Groups and Symmetries in Numerical Linear Algebra
Part 2a

Hans Munthe-Kaas

[1] Department of Mathematics
University of Bergen
Norway
http://hans.munthe-kaas.no
mailto:hans@math.uib.no

CIME-EMS Summer School
Cetraro, Italy, 2015

Menu Wednesday:
Abelian groups, lattices and kaleidoscopes

Locally Compact Abelian groups (LCA)

$$G = \{G, +, 0\}, \quad a + b = b + a$$

Classical groups:
(all you get from $\mathbb{R}$ and $\mathbb{Z}$ by products, subgroups and quotients)

Reals	$\mathbb{R}$
Circle	$T = \mathbb{R}/\mathbb{Z}$
Integers	$\mathbb{Z}$
Discrete circle	$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$
FGA	$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \times \mathbb{Z}^m$

Theorem

- *All Finitely Generated Abelian groups are of the form FGA .*
- *All Finite Abelian groups (FA) are of the form FGA with* $m = 0$.
- $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ *if and only if* $\gcd(m, n) = 1$.

Isomorphism: $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ iff $\gcd(m, n) = 1$

Euclidean algorithm

$$(g, a, b) = \gcd(m, n)$$

finds

$$am + bn = g = \gcd(m, n).$$

Isomorphisms: (Chinese remainder theorem)

$$(b, a)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = bx_1 + ax_2 : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_{mn}$$

$$\begin{pmatrix} n \\ m \end{pmatrix} y = \begin{pmatrix} ny \\ my \end{pmatrix} : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$$

Pontryagin duality

All LCAs come in *dual pairs*, G, $\widehat{G}$ s.t. $\widehat{\widehat{G}} \simeq G$.

Definition

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times})$$

(The irreducible representations of G).

Means: $\widehat{G} = \{\, \chi\colon G \to \mathbb{C}^{\times} \mid \chi(j+k) = \chi(j)\chi(k), \quad \chi \text{ continuous} \,\}$.

Note: $\widehat{G}$ is an LCA with multiplicative group with product

$$(\chi \cdot \chi')(j) = \chi(j)\chi'(j)$$

(and compact-open topology).

Dual pairs of LCAs

Definition

G and $\widehat{G}$ form a dual pair, if there exist a *pairing* $\langle\cdot,\cdot\rangle\colon \widehat{G}\times G\to\mathbb{C}^\times$ s.t.

$$\left\{\langle k,\cdot\rangle \mid k\in\widehat{G}\right\} = \mathrm{Hom}(G,\mathbb{C}^\times)$$

$$\{\langle\cdot,j\rangle \mid j\in G\} = \mathrm{Hom}(\widehat{G},\mathbb{C}^\times)$$

G	$\widehat{G}$	$(\cdot,\cdot)$	$\widehat{f}(\cdot)$	$f(\cdot)$
$x\in\mathbb{R}$	$\omega\in\mathbb{R}$	$e^{2\pi i\omega x}$	$\int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)dx$	$\int_{-\infty}^{\infty} e^{2\pi i\omega x}\widehat{f}(\omega)d\omega$
$x\in T$	$k\in\mathbb{Z}$	$e^{2\pi ikx}$	$\int_0^1 e^{-2\pi ikx} f(x)dx$	$\sum_{k=-\infty}^{\infty} e^{2\pi ikx}\widehat{f}(k)$
$j\in\mathbb{Z}_n$	$k\in\mathbb{Z}_n$	$e^{\frac{2\pi ikj}{n}}$	$\sum_{j=0}^{n-1} e^{\frac{-2\pi ikj}{n}} f(j)$	$\frac{1}{n}\sum_{k=0}^{n-1} e^{\frac{2\pi ikj}{n}}\widehat{f}(k)$

The Fourier transform

Haar measure

Groups have a unique translation invariant measure,

$$\mathbb{R}: \int_{-\infty}^{\infty} f(x)dx, \quad \mathbb{Z}: \sum_{j=-\infty}^{\infty} f(j).$$

Inner product on $L^2(G)$: $(f,g) = \int_G \overline{f(x)}g(x)dx$.

The Fourier tranform $\boldsymbol{F}\colon \mathbb{C}G \to \mathbb{C}\widehat{G}$

$$\boldsymbol{F}(f)(\xi) \equiv \widehat{f}(\xi) = \int_{x\in G} \overline{\langle \xi, x\rangle} f(x)dx$$

$$\boldsymbol{F}^{-1}(\widehat{f})(x) = \frac{1}{c}\int_{\xi\in\widehat{G}} \langle \xi, x\rangle \widehat{f}(\xi)d\xi \quad \text{for some } c.$$

$\boldsymbol{F}\colon L^2G \to L^2(\widehat{G})$ is unitary and $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

Group homomorphisms, lattices and sampling

$$\mathrm{Hom}(G,H) = \{\, \phi\colon G \to H \mid \phi(j+k) = \phi(j) + \phi(k), \phi \text{ continuous} \,\}$$

Dual homomorphism

Any $\phi \in \mathrm{Hom}(G,H)$ has a dual $\widehat{\phi} \in \mathrm{Hom}(\widehat{H}, \widehat{G})$ defined such that

$$\langle \widehat{\phi}(\xi), x\rangle_G = \langle \xi, \phi(x)\rangle_H, \quad \text{for all } x \in H \text{ and } \xi \in \widehat{G}.$$

Example:

$\phi\colon \mathbb{Z} \to \mathbb{R}$, $\phi(j) = j\delta x$, $\delta x \in \mathbb{R}$.

$$\langle \xi, \phi(j)\rangle = e^{2\pi i \xi j \delta x}$$
$$\langle \widehat{\phi}(\xi), j\rangle_{\mathbb{Z}} = e^{2\pi i \phi(\xi) j}$$
$$\Rightarrow \widehat{\phi}(\xi) = \frac{\xi}{\delta x}$$

A wee bit homological algebra

For $\phi \in \mathrm{Hom}(H, G)$, define

$$\ker(\phi) = \{\, h \in H \mid \phi(h) = 0 \,\}$$
$$\mathrm{Im}(\phi) = \{\, g \in G \mid g = \phi(h) \,\}$$

A *chain complex* is a sequence of homomorphisms

$$G_1 \stackrel{\phi_1}{\to} G_2 \stackrel{\phi_2}{\to} G_3 \stackrel{\phi_3}{\to} \cdots$$

such that $\mathrm{Im}(\phi_i) \subset \ker(\phi_{i+1})$, or $\phi_{i+1} \circ \phi_i \equiv 0$ for all i.
The sequence is *exact* if $\mathrm{Im}(\phi_i) = \ker(\phi_{i+1})$.

Exact sequences, examples.

$$0 \to H \xrightarrow{\phi} G \quad \text{exact means } \phi \text{ is injective}$$

$$G \xrightarrow{\phi} K \to 0 \quad \text{exact means } \phi \text{ is surjective}$$

$$0 \to G \to G' \to 0 \quad \text{exact means } G \simeq G'$$

$$0 \to H \to G \to K \to 0 \quad \text{exact means } H < G \text{ and } K = G/H.$$

Definition

A *lattice* $H < G$ is a short exact sequence

$$0 \to H \to G \to K \to 0$$

such that H is discrete and K is compact.

Example: Sampling of 'music'. Sampling points form a lattice.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot \delta t} \mathbb{R} \xrightarrow{/\delta t (\text{ mod } 1)} \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

Duality subgroup and quotient

Theorem

The sequence

$$0 \longrightarrow H \xrightarrow{\phi_1} G \xrightarrow{\phi_2} K \longrightarrow 0$$

is exact if and only if the dual sequence

$$0 \longleftarrow \widehat{H} \xleftarrow{\widehat{\phi_1}} \widehat{G} \xleftarrow{\widehat{\phi_2}} \widehat{K} \longleftarrow 0$$

is exact.

We see: $H < G$, $K = G/H$, $\widehat{K} < \widehat{G}$, $\widehat{H} = \widehat{G}/\widehat{K}$.

Reciprocal lattice

Theorem

An LCA H is discrete if and only if the dual $\widehat{H}$ is compact.

Reciprocal lattice

$H < G$ is a lattice if and only if $\widehat{K} < \widehat{G}$ is a lattice.
We write $\widehat{K} = H^{\perp}$ for reciprocal lattices.

Lemma

The characters in $H^{\perp}$ alias on H, i.e.

$$\langle \widehat{\phi_2}(\xi), \phi_1(h) \rangle_G = 1 \quad \textit{for all } h \in H, \xi \in H^{\perp}$$

Proof:

$$\langle \widehat{\phi_2}(\xi), \phi_1(h) \rangle_G = \langle \xi, \phi_2 \circ \phi_1(h) \rangle_K = \langle \xi, 0 \rangle_K = 1$$

Downsampling and periodisation

Definition

Downsampling: $\phi_1^* : \mathbb{C}G \to \mathbb{C}H$ is defined as:

$$\phi_1^* f = f \circ \phi_1 .$$

Definition

Periodisation: $\widehat{\phi_1}_* : \mathbb{C}\widehat{G} \to \mathbb{C}\widehat{H}$ is defined as:

$$\widehat{\phi_1}_* \widehat{f}(\widehat{\phi_1}(\xi)) = \sum_{\xi' \in \ker(\widehat{\phi_1})} \widehat{f}(\xi + \xi').$$

Duality downsampling $\leftrightarrow$ periodisation

Theorem

For nice functions $f \in \mathbb{C}G$ (e.g. Schwartz functions) we have

$$\boldsymbol{F}_H(\phi_1^* f) = \widehat{\phi_1}_* \boldsymbol{F}_G(f).$$

Topics to be discussed in the final notes:

- Band limited functions, Nyquist reconstruction.
- The FFT and symmetric FFTs.
- Lattice rules for integration and FFT.
- Cyclic reduction.
- Boundary conditions, mirrors and caleidoscopes.
- Preconditioning by projection to space of circulants.

The FFT

Special case: Twiddle-free FFT

If $G \simeq H \times K$, then

$$
\begin{array}{ccccccc}
0 & \longleftarrow & \widehat{H} \times \widehat{K} & \underset{\widehat{\phi}}{\longleftarrow} & \widehat{G} & \longleftarrow & 0 \\
 & & | & & | & & \\
0 & \longrightarrow & H \times K & \overset{\phi}{\longrightarrow} & G & \longrightarrow & 0
\end{array}
$$

FFT factorization: $\boldsymbol{F}_G = \widehat{\phi}^* \circ (\boldsymbol{F}_H \otimes \boldsymbol{F}_K) \circ \phi^*$.

The general case $H < G$

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \widehat{H} & \longleftarrow & \widehat{G} & \longleftarrow & H^{\perp} & \longleftarrow & 0 \\
 & & | & & | & & | & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0
\end{array}
$$

Details in final notes; (Weyl-Brezin-Zak transform, Heissenberg group).

Fourier transforms on $\mathbb{R}^d$. Lattice rules.

Sampling lattice: $0 \to \mathbb{Z}^d \xrightarrow{S} \mathbb{R}^d \to 0$.
Periodic lattice: $0 \to \mathbb{Z}^d \xrightarrow{SA} \mathbb{R}^d \to 0$, where $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$.
The periodic lattice is a sub-lattice of the sampling lattice.

Smith normal form

An integer matrix $A \in \mathbb{Z}^{d\times d}$ can be factorized as $A = UNV^T$, where $U, V \in \mathbb{Z}^{d\times d}$ are unimodular and $N = \text{diag}(n_1, n_2, \ldots)$ s. t. $n_i | n_{i+1}$.

Let k denote the number of n_i such that $n_i > 1$, and let $\mathbf{n}$ be the vector containing these last k diagonal elements, defining the FAG $\mathbb{Z}_{\mathbf{n}}$. Let $U_k \in \mathbb{Z}^{k\times d}$ denote the last k rows of U^{-1} and let $V_k^T \in \mathbb{Z}^{d\times k}$ denote the last k columns of V^{-T}. Finally, let $N_k = \text{diag}(\mathbf{n}) \in \mathbb{Z}^{k\times k}$ and $N_k^{-1} \in \mathbb{R}^{k\times k}$.

Fourier transforms on $\mathbb{R}^d$. Lattice rules.

This diagram describes double sampling with a rank-k lattice rule. A function $f \in \mathbb{C}\mathbb{R}^d$ is discretized in $\mathbb{C}\mathbb{Z}_{\mathbf{n}}$, where $\mathbb{Z}_{\mathbf{n}} = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ is a finite abelian group.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}^d & = & \mathbb{Z}^d & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle SA} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^d & \xrightarrow{S} & \mathbb{R}^d & \xrightarrow{S^{-1}} & T^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U_k} & & \downarrow{\scriptstyle (SA)^{-1}} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}_{\mathbf{n}} & \xrightarrow{V_k^T N_k^{-1}} & T^d & \xrightarrow{A} & T^d & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

All rows and columns are exact.

Dual space

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longleftarrow & T^d & = & T^d & \longleftarrow & 0 & & \\
 & & \uparrow{\scriptstyle A^T} & & \uparrow{\scriptstyle (SA)^T} & & \uparrow & & \\
0 & \longleftarrow & T^d & \xleftarrow[S^T]{} & \mathbb{R}^d & \xleftarrow[S^{-T}]{} & \mathbb{Z}^d & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle U_k^T N_k^{-1}} & & \uparrow{\scriptstyle (SA)^{-T}} & & \| & & \\
0 & \longleftarrow & \mathbb{Z}_{\mathbf{n}} & \xleftarrow[V_k]{} & \mathbb{Z}^d & \xleftarrow[A^T]{} & \mathbb{Z}^d & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$