Low-rank techniques for high-dimensional problems

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What is a tensor?

Tensor is a multi-dimensional array of numbers.

A d-th order tensor $\mathcal{X}$ of size $n_1 \times n_2 \times \cdots \times n_d$ is a d-dimensional array with entries

$$\mathcal{X}_{i_1,i_2,\ldots,i_d}, \qquad i_\mu \in \{1,\ldots,n_\mu\} \text{ for } \mu = 1,\ldots,d.$$

$$\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}.$$

Multi-index notation:

$$\mathfrak{I} = \{1,\ldots,n_1\} \times \{1,\ldots,n_2\} \times \cdots \times \{1,\ldots,n_d\}.$$

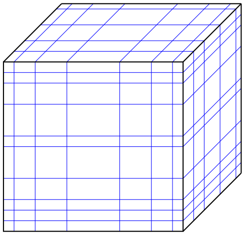
Then $i \in \mathfrak{I}$ is a tuple of d indices:

$$i = (i_1, i_2, \ldots, i_d).$$

Allows to write entries of $\mathcal{X}$ as $\mathcal{X}_i$ for $i \in \mathfrak{I}$.

Tensors from multivariate functions

Discretization of function $f(\xi_1, \dots, \xi_d)$
on hypercube $\rightsquigarrow$ tensor of order d

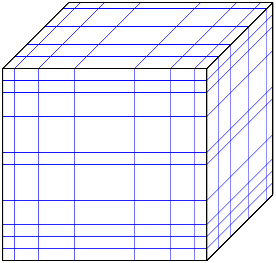


High-dimensional elliptic PDEs: 3D model problem

- Consider

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

  on unit cube $\Omega = [0,1]^3$.



- Discretize on tensor grid.
  Uniform grid for simplicity:

$$\xi_\mu^{(j)} = jh, \quad h = \frac{1}{n+1}$$

  for $\mu = 1, 2, 3$.
- Approximate solution tensor $\mathcal{U} \in \mathbb{R}^{n\times n\times n}$:

$$\mathcal{U}_{i_1,i_2,i_3} \approx u\big(\xi_1^{(i_1)}, \xi_2^{(i_2)}, \xi_3^{(i_3)}\big).$$

High-dimensional elliptic PDEs: Arbitrary dimensions

Finite difference discretization of model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

for $\Omega = [0,1]^d$ takes the form

$$\Big(\sum_{j=1}^{d} I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I\Big)\mathbf{u} = \mathbf{f}.$$

To obtain such Kronecker structure in general:

- tensorized domain;
- highly structured grid;
- coefficients that can be written/approximated as sum of separable functions.

Low-rank approximation of solution tensor:
Highly active research area.

Tensors from Taylor expansion

$f : \mathbb{R}^m \to \mathbb{R}^n$ sufficiently smooth:

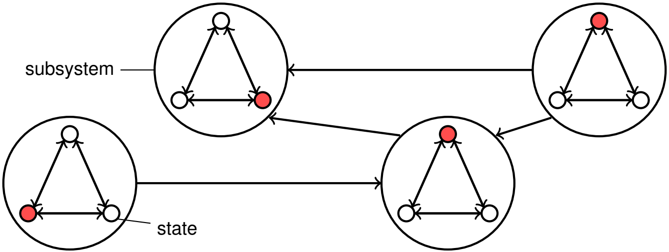
$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{1}{2} f''(x)[h,h] + \frac{1}{3!} f^{(3)}(x)[h,h,h] \\ &\quad + \frac{1}{4!} f^{(4)}(x)[h,h,h,h] + \frac{1}{5!} f^{(5)}(x)[h,h,h,h,h] + \cdots \end{aligned}$$

with multilinear map $f^{(k)} : \mathbb{R}^{m\times\cdots\times m} \to \mathbb{R}^n$ represented by $(k+1)$-dimensional tensor.

Low-rank approximation of $f^{(k)}$ used in:

- Model reduction and simulation of nonlinear circuits [Liu/Daniel/Wong'2015]
- Approximate representation and computation of random fields in UQ [DK/Kumar/Nobile/Tobler'2015], [Bonizzoni/Nobile/DK'2014].

Tensors from interconnected systems



- Set of subsystems with finite number of states interacting with each other.
- State of whole system characterized by combined states of subsystems.
- d subsystems with n states $\rightsquigarrow$ n^d states in total!

Low rank tensors in reliability analysis:

- Approximation of d-dimensional tensors containing joint probability distributions.
- [Buchholz'2011], [DK/Macedo'2014].

Low-rank tensor techniques

- Emerged during last 5 – 10 years in numerical analysis.
- Successfully applied to:
  - parameter-dependent / multi-dimensional integrals;
  - electronic structure calculations: Hartree-Fock / DFT;
  - stochastic and parametric PDEs;
  - high-dimensional Boltzmann / chemical master / Fokker-Planck / Schrödinger equations;
  - micromagnetism;
  - rational approximation problems;
  - computational homogenization;
  - computational finance;
  - multivariate regression and machine learning;
  - queuing models;
  - ...
- For references on these applications, see
  - W. Hackbusch (2012). Tensor Spaces and Numerical Tensor Calculus, Springer.
  - L. Grasedyck, DK, Ch. Tobler (2013). A literature survey of low-rank tensor approximation techniques. GAMM-Mitteilungen, 36(1).

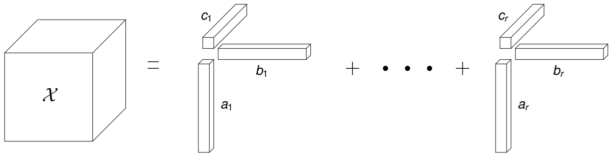
Rank of a tensor

Rank-1 tensor = outer product of vectors:

$$\mathcal{X} = a \circ b \circ c.$$

CANDECOMP/PARAFAC (CP) decomposition =
(seemingly) natural extension:

$$\mathcal{X} = a_1 \circ b_1 \circ c_1 + a_2 \circ b_2 \circ c_2 + \cdots + a_r \circ b_r \circ c_r.$$



CP decomposition

- CP decomposition offers low data-complexity; for constant r:

  linear complexity in d.

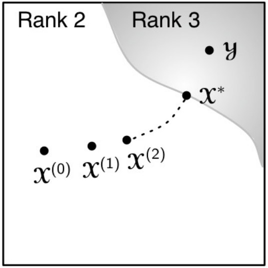
Theoretical issues:

For tensors of order $d \geq 3$:

- tensor rank r is **not** upper semi-continuous $\rightsquigarrow$

  lack of closedness

- successive rank-1 approximations fail
- all algorithms based on optimization techniques (ALS, Gauss-Newton)



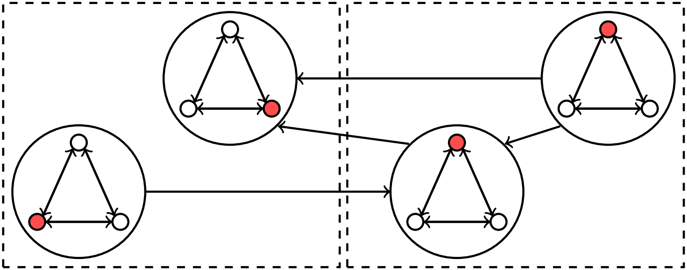
Picture taken from [Kolda/Bader'2009].

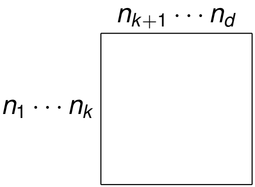
Practical issues:

- No SVD-based compression possible.
- CP ignores locality of interactions!

SVD-based formats

Aggregate interconnected system into two sub-systems
$\hat{=}$ **matricization of tensor** $\mathcal{X}$.

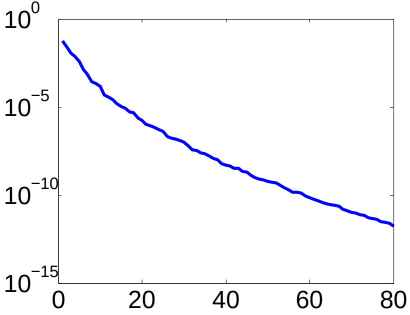




Aggregation of $1, ..., k$, and $k+1, ..., d$.

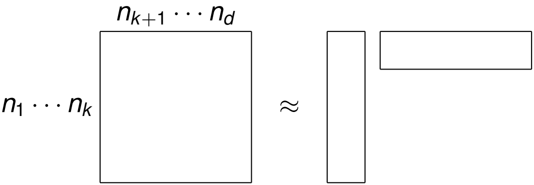
SVD-based formats

- Singular value decay of matricization for 'large overflow model'.
- First two systems mapped to rows $\Rightarrow \mathbb{R}^{n_1 n_2 \times n_3 n_4}$ matrix.
- $n_1 = n_2 = n_3 = n_4 = 40$, plot first 80 (from a total of 1600) singular values.



SVD-based formats

Low-rank approximation of matricization:



Insufficient for high dimensions.

Solution:
Consider tensor format based on *aggregations into* $\{1, ..., k\}$ *and* $\{k+1, ..., d\}$, *for all* $k = 1, ..., d$.

$\rightsquigarrow$ TT format.

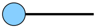
Tensor network diagrams

Tensor network diagrams

- Introduced by Roger Penrose.
- Heavily used in quantum mechanics (spin networks).

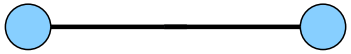
This is a scalar $\gamma \in \mathbb{R}$

This is a vector $x \in \mathbb{R}^n$



These are two vectors $x, y \in \mathbb{R}^n$

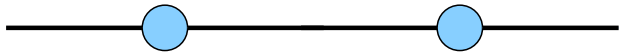
This is the inner product between $x, y \in \mathbb{R}^n$



$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

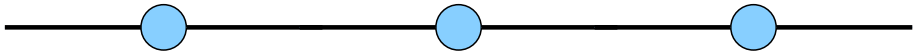
These are two matrices A, B

This is the matrix product $C = AB$



$$C_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}$$

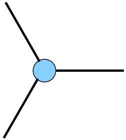
This is the matrix product $C = U\Sigma V^T$



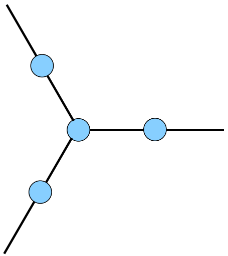
$$C_{ij} = \sum_{k=1}^{r}\sum_{\ell=1}^{r} U_{ik}\Sigma_{k\ell}V_{j\ell}$$

If $r \ll n$: Implicit representation of C via smaller matrices U, V, Σ.

This is a tensor $\mathcal{X}$ of order 3



This is a tensor $\mathcal{X}$ of order 3 in Tucker decomposition



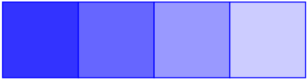
$$\mathcal{X}_{ijk} = \sum_{\ell_1=1}^{r_1} \sum_{\ell_2=1}^{r_2} \sum_{\ell_3=1}^{r_3} \mathcal{C}_{\ell_1 \ell_2 \ell_3} U_{i\ell_1} V_{j\ell_2} W_{k\ell_3}$$

Implicit representation of $\mathcal{X}$ via

- $r_1 \times r_2 \times r_3$ core tensor $\mathcal{C}$
- $n_1 \times r_1$ matrix U spans first mode
- $n_2 \times r_2$ matrix V spans second mode
- $n_3 \times r_3$ matrix W spans third mode.

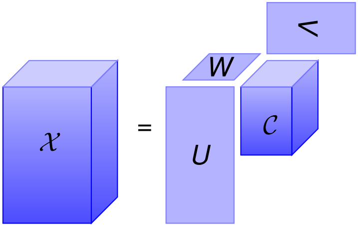
Tucker decomposition & multilinear rank

Reshape tensor into matrix by slicing, e.g. for first dimension:

$\mathcal{X} =$  $\rightsquigarrow \quad X_{(1)} =$  $\in \mathbb{R}^{n_1 \times (n_2 \cdot n_3)}$

Multilinear rank of tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ defined by tuple

$$\mathbf{r} = (r_1, r_2, r_3), \quad \text{with} \quad r_i = \operatorname{rank}\big(X_{(i)}\big).$$



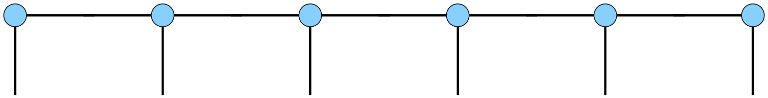
Representation of rank-**r**-tensor:

Tucker decomposition:

$$\mathcal{X} = \mathcal{C} \times_1 U \times_2 V \times_3 W$$

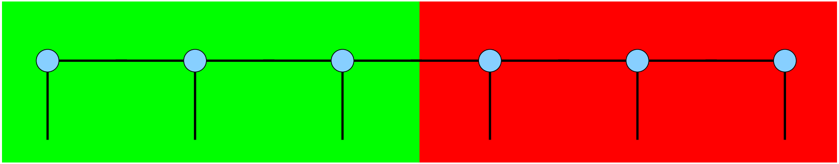
$U \in \mathbb{R}^{n_1 \times r_1}$, $V \in \mathbb{R}^{n_2 \times r_2}$, $W \in \mathbb{R}^{n_3 \times r_3}$, and core tensor $\mathcal{C} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$

Six-dimensional tensor $\mathcal{X}$ in TT format



- $\mathcal{X}$ implicitly represented by four $r \times n \times r$ tensors and two $n \times r$ matrices
- Quantum mechanics: MPS (matrix product states)
- Matrix-based tensor formats introduced in numerical analysis by Grasedyck, Hackbusch, Kühn, Oseledets, Tyrtishnikov.

Six-dimensional tensor $\mathcal{X}$ in TT format



This partition corresponds to low-rank factorization

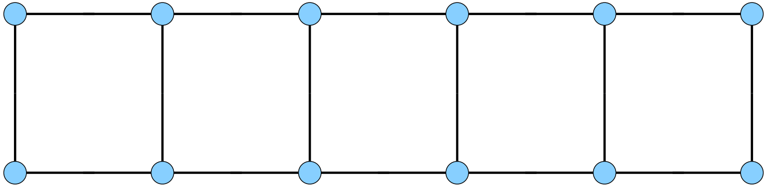
$$X^{(1,2,3)} = UV^T, \qquad X^{(1,2,3)} \in \mathbb{R}^{n_1 n_2 n_3 \times n_4 n_5 n_6},\ U \in \mathbb{R}^{n_1 n_2 n_3 \times r_3},\ V \in \mathbb{R}^{n_4 n_5 n_6 \times r_3}$$

$X^{(1,2,3)}$ is matricization of $\mathcal{X}$:

Merge multi-indices $(1,2,3)$ into row indices and
multi-indices $(4,5,6)$ into column indices

The ranks of $X^{(1,\dots,\mu)}$ for $\mu = 1,\dots,d-1$ are the TT ranks of $\mathcal{X}$.

Inner product of two tensors in TT format



- Carrying out contractions requires $O(dnr^4)$ instead of $O(n^d)$ operations for tensors of order d.

Operations with tensors in TT format

Easy:

- inner product, 2-norm
- multiplication with Kronecker structured $\mathcal{A}$
- recompression/truncation
- (partial) contractions

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Hard:

- almost everything else

Operations with tensors in TT format

Easy:

- inner product, 2-norm
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- recompression/truncation
- (partial) contractions

Hard:

- almost everything else

2 classes of algorithms for solving (linear algebra) problems:

- iterate and truncate
- constrain and optimize

All under the assumption that ranks stay small!

TT format for function-related tensors

- When to expect good low-rank approximations?
- Approximation error from separation wrt to $\{x_1, \dots, x_a\}$:

$$f(x_1, \dots, x_a, x_{a+1}, \dots, x_d) \approx \sum_{k=1}^{r} g_k(x_1, \dots, x_a) h_k(x_{a+1}, \dots, x_d)$$

  for $a = 1, \dots, d-1$.
- For analytic functions

$$\text{error} \lesssim \exp(-r^{\max\{1/a, 1/(d-a)\}}).$$

- [Temlyakov'1992, Uschmajew/Schneider'2013]: For $f \in B^{s,\text{mix}}$

$$\text{error} \lesssim r^{-2s} (\log r)^{2s(\max\{a, d-a\}-1)}.$$

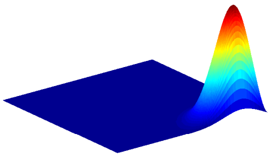
Smoothness is neither sufficient nor necessary for high dimensions!

ALS-based algorithms

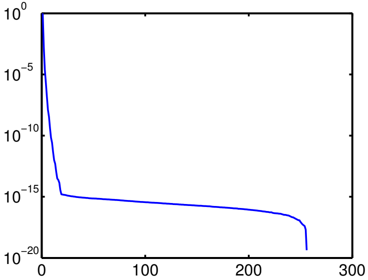
2D eigenvalue problem

- $-\triangle u(x) + V(x)u = \lambda u(x)$ in $\Omega = [0,1] \times [0,1]$ with Dirichlet b.c. and Henon-Heiles potential V
- Regular discretization
- Reshaped ground state into matrix

Ground state



Singular values



Excellent rank-10 approximation possible

Rayleigh quotients wrt low-rank matrices

Consider symmetric $n^2 \times n^2$ matrix $\mathcal{A}$. Then

$$\lambda_{\min}(\mathcal{A}) = \min_{x \neq 0} \frac{\langle x, \mathcal{A}x \rangle}{\langle x, x \rangle}.$$

We now...

- reshape vector x into $n \times n$ matrix X;
- reinterpret $\mathcal{A}x$ as linear operator $\mathcal{A} : X \mapsto \mathcal{A}(X)$.

Rayleigh quotients wrt low-rank matrices

Consider symmetric $n^2 \times n^2$ matrix $\mathcal{A}$. Then

$$\lambda_{\min}(\mathcal{A}) = \min_{X \neq 0} \frac{\langle X, \mathcal{A}(X) \rangle}{\langle X, X \rangle}$$

with matrix inner product $\langle \cdot, \cdot \rangle$. We now...

- restrict X to low-rank matrices.

Rayleigh quotients wrt low-rank matrices

Consider symmetric $n^2 \times n^2$ matrix $\mathcal{A}$. Then

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X = UV^T \neq 0} \frac{\langle X, \mathcal{A}(X) \rangle}{\langle X, X \rangle}.$$

- Approximation error governed by low-rank approximability of X.
- Solved by Riemannian optimization techniques or ALS.

ALS

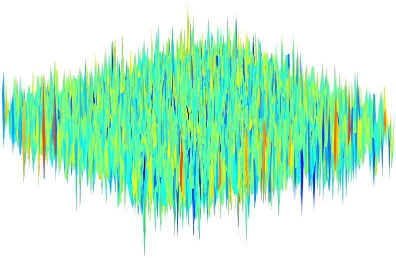
ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X) \rangle}{\langle X, X \rangle}.$$

Initially:

- fix target rank r
- $U \in \mathbb{R}^{m \times r}$, $V^{n \times r}$ randomly, such that V is ONB

$\tilde{\lambda} - \lambda = 6 \times 10^3$
residual $= 3 \times 10^3$



ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X)\rangle}{\langle X, X\rangle}.$$

Fix V, optimize for U.

$$\begin{aligned}
\langle X, \mathcal{A}(X)\rangle &= \mathrm{vec}(UV^T)^T \mathcal{A}\,\mathrm{vec}(UV^T) \\
&= \mathrm{vec}(U)^T (V \otimes I)^T \mathcal{A} (V \otimes I) \mathrm{vec}(U)
\end{aligned}$$

$\rightsquigarrow$ Compute smallest eigenvalue of reduced matrix ($rn \times rn$) matrix

$$(V \otimes I)^T \mathcal{A} (V \otimes I).$$

Note: Computation of reduced matrix benefits from Kronecker structure of $\mathcal{A}$.

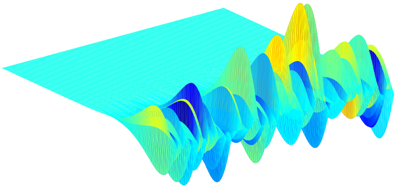
ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X)\rangle}{\langle X, X\rangle}.$$

Fix V, optimize for U.

$\tilde{\lambda} - \lambda = 2 \times 10^3$
residual $= 2 \times 10^3$



ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X)\rangle}{\langle X, X\rangle}.$$

Orthonormalize U, fix U, optimize for V.

$$\begin{aligned}\langle X, \mathcal{A}(X)\rangle &= \mathrm{vec}(UV^T)^T \mathcal{A}\, \mathrm{vec}(UV^T) \\ &= \mathrm{vec}(V^T)(I \otimes U)^T \mathcal{A}(I \otimes U)\mathrm{vec}(V^T)\end{aligned}$$

$\rightsquigarrow$ Compute smallest eigenvalue of reduced matrix ($rn \times rn$) matrix

$$(I \otimes U)^T \mathcal{A}(I \otimes U).$$

Note: Computation of reduced matrix benefits from Kronecker structure of $\mathcal{A}$.

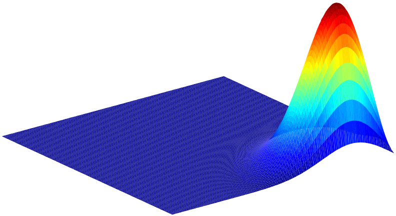
ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X = UV^T \neq 0} \frac{\langle X, \mathcal{A}(X) \rangle}{\langle X, X \rangle}.$$

Orthonormalize U, fix U, optimize for V.

$\tilde{\lambda} - \lambda = 1.5 \times 10^{-7}$
residual $= 7.7 \times 10^{-3}$



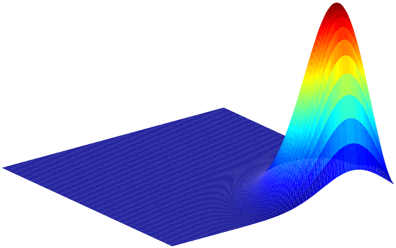
ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X)\rangle}{\langle X, X\rangle}.$$

Orthonormalize V, fix V, optimize for U.

$\tilde{\lambda} - \lambda = 1 \times 10^{-12}$
residual $= 6 \times 10^{-7}$



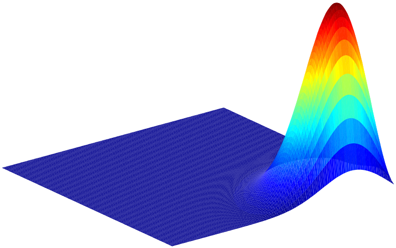
ALS

ALS for solving

$$\lambda_{\min}(\mathcal{A}) \approx \min_{X=UV^T \neq 0} \frac{\langle X, \mathcal{A}(X)\rangle}{\langle X, X\rangle}.$$

Orthonormalize U, fix U, optimize for V.

$\tilde{\lambda} - \lambda = 7.6 \times 10^{-13}$
residual $= 7.2 \times 10^{-8}$

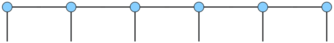


ALS for TT format

Originates from quantum mechanics = one-site DMRG.

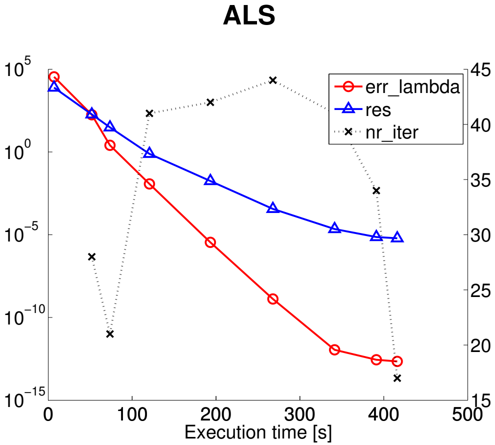
Goal:

$$\min\left\{\frac{\langle \mathcal{X}, \mathcal{A}(\mathcal{X})\rangle}{\langle \mathcal{X}, \mathcal{X}\rangle} : \mathcal{X} \in \mathcal{M}_{\mathbf{r}},\ \mathcal{X} \neq 0\right\}$$

$\mathcal{M}_{\mathbf{r}} =$ 

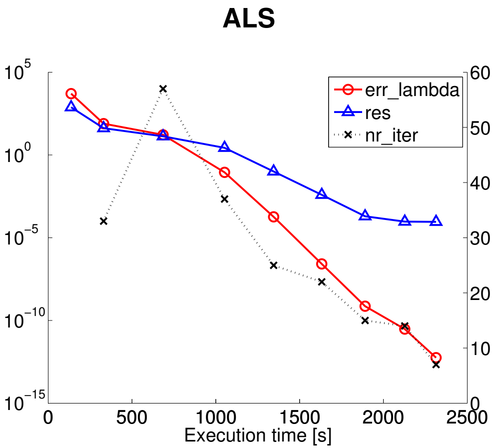
ALS: Choose one node t, fix all other nodes, set new tensor at node t to minimize Rayleigh quotient $\frac{\langle \mathcal{X}, \mathcal{A}(\mathcal{X})\rangle}{\langle \mathcal{X}, \mathcal{X}\rangle}$. This is done for all nodes (a sweep), and sweeps are continued until convergence.

Numerical Experiments - Sine potential, $d = 10$



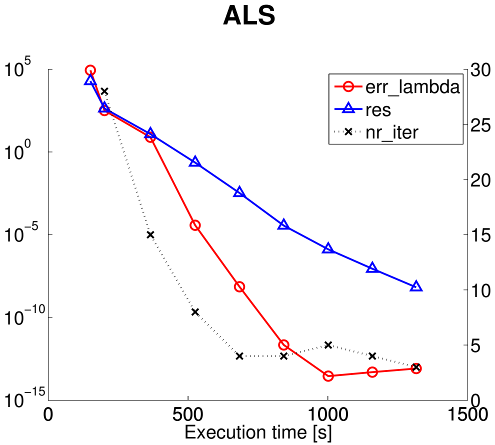
Size = $128^{10} \approx 10^{21}$. Maximal TT rank 40.

Numerical Experiments - Henon-Heiles, $d = 20$



Size = $128^{20} \approx 10^{42}$. Maximal TT rank 40.

Numerical Experiments - $1/\|\xi\|_2$ potential, $d = 20$

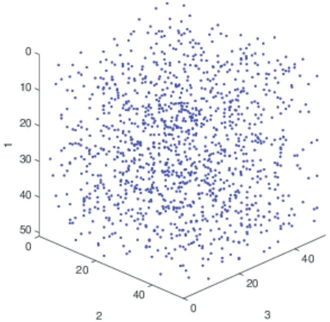


Size = $128^{20} \approx 10^{42}$. Maximal TT rank 30.

Manifold-based algorithms

Robust low-rank tensor completion

$$\begin{array}{ll} \underset{\mathcal{X}}{\text{minimize}} & f(\mathcal{X}) := \frac{1}{2}\|P_\Omega(\mathcal{X} - \mathcal{A})\|^2 \\ \text{subject to} & \mathcal{X} \in \mathcal{M}_{\mathbf{r}} \end{array}$$



Applications:

- Completion of multidimensional data, e.g. hyperspectral images, CT Scans
- Compression of multivariate functions with singularities
- Non-intrusive methods for stochastic PDEs
- Context-aware recommender systems
- ...

Manifold of low-rank tensors

$$\mathcal{M}_{\mathbf{r}} := \left\{ \mathcal{X} \in \mathbb{R}^{n_1 \times \cdots \times n_d} : \mathrm{rank}(\mathcal{X}) = \mathbf{r} \right\}$$

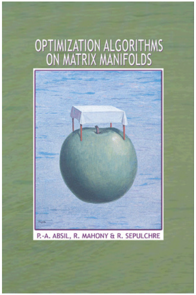
- $\mathcal{M}_{\mathbf{r}}$ is a smooth manifold for TT format and its variants.
  [Holtz/Rohwedder/Schneider'2012], [Uschmajew/Vandereycken'2012]
- Riemannian with metric induced by standard inner product
  $\langle \mathcal{X}, \mathcal{Y} \rangle = \langle \mathcal{X}_{(1)}, \mathcal{Y}_{(1)} \rangle$ *(sum of element-wise product)*

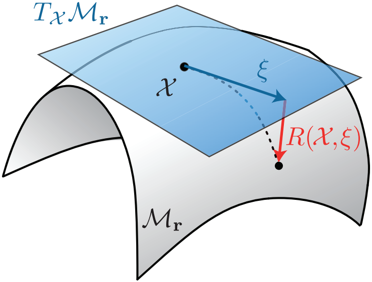
Manifold structure used in

- dynamical low-rank approximation
  [Koch/Lubich'2010], [Arnold/Jahnke'2012],
  [Lubich/Rohwedder/Schneider/Vandereycken'2012],
  [Khoromskij/Oseledets/Schneider'2012], ...
- best multilinear approximation [Eldén/Savas'2009], [Ishteva/Absil/Van Huffel/De Lathauwer'2011], [Curtef/Dirr/Helmke'2012]
- robust tensor completion [DK/Steinlechner/Vandereycken'2013]

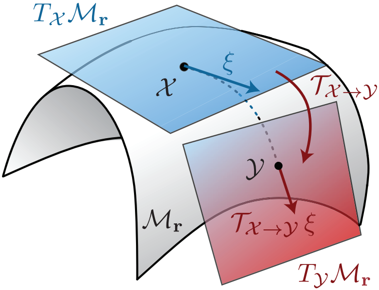
Riemannian optimization in a nutshell

- optimize in direction of Riemannian gradient
- combine different directions using vector transport





Retraction



Vector transport

Geometric nonlinear CG for tensor completion

Input: Initial guess $\mathcal{X}_0 \in \mathcal{M}_{\mathbf{r}}$.

$\eta_0 \leftarrow -\mathrm{grad}\, f(\mathcal{X}_0)$

$\alpha_0 \leftarrow \mathrm{argmin}_\alpha\, f(\mathcal{X}_0 + \alpha\eta_0)$

$\mathcal{X}_1 \leftarrow R_{\mathcal{X}_0}(\alpha_0\eta_0)$

for $i = 1, 2, \dots$ **do**

Compute gradient:

$\xi_i \leftarrow \mathrm{grad}\, f(\mathcal{X}_i)$

Conjugate direction by PR+ updating rule:

$\eta_i \leftarrow -\xi_i + \beta_i \mathcal{T}_{\mathcal{X}_{i-1}\to\mathcal{X}_i} f(\eta_{i-1})$

Initial step size from linearized line search:

$\alpha_i \leftarrow \mathrm{argmin}_\alpha\, f(\mathcal{X}_i + \alpha\eta_i)$

Armijo backtracking for sufficient decrease:

Find smallest integer $m \geq 0$ such that

$f(\mathcal{X}_i) - f(R_{\mathcal{X}_i}(2^{-m}\alpha_i\eta_i)) \geq -1 \cdot 10^{-4}\langle \xi_i, 2^{-m}\alpha_i\eta_i \rangle$

Obtain next iterate:

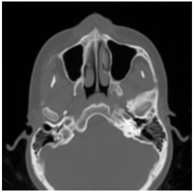
$\mathcal{X}_{i+1} \leftarrow R_{\mathcal{X}_i}(2^{-m}\alpha_i\eta_i)$

end for Cost/iteration: $O(nr^d + |\Omega| r^{d-1})$ ops.

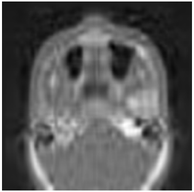
Reconstruction of CT Scan

$199 \times 199 \times 150$ tensor from scaled CT data set "INCISIX",
*(taken from OSIRIX MRI/CT data base
[www.osirix-viewer.com/datasets/])*

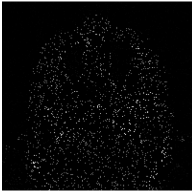
Slice of original tensor



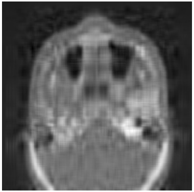
HOSVD approx. of rank 21



Sampled tensor (6.7%)



Low-rank completion of rank 21



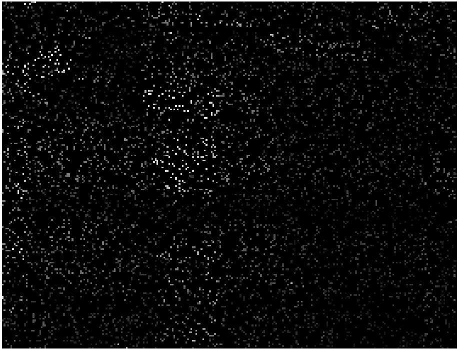
Compares very well with existing results w.r.t. low-rank recovery and speed, e.g., [Gandy/Recht/Yamada/'2011].

Hyperspectral Image

Set of photographs, (204×268 px) taken across a large range of wavelengths. 33 samples from ultraviolet to infrared [Image data: Foster et al.'2004]

Stacked into a tensor of size $204 \times 268 \times 33$

10% of the Original Hyperspectral Imega Tensor, 16th Slice
Size of Tensor is [204, 268, 33]



Completed Tensor, 16th Slice
Final Rank is k = [50 50 6]

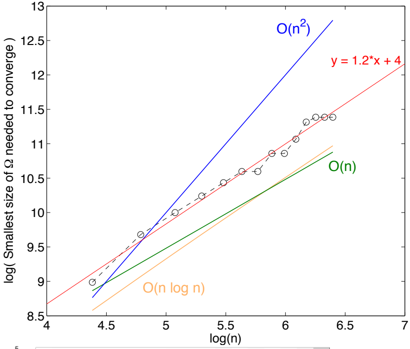
Here: Only 10% of entries known; [Signoretti et al.'2011] use 50%.

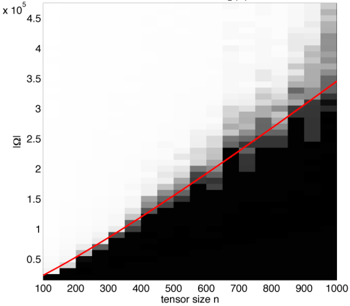
How many samples do we need?

Matrix case:
$O(n \cdot \log^{\beta} n)$ samples suffice!
[Candès/Tao'2009]
$\Rightarrow$ *Completion of tensor by applying matrix completion to matricization:* $O(n^2 \log(n))$. Gives upper bound!

Tensor case:
Certainly: $|\Omega| \ll O(n^2)$
In all cases of convergence
$\rightsquigarrow$ exact reconstruction.

Conjecture: $|\Omega| = O(n \cdot \log^{\beta} n)$





Conclusions and Outlook

- Scientific computing with low-rank tensors rapidly evolving field and highly technical.
- Low-rank tensors successfully solve certain classes of high-dimensional problems.
- Precise scope of applications far from clear; many applications remain to be explored. More analysis needed!

Software packages:

- MATLAB: Tensor toolbox, *N*-way toolbox, Tensorlab, TT-toolbox, `htucker`
- Python: `ttpy`
- C/C++: ALPS, TensorCalculus library
- Julia: TensorOperations.jl
- ...