A multivariate generalization of Prony's method

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Outline

d-variate exponential sums

$d \in \mathbb{N}$.

d-variate exponential sum:

$$f : \mathbb{Z}^d \longrightarrow \mathbb{C}$$
$$k \longmapsto \sum_{j=1}^{M} f_j z_j^k = \sum_{j=1}^{M} f_j \prod_{\ell=1}^{d} z_{j,\ell}^{k_\ell}$$

$f_1, \ldots, f_M \in \mathbb{C} \setminus \{0\}$
$z_1, \ldots, z_M \in (\mathbb{C} \setminus \{0\})^d$ pairwise distinct.

Further notation:

$$\Omega := \{z_1, \ldots, z_M\}.$$

Reconstruction problem

Problem: Reconstruct all f_j and z_j from samples of f.

Solution for $d = 1$: de Prony (1795).

- Reconstruction from $2M + 1$ equidistant samples.
- Reconstruction of Ω *independently* of the coefficients f_j.

Prony method for univariate exponential sums

$f : \mathbb{Z} \to \mathbb{C}$ univariate exponential sum, i. e. $d = 1$.

Classical Prony method:

Given: $f(-M), \dots, f(M)$.

- $$T := (f(k-m))_{\substack{m=0,\dots,M \\ k=0,\dots,M}} \in \mathbb{C}^{M+1 \times M+1}.$$

  Fact: $\dim \ker T = 1$.

  $$(p_0, \dots, p_M) \in \ker T \setminus \{0\},\ p_j \in \mathbb{C}.$$

- z_j — Roots of polynomial $p := \sum_{k=0}^{M} p_k \mathrm{X}^k \in \mathbb{C}[\mathrm{X}]$.
- $(f_1, \dots, f_M) \in \mathbb{C}^M$ — Solution to a regular linear system.

$d = 1$ revisited

Construct $T \in \mathbb{C}^{M+1 \times M+1}$ with

$$\ker T = \operatorname{span}\{p\}$$

and

$$\Omega = \{z \in \mathbb{C} \mid p(z) = 0\}.$$

Thus

$$\Omega = \{z \in \mathbb{C} \mid q(z) = 0 \text{ for all } q \in \ker T\} = \mathrm{V}(\ker T).$$

Multivariate Prony method

Now: $d \in \mathbb{N}$ arbitrary, so $f : \mathbb{Z}^d \to \mathbb{C}$.

Notation:

$$\Pi_n := \{q \in \mathbb{C}[X_1, \dots, X_d] \mid \underbrace{\max\{\|k\|_\infty \mid k \in \mathbb{N}, q_k \neq 0\}}_{=:\mathrm{maxdeg}\, q} \leq n\},$$

$$N := \dim \Pi_n = (n+1)^d.$$

Idea: Identify Π_n with $\mathbb{C}^N$ and construct a matrix

$$T_n \in \mathbb{C}^{N \times N}$$

with

$$V(\ker T_n) = \Omega.$$

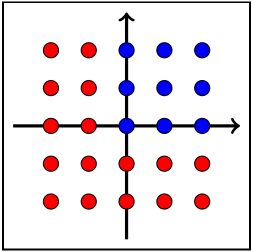
See also Peter-Plonka-Schaback, 2015 (submitted).

Multivariate Prony method

For $n \in \mathbb{N}$ let

$$T_n := (f(k-m))_{\|m\|_\infty, \|k\|_\infty \leq n} \in \mathbb{C}^{N \times N}.$$

(Block Toeplitz with Toeplitz blocks!)



$d = n = 2$ sampling grid; $(2n+1)^2 = 25$ samples

Multivariate Prony method

Let

$$\mathcal{A}_n \colon \Pi_n \longrightarrow \mathbb{C}^M$$
$$p \longmapsto (p(z_1), \ldots, p(z_M))$$

be the evaluation homomorphism at $\Omega = \{z_1, \ldots, z_M\}$,

$$A_n \in \mathbb{C}^{M \times N}$$

the representation matrix of $\mathcal{A}_n$.

Then

$$T_n = P_n A_n^\top D_n A_n,$$

with $D_n := \operatorname{diag}(z_1^{-n} f_1, \ldots, z_M^{-n} f_M)$. Thus

$$\ker T_n \supseteq \ker A_n, \quad \text{so} \quad \mathrm{V}(\ker T_n) \subseteq \mathrm{V}(\ker A_n)$$

Remarks

So far

$$\mathrm{V}(\ker T_n) \subseteq \mathrm{V}(\ker \mathcal{A}_n) \supseteq \Omega.$$

Is $\mathrm{V}(\ker T_n) \neq \Omega$ possible?

Yes, $\mathrm{V}(\ker T_n)$ can be *infinite*, e. g. if $\dim \ker T_n = 1$.

So, how to choose n?

Multivariate polynomials $\rightsquigarrow$ algebraic tools!

$\mathrm{V}(\ker T_n) = \Omega$?

Theorem

Let f be a d-variate exponential sum with parameters $z_1, \ldots, z_M$. If $n \geq M$, then $\mathrm{V}(\ker T_n) = \Omega := \{z_1, \ldots, z_M\}$.

Proof.

Show:

- $\mathrm{V}(\ker \mathcal{A}_n) = \Omega$.
- $\operatorname{rank} A_n = M$.
- $\ker A_n = \ker T_n$.

Example 1

$$f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$$
$$k \longmapsto (1,1)^k + (-1,-1)^k.$$

Choose $n = 2$. Then

$$T_2 = \begin{pmatrix} T & T' & T \\ T' & T & T' \\ T & T' & T \end{pmatrix}, \; T := \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \; T' := \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

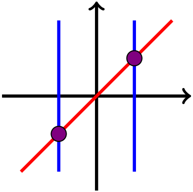
$$\dim \ker T_2 = 7 \text{ and } \langle \ker T_2 \rangle = \langle -1 + \mathsf{X}^2, \; -\mathsf{X} + \mathsf{Y} \rangle,$$

hence

$$\mathsf{V}(\ker T_2) = \mathsf{V}(-1 + \mathsf{X}^2, \; -\mathsf{X} + \mathsf{Y}) = \{(1,1), \; (-1,-1)\}.$$

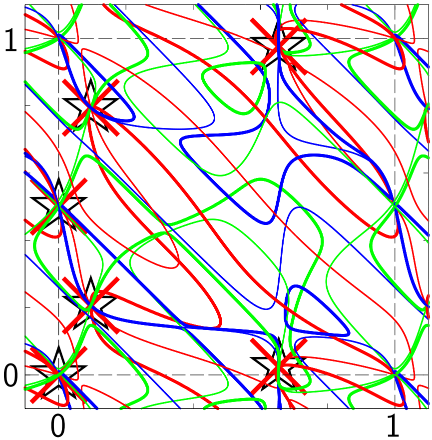
Example 1

Visually:



$$
\begin{aligned}
p_1 &= -1 + X^2, \\
p_2 &= -X + Y, \\
V(p_1, p_2) &= \{(-1,-1),(1,1)\} = \Omega,
\end{aligned}
$$

Example 2



Zero loci on $\mathbb{T}^2 \cong [0,1[$ of real- and imaginary parts of $\mathbb{C}$-basis of ker T_2 for $M = 6$, $z_j = \exp(\pi i(1 + \cos((j-1)\pi/2), 1 + \sin((j-1)\pi/2)))$, $j = 1, \ldots, 5$, $z_5 = (1, 1)$, $f_j = 1$, and $n = 2 < M$ (computed with `Octave` resp. `Bertini`)

Summary

- Parameter reconstruction for *d*-variate exponential sums
  - Deterministic
  - Sampling set known a priori

- $T_n \in \mathbb{C}^{N \times N}$, $N = (n+1)^d$
- $\ker T_n \subseteq \mathbb{C}[\mathsf{X}_1, \ldots, \mathsf{X}_d]$
- $n \geq M \Rightarrow \mathsf{V}(\ker T_n) = \Omega$

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Thank you for your attention!

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