.14167	401	
3.10e-02	7.33e+03	0.10462	294	1.11249	1824†	0.47047	1363	
2.40e-01	6.92e+03	0.12123	286	1.10577	1824†	0.62818	1824	
		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		$AINV_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
0.00e+00	6.63e+06	**0.11800**	348	0.11991	348	227.82443	0.11829	348
1.49e-05	3.99e+05	**0.06363**	184	0.06929	183	227.92061	0.06174	183
2.38e-04	5.22e+04	**0.03801**	111	0.04261	111	226.65847	0.02739	84
1.50e-03	1.70e+04	**0.04110**	120	0.04575	119	226.58604	0.01801	55
3.10e-02	7.33e+03	0.09412	277	**0.07543**	203	225.86866	0.01579	53
2.40e-01	6.92e+03	0.19569	560	**0.01949**	64	225.43537	0.01892	61

Table: Matrix Nasa/nasa1824 - Preconditioner Update.

Numerical Results

The symmetric case

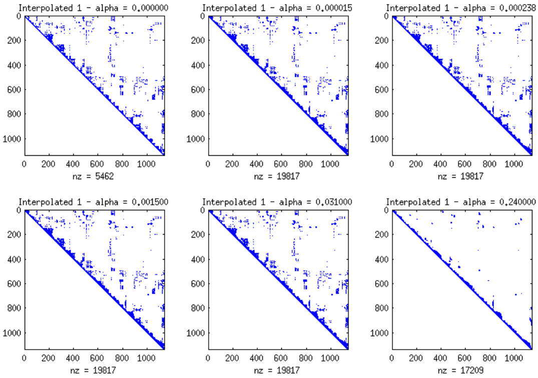


Figure: Pattern of the $\tilde{Z}_\alpha$ matrix.

The non definite and non symmetric case

Preliminary Results

The matrix $A = \hat{A} + \beta I$, with I the identity matrix and $\beta \in \mathbb{R}$ such that **some of the eigenvalues** of $\hat{A}$ has **negative real part**, see Table 4. The sequence of matrices is given by,

$$A_\alpha \mathbf{x} = \mathbf{b}_\alpha, \; A_\alpha = A + \alpha B,$$

and the B matrix is taken as the tridiagonal discretization of the laplacian operator.

Matrix name	$nnz(A)$	Size	κ	Results	Spectrum
DRIVCAV/cavity06	29675	1182	3.28e+03	5	4
DRIVCAV/cavity06 (D)	29675	1182	3.28e+03	6	5
HB/jpwh-991	6021	991	3.76e+04	7	6
HB/jpwh-991 (D)	6021	991	5.62e+04	8	7

Table: Numerical experiments for non symmetric matrices.

The non definite and non symmetric case

Preliminary Results

Driven Cavity 10 x 10, Reynolds number: 200

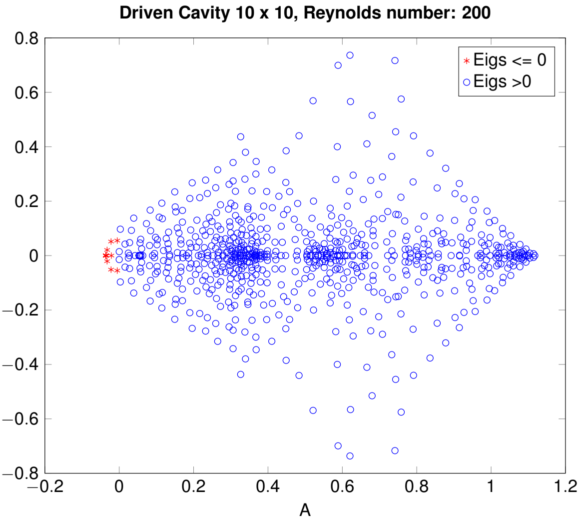
		GMRES		$ILU(0)$		$AINV$	
α	$\kappa(A)$	T	IT	T	IT	T	IT
0.00e+00	3.57e+03	16.84182†	1120 40	22.06934†	1032 40	**0.04598**	3 20
1.49e-07	3.57e+03	16.92518†	983 40	21.76088†	210 40	0.04580	3 20
2.38e-06	3.56e+03	16.86784†	288 40	21.88902†	143 40	0.04609	3 20
1.50e-05	3.56e+03	16.72715†	132 40	21.77816†	534 40	0.04656	3 21
3.10e-04	3.44e+03	16.89740†	1051 40	21.78865†	588 40	0.04708	3 22
2.40e-03	3.42e+03	16.84590†	60 40	0.59367	32 39	0.03776	2 38

		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		$AINV_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
0.00e+00	3.57e+03	0.06119	3 20	0.04599	3 20	54.70533	0.04726	3 20
1.49e-07	3.57e+03	0.05981	3 20	**0.04548**	3 20	54.70862	0.04720	3 17
2.38e-06	3.56e+03	0.05987	3 20	**0.04560**	3 20	54.68270	0.04753	3 15
1.50e-05	3.56e+03	0.06044	3 21	**0.04523**	3 20	54.82130	0.04784	3 18
3.10e-04	3.44e+03	0.05939	3 19	**0.04477**	3 18	54.73084	0.04184	3 9
2.40e-03	3.42e+03	0.04849	2 38	**0.03808**	2 39	54.67057	0.04049	2 39

Table: Results for matrix DRIVCAV/cavity06.

The non definite and non symmetric case

Preliminary Results



The non definite and non symmetric case

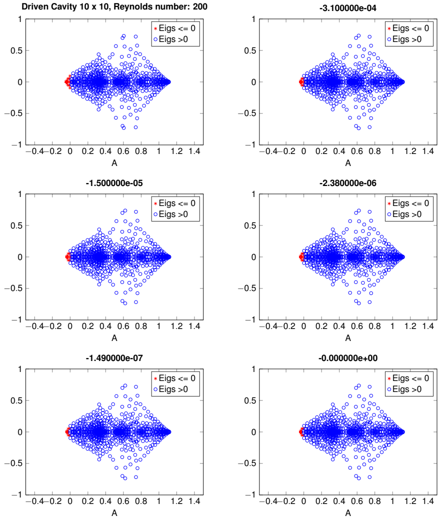
Preliminary Results

Driven Cavity 10 x 10, Reynolds number: 200								
		GMRES		*ILU*(0)		*AINV*		
α	$\kappa(A)$	T	IT	T	IT	T	IT	
-2.40e-03	5.64e+03	16.83142†	885 40	21.76976†	1105 40	0.05713	3 37	
-3.10e-04	3.74e+03	17.00274†	717 40	21.79849†	613 40	0.04678	3 21	
-1.50e-05	3.57e+03	17.08578†	352 40	21.78114†	620 40	0.04632	3 20	
-2.38e-06	3.57e+03	17.07111†	132 40	21.97115†	1075 40	0.04597	3 20	
-1.49e-07	3.57e+03	17.06821†	988 40	21.82080†	173 40	0.04632	3 20	
0.00e+00	3.57e+03	16.96038†	1120 40	21.71738†	1032 40	0.04585	3 20	
		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		$AINV_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
-2.40e-03	5.64e+03	0.07605	3 37	**0.05553**	3 37	54.57021	0.06044	3 38
-3.10e-04	3.74e+03	0.05888	3 21	**0.04549**	3 21	54.56038	0.05074	3 21
-1.50e-05	3.57e+03	0.05813	3 20	**0.04487**	3 20	54.45571	0.04200	3 11
-2.38e-06	3.57e+03	0.05887	3 20	**0.04483**	3 20	54.41646	0.05025	3 24
-1.49e-07	3.57e+03	0.05836	3 20	**0.04495**	3 20	54.58939	0.04622	3 20
0.00e+00	3.57e+03	0.05868	3 20	**0.04477**	3 20	54.68699	0.04751	3 20

Table: Results for matrix DRIVCAV/cavity06 with $\alpha < 0$.

The non definite and non symmetric case

Preliminary Results



The non definite and non symmetric case

Preliminary Results

UNSYMMETRIC MATRIX FROM PHILIPS LTD, J.P.WHELAN,1978.								
		GMRES		$ILU(0)$		$AINV$		
α	$\kappa(A)$	T	IT	T	IT	T	IT	
0.00e+00	3.76e+04	4.07420	309 20	0.05846	4 18	0.02263	1 38	
1.49e-07	3.77e+04	4.10995	310 31	0.05518	4 18	0.02405	1 38	
2.38e-06	3.78e+04	3.44219	262 35	0.05557	4 18	0.02324	1 38	
1.50e-05	3.86e+04	1.82131	139 36	0.05373	4 17	0.02258	1 38	
3.10e-04	9.03e+04	1.85225	142 40	0.04806	3 37	0.02299	1 38	
2.40e-03	9.30e+03	0.18288	14 33	0.02685	2 28	0.01955	1 33	
		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		$AINV_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
0.00e+00	3.76e+04	0.02975	1 38	**0.02249**	1 38	36.76922	0.02439	1 38
1.49e-07	3.77e+04	0.02898	1 38	**0.02343**	1 38	36.68682	0.02783	1 38
2.38e-06	3.78e+04	0.02881	1 38	**0.02235**	1 38	37.05723	0.02867	1 39
1.50e-05	3.86e+04	0.02943	1 38	**0.02248**	1 38	36.67420	0.02854	1 39
3.10e-04	9.03e+04	0.02885	1 38	**0.02211**	1 38	36.91389	0.02748	1 38
2.40e-03	9.30e+03	0.02306	1 32	**0.01686**	1 32	36.66777	0.02177	1 32

Table: Results for matrix HB/jpwh-991.

The non definite and non symmetric case

Preliminary Results

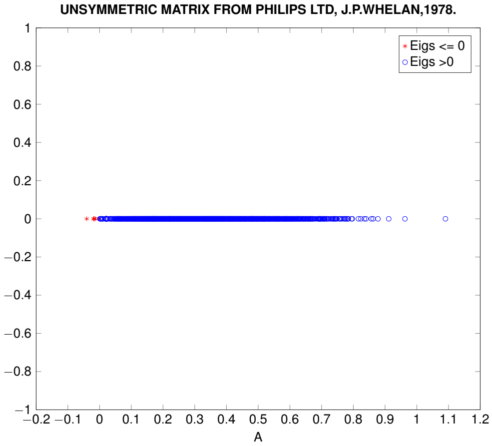


Figure: Specturm of the matrix HB/jpwh-991.

The non definite and non symmetric case

Preliminary Results

UNSYMMETRIC MATRIX FROM PHILIPS LTD, J.P.WHELAN,1978.

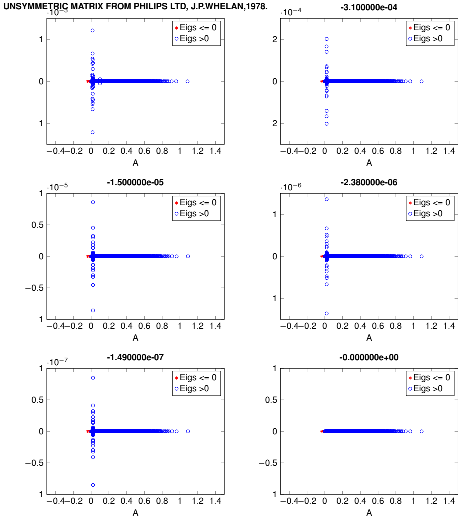
		GMRES		$ILU(0)$		$AINV$	
α	$\kappa(A)$	T	IT	T	IT	T	IT
-2.40e-03	5.62e+04	2.14150	169 37	0.20322	13 26	0.03826	2 27
-3.10e-04	6.94e+04	4.91671	390 1	0.07041	5 15	0.02445	1 40
-1.50e-05	2.30e+04	2.99612	232 3	0.05641	4 21	0.02406	1 39
-2.38e-06	3.75e+04	3.55586	275 28	0.05669	4 20	0.02247	1 38
-1.49e-07	3.76e+04	4.00763	307 35	0.05733	4 19	0.02276	1 38
0.00e+00	3.76e+04	4.00790	309 20	0.05592	4 18	0.02270	1 38

		$M_{\alpha,1}^{-1}$ shift		$M_{\alpha,1}^{-1}$ interp		$AINV_{rc}$		
α	$\kappa(A)$	T	IT	T	IT	T	T	IT
-2.40e-03	5.62e+04	0.04714	2 25	**0.03617**	2 25	37.13643	0.04999	2 30
-3.10e-04	6.94e+04	0.03075	2 1	**0.02440**	2 1	36.45920	0.02890	2 1
-1.50e-05	2.30e+04	0.02894	1 39	**0.02265**	1 39	36.60496	0.02439	1 39
-2.38e-06	3.75e+04	0.02745	1 38	**0.02193**	1 38	37.00270	0.02841	1 39
-1.49e-07	3.76e+04	0.02800	1 38	**0.02187**	1 38	39.16368	0.02576	1 38
0.00e+00	3.76e+04	0.02770	1 38	**0.02180**	1 38	42.80209	0.03300	1 38

Table: Results for matrix HB/jpwh-991 with $\alpha < 0$..

The non definite and non symmetric case

Preliminary Results



Thanks for your attention!