On unreduced KKT systems arising from Interior Point methods

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The problem

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Computational Fluid Dynamics (Elman, Silvester, Wathen 2005)
- Elasticity problems
- Mixed (FE) formulations of II and IV order elliptic PDEs
- Linearly Constrained Programs
- Linear Regression in Statistics
- Image restoration
- ... Survey: Benzi, Golub and Liesen, Acta Num 2005

The setting

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Iterative solution by means of Krylov subspace methods
- Structural properties of interest to our context:
  - A symmetric positive (semi)definite
  - B^T tall, possibly rank deficient
  - C symmetric positive (semi)definite

Spectral properties

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

$$\begin{array}{lr} 0 < \lambda_n \leq \cdots \leq \lambda_1 & \text{eigs of } A \\ 0 = \sigma_m \leq \cdots \leq \sigma_1 & \text{sing. vals of } B \\ \lambda_{\max}(C) > 0, \quad BB^T + C \quad \text{full rank} & \end{array}$$

$$\operatorname{spec}(\mathcal{M}) \subset [-a, -b] \cup [c, d], \quad a, b, c, d > 0$$

$\Rightarrow$ A large variety of results on the spectrum of $\mathcal{M}$, also for indefinite and singular A

$\Rightarrow$ Search for good preconditioning strategies...

General preconditioning strategy

- Find $\mathcal{P}$ such that

$$\mathcal{M}\mathcal{P}^{-1}\hat{u} = b \qquad \hat{u} = \mathcal{P}u$$

is easier (faster) to solve than $\mathcal{M}u = b$

- A look at efficiency:
  - Dealing with $\mathcal{P}$ should be cheap
  - Storage requirements for $\mathcal{P}$ should be low
  - Properties (algebraic/functional) should be exploited

Mesh/parameter independence

Structure preserving preconditioners

Block diagonal Preconditioner

⋆ A nonsing., $C = 0$:

$$\mathcal{P}_0 = \begin{bmatrix} A & 0 \\ 0 & BA^{-1}B^T \end{bmatrix}$$

$$\Rightarrow \qquad \mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}} = \begin{bmatrix} I & A^{-\frac{1}{2}} B^T (BA^{-1}B^T)^{-\frac{1}{2}} \\ (BA^{-1}B^T)^{-\frac{1}{2}} BA^{-\frac{1}{2}} & 0 \end{bmatrix}$$

MINRES converges in at most 3 iterations.

$\mathrm{spec}(\mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}}) = \left\{1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}\right\}$

A more practical choice:

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \qquad \text{spd.} \quad \widetilde{A} \approx A \qquad \widetilde{S} \approx BA^{-1}B^T$$

eigs of $\mathcal{M}\mathcal{P}^{-1}$ in $\qquad [-a, -b] \cup [c, d], \qquad a, b, c, d > 0$

Still an Indefinite Problem

Giving up symmetry ...

- Change the preconditioner: *Mimic the LU factors*

$$\mathcal{M} = \begin{bmatrix} I & O \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B^T \\ O & BA^{-1}B^T + C \end{bmatrix} \quad \Rightarrow \mathcal{P} \approx \begin{bmatrix} A & B^T \\ O & BA^{-1}B^T + C \end{bmatrix}$$

- Change the preconditioner: *Mimic the Structure*

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \quad \Rightarrow \mathcal{P} \approx \mathcal{M}$$

- Change the matrix: *Eliminate indef.*

$$\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}$$

- Change the matrix: *Regularize* $(C = 0)$

$$\mathcal{M} \;\Rightarrow\; \mathcal{M}_\gamma = \begin{bmatrix} A & B^T \\ B & -\gamma W \end{bmatrix} \text{ or } \mathcal{M}_\gamma = \begin{bmatrix} A + \frac{1}{\gamma} B^T W^{-1} B & B^T \\ B & O \end{bmatrix}$$

Constraint (Indefinite) Preconditioner

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ B & -C \end{bmatrix} \quad \mathcal{M}\mathcal{P}^{-1} = \begin{bmatrix} A\widetilde{A}^{-1}(I - \Pi) + \Pi & \star \\ O & I \end{bmatrix}$$

with $\Pi = B(B\widetilde{A}^{-1}B^T + C)^{-1}B\widetilde{A}^{-1}$

- Constraint equation satisfied at each iteration
- If C nonsing $\Rightarrow$ all eigs real and positive
- If $B^TC = 0$ and $BB^T + C > 0 \Rightarrow$ all eigs real and positive

$\Rightarrow$ More general cases, $\widetilde{B} \approx B$, $\widetilde{C} \approx C$

Block triangular preconditioner

$$A \text{ spd}, \quad \mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ 0 & -\widetilde{C} \end{bmatrix} \qquad \widetilde{A} \approx A, \quad \widetilde{C} \approx BA^{-1}B^T + C$$

Ideal case: $\widetilde{A} = A,\ \widetilde{C} = BA^{-1}B^T + C \quad \Rightarrow \quad \mathcal{M}\mathcal{P}^{-1} = \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix}$

Recovering symmetry?

- If $\widetilde{C} = C$ nonsing., then $\sigma(\mathcal{M}\mathcal{P}^{-1})$ in $\mathbb{R}^+$
- If $\widetilde{A} < A$ then $\sigma(\mathcal{M}\mathcal{P}^{-1})$ in $\mathbb{R}^+$ with

$$\lambda \in [\chi_1, \chi_2] \ni 1, \qquad \chi_j = \chi_j((B^T\widetilde{A}^{-1}B + C)\widetilde{C}^{-1}, \widetilde{A}^{-1}A)$$

Regularized problem and Augmented preconditioners

Augmented Lagrangian approach:

$$\mathcal{M}_\gamma = \begin{bmatrix} A + \frac{1}{\gamma} B^T W^{-1} B & B^T \\ B & O \end{bmatrix}$$

Particularly interesting for A indefinite or singular
⋆ Any of the above preconditioners may be used.

Somehow related preconditioner for $\mathcal{M} = \begin{bmatrix} A & B^T \\ B & O \end{bmatrix}$:

$$\mathcal{P} = \begin{bmatrix} A + B^T W^{-1} B & B^T \\ O & W \end{bmatrix}$$

Application. Convex Quadratic Programming (QP) Pbs

We focus on the linear algebra phase of Interior-Point methods applied to convex QP problems.

Primal-dual pair of convex QP problems in standard form:

$$\min_{x} \quad c^T x + \tfrac{1}{2} x^T H x \qquad \text{subject to} \quad Jx = b, \; x \geq 0$$
$$\max_{x,y,z} \quad b^T y - \tfrac{1}{2} x^T H x \qquad \text{subject to} \quad J^T y + z - Hx = c, \; z \geq 0$$

- $H \in \mathbb{R}^{n \times n}$, symmetric and positive semidefinite
- $J \in \mathbb{R}^{m \times n}$, $m \leq n$ is full-row rank
- $x, \; z, \; c \in \mathbb{R}^n$, $y, \; b \in \mathbb{R}^m$

Interior Point (IP) methods

At a generic IP iteration k, the primal-dual Newton direction solves, possibly approximately, the linear system of dimension $2n+m$ with direction $(\Delta x_k, \Delta y_k, \Delta z_k)$:

$$\underbrace{\begin{bmatrix} H & J^T & -I_n \\ J & 0 & 0 \\ -Z_k & 0 & -X_k \end{bmatrix}}_{K_3} \begin{bmatrix} \Delta x_k \\ -\Delta y_k \\ \Delta z_k \end{bmatrix} = \begin{bmatrix} -c - Hx_k + J^T y_k + z_k \\ b - Jx_k \\ \tau_k e - X_k Z_k e \end{bmatrix}$$

where

$$X_k = \mathrm{diag}(x_k), \qquad Z_k = \mathrm{diag}(z_k) \quad \text{and} \quad (x_k,\, z_k) > 0$$

$e = (1, \ldots, 1)^T$,

$\tau_k = x_k^T z_k / n$: barrier parameter (controls distance to optimality). Gradually reduced through the IP iterations

Block eliminations approaches

Unreduced matrix: K_3 of dimension $2n+m$.

Reduced matrix:

$$K_3 = \begin{bmatrix} H & J^T & -I \\ J & 0 & 0 \\ -Z & 0 & -X \end{bmatrix} \quad \Longrightarrow \quad K_2 = \begin{bmatrix} H + X^{-1}Z & J^T \\ J & 0 \end{bmatrix}$$

- K_2 is symmetric and has dimension $n+m$; inexpensive to form since X and Z are diagonal.

Condensed matrix:

$$K_2 = \begin{bmatrix} H + X^{-1}Z & J^T \\ J & 0 \end{bmatrix} \quad \Longrightarrow \quad K_1 = J(H + X^{-1}Z)^{-1}J^T$$

Regularized matrices

Given $\delta \geq 0$ and $\rho \geq 0$, consider the regularized problem

$$\min_{x,r} c^T x + \frac{1}{2} x^T H x + \frac{1}{2}\rho\|x\|^2 + \frac{1}{2}\|r\|^2 \quad \text{subject to } Jx + \delta r = b, \;\; x \geq 0$$

Then

$$K_{3,\text{reg}} = \begin{bmatrix} H + \rho I_n & J^T & -I_n \\ J & -\delta I_m & 0 \\ -Z & 0 & -X \end{bmatrix}$$

$$K_{2,\text{reg}} = \begin{bmatrix} H + \rho I_n + X^{-1} Z & J^T \\ J & -\delta I_m \end{bmatrix}$$

Eigenvalues of H and singular values of J are shifted away from zero.

[Altman and Gondzio 1999], [Friedlander and Orban 2012], [Gondzio 2012], [Saunders, 1996].

Main features of reduced and unreduced matrices

- For X and Z positive definite, $K_{2,\text{reg}}$ and $K_{3,\text{reg}}$ are nonsingular.

- If $(\bar{x},\, \bar{y},\, \bar{z})$ solves the QP pair then $\bar{x},\, \bar{z} \geq 0$ and

$$\bar{x}_i \bar{z}_i = 0, \quad i = 1, \ldots, n.$$

  $K_{2,\text{reg}}$ becomes increasingly ill-conditioned as the IP iterates approach the solution due to $X^{-1}Z$.

- $K_{3,\text{reg}}$ can be convenient in terms of eigenvalues and conditioning throughout the IP iterations, [Forsgren, 2002], [Forsgren, Gill and M. Wright, 2002], [M. Wright, 1998].

Nonsingularity of $K_{3,\text{reg}}$

Greif, Moulding and Orban have recently provided spectral bounds for $K_{3,\text{reg}}$ and claimed that in terms of eigenvalues and conditioning, it may be beneficial to use the unreduced formulation.

Theorem (Greif, Moulding and Orban, 2014)

$K_{3,\text{reg}}$ is nonsingular at $(\bar{x}, \bar{y}, \bar{z})$ if and only if

1. $\bar{x}$ and $\bar{z}$ are strictly complementary, $\bar{x}_i = 0 \Longrightarrow \bar{z}_i > 0 \ \forall\, i$
2. If $\rho = 0$, $\ker(H) \cap \ker(J) \cap \ker(\bar{Z}) = \{0\}$ where $\bar{Z} = \operatorname{diag}(\bar{z})$.
3. If $\delta = 0$, the Linear Independence Constraint Qualification (LICQ) is satisfied at $\bar{x}$, i.e. for $\mathcal{A} = \{ i \mid \bar{x}_i = 0 \}$, the matrix
$$\begin{bmatrix} J^T & -I_{\mathcal{A}} \end{bmatrix}$$
has full column rank.

Spectral Properties of the $K_{3,\text{reg}}$

- $K_{3,\text{reg}}$ is symmetrizable and has real eigenvalues, [Forsgren, 2002], [Saunders, 1998]. Let

$$R = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & Z^{\frac{1}{2}} \end{bmatrix}$$

By the similarity transformation associated with R we obtain

$$\begin{aligned} K_{3,\text{sym}} &= R^{-1} K_{3,\text{reg}} R \\ &= \begin{bmatrix} H + \rho I_n & J^T & -Z^{\frac{1}{2}} \\ J & -\delta I_m & 0 \\ -Z^{\frac{1}{2}} & 0 & -X \end{bmatrix} \end{aligned}$$

- There are other ways to symmetrize $K_{3,\text{reg}}$.
Here $K_{3,\text{sym}}$ remains nonsingular in the limit but R becomes ill-conditioned.

Spectral bounds for nonsingular $K_{3,\text{reg}}$

Theorem (Greif, Moulding and Orban, 2014)

The eigenvalues θ of K_3 $(\delta = \rho = 0)$ satisfy

$$\theta \in [\theta_1, 0) \cup [\theta_3,\, \theta_4]$$

The eigenvalues θ of $K_{3,\text{reg}}$ $(\delta, \rho > 0)$ satisfy

$$\theta \in [\theta_1, -\delta] \cup [\theta_3,\, \theta_4], \quad \theta_3 \geq \rho$$

Drawbacks in the unregularized case:

- a meaningful upper bound on negative eigenvalues is not provided;
- if H is positive semidefinite, θ_3 goes to 0 as $x \to \bar{x}$ and $z \to \bar{z}$, even though, in the limit, K_3 may be nonsingular.

Our focus: Assess the potentials of the use of the unreduced formulation by providing new results on spectral analysis and its solution.

Refined spectral estimates for nonsingular $K_{3,\mathrm{reg}}$

$$x_{\min} = \min_i x_i \qquad z_{\max} = \max_i z_i$$

$$\lambda_{\min} = \lambda_{\min}(H) \qquad \lambda_{\max} = \lambda_{\max}(H) \qquad \sigma_{\min} = \sigma_{\min}(J) \qquad \sigma_{\max} = \sigma_{\max}(J)$$

Theorem

Let θ^- be a negative eigenvalue of K_3. It holds

- $\theta^- \leq \theta_2$ where θ_2 is the greatest negative root of the cubic polynomial

  $$\pi(\theta) = \theta^3 + (x_{\min} - \lambda_{\max})\theta^2 - (x_{\min}\lambda_{\max} + \sigma_{\min}^2 + z_{\max})\theta - \sigma_{\min}^2 x_{\min}$$

  and is s.t. $\theta_2 > -x_{\min}$.

- If $(\bar{x}, \bar{z})$ is approached, $\mathcal{A}$ and $\mathcal{I}$ are the index sets of active and inactive bounds at $\bar{x}$, $G^T = \begin{bmatrix} J_{\mathcal{A}} & J_{\mathcal{I}} \\ -Z_{\mathcal{A}}^{\frac{1}{2}} & 0 \end{bmatrix}$, then

  $$\theta^- \leq \mu_2 = \max\left\{-(x_{\mathcal{I}})_{\min},\ \frac{1}{2}\left(\lambda_{\max} - \sqrt{\lambda_{\max}^2 + 4\sigma_{\min}^2(G)}\right)\right\} + \sqrt{(z_{\mathcal{I}})_{\max}}$$

Note: if $\mathcal{A} \neq \emptyset$ then θ_2 goes to 0 as $(\bar{x}, \bar{z})$ is approached.

Refined spectral estimates for nonsingular $K_{3,\text{reg}}$

Theorem

Let θ^+ be a positive eigenvalue of K_3.

If $(\bar{x}, \bar{z})$ is approached, $\mathcal{A}$ and $\mathcal{I}$ are the index sets of active and inactive bounds at $\bar{x}$, $G^T = \begin{bmatrix} J_{\mathcal{A}} & J_{\mathcal{I}} \\ -Z_{\mathcal{A}}^{\frac{1}{2}} & 0 \end{bmatrix}$, then

$$\theta^+ \geq \mu_3 = \tilde{\mu}_3 - (x_{\mathcal{A}})_{\max}$$

where $\tilde{\mu}_3$ is the smallest positive root of the cubic polynomial

$$q(\mu) = \mu^3 - (\lambda_{\max} + \lambda_{\min})\mu^2 + (\lambda_{\min}^2 - \sigma_{\min}^2(G))\mu + \lambda_{\min}\sigma_{\min}^2(G)$$

Numerical experiments

CONT-050 problem (Maros-Mezaros Collection), $n = 2597$, $m = 2401$. No regularization.

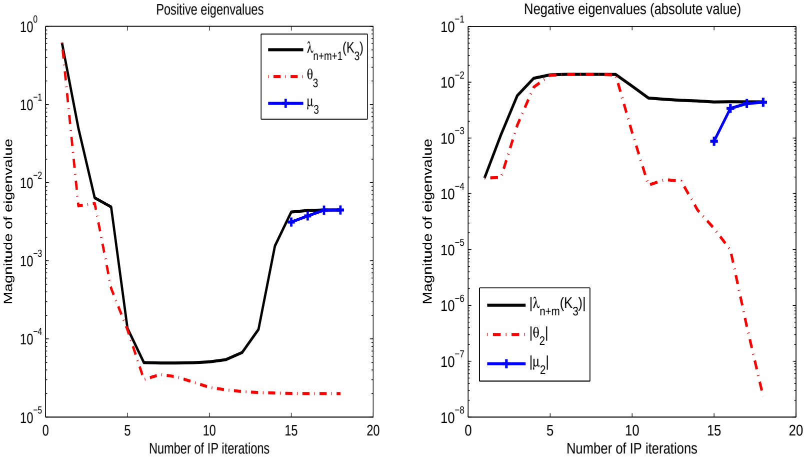

Figure : Problem CONT-050: eigenvalues of K_3 closest to zero (solid line) and their bounds at every iteration. Left: positive eigenvalues. Right: negative eigenvalues.

On the use of the reduced and unreduced systems

- Direct solvers: the effect of ill-conditioning in $K_{2,\text{reg}}$ is *benign* [Poncelon, 1991], [S. Wright 1995], [Forsgren, Gill, Shinnerl, 1996], [M. Wright, 1998].
- Iterative solvers: preconditioning is required

$$P_2^{-1} K_{2,\text{reg}}\, \Delta_2 = P_2^{-1} f_2$$

$$\widehat{P}_3^{-1} K_{3,\text{reg}}\, \Delta_3 = \widehat{P}_3^{-1} \widehat{f}_3 \qquad \text{3x3 unsymmetric}$$

$$P_3^{-1} K_{3,\text{sym}}\, \Delta_3 = P_3^{-1} f_3 \qquad \text{3x3 symmetric}$$

  Preconditioners analyzed: constraint, augmented diagonal and triangular preconditioners.

- Our conclusions:
  1. Connections between the spectra of the 2x2 and 3x3 preconditioned matrices hold.
  2. Equivalences between blocks of the 3x3 preconditioned systems and the 2x2 preconditioned systems hold.
  3. As long as IP implementations with reduced and unreduced systems are successful, CPU times are in favor of the former due to their smaller dimensions.

Relations between unreduced and reduced matrices

- Unsymmetric formulation. Let

$$\widehat{L}_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ X^{-1} & 0 & I \end{bmatrix}, \quad \widehat{L}_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ X^{-1}Z & 0 & I \end{bmatrix}$$

Then

$$K_{3,\text{reg}} = \widehat{L}_1^T \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$K_{2,\text{reg}}$}} & 0 \\ & & 0 \\ 0 & 0 & -X \end{bmatrix} \widehat{L}_2$$

- Symmetric formulation. Let

$$L = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ X^{-1}Z^{\frac{1}{2}} & 0 & I \end{bmatrix}$$

then

$$K_{3,\text{sym}} = L^T \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$K_{2,\text{reg}}$}} & 0 \\ & & 0 \\ 0 & 0 & -X \end{bmatrix} L$$

(congruence transformation).

Constraint Preconditioners

$$P_{2,\mathcal{C}} = \begin{bmatrix} \mathrm{diag}(H + \rho I_n + X^{-1}Z) & J^T \\ J & -\delta I_m \end{bmatrix},$$

$$\widehat{P}_{3,\mathcal{C}} = \begin{bmatrix} \mathrm{diag}(H + \rho I_n) & J^T & -I_n \\ J & -\delta I_m & 0 \\ -Z & 0 & -X \end{bmatrix} = \widehat{L}_1^T \begin{bmatrix} P_{2,\mathcal{C}} & & 0 \\ & & 0 \\ 0 & 0 & -X \end{bmatrix} \widehat{L}_2, \quad \text{unsymmetric 3x3}$$

$$P_{3,\mathcal{C}} = \begin{bmatrix} \mathrm{diag}(H + \rho I_n) & J^T & -Z^{\frac{1}{2}} \\ J & -\delta I_m & 0 \\ -Z^{\frac{1}{2}} & 0 & -X \end{bmatrix} = L^T \begin{bmatrix} P_{2,\mathcal{C}} & & 0 \\ & & 0 \\ 0 & 0 & -X \end{bmatrix} L, \quad \text{symmetric 3x3}$$

Theorem

1. $\widehat{P}_{3,\mathcal{C}}$ and $P_{3,\mathcal{C}}$ remain invertible in the limit (and possibly well-conditioned).
2. For the unsymmetric 3×3 system:

$$\theta \in \Lambda\left(\widehat{P}_{3,\mathcal{C}}^{-1} K_{3,\mathrm{reg}}\right) \iff \theta \in \{1\} \cup \Lambda\left(P_{2,\mathcal{C}}^{-1} K_{2,\mathrm{reg}}\right)$$

   The first two block equations of $\widehat{P}_{3,\mathcal{C}}^{-1} K_3 \Delta_3 = \widehat{P}_{3,\mathcal{C}}^{-1} f_3$ are equivalent to $P_{2,\mathcal{C}}^{-1} K_{2,\mathrm{reg}} \Delta_2 = P_{2,\mathcal{C}}^{-1} f_2$, the third block equation is equivalent to the third equation in $K_3 \Delta_3 = f_3$.

3. The same results hold for the symmetric 3×3 formulation
4. Similar results hold for certain block triangular preconditioners

Augmented diagonal preconditioners

Let

$$P_{2,\mathcal{D}} = \begin{bmatrix} H + \rho I_n + X^{-1}Z + \delta^{-1}J^TJ & 0 \\ 0 & \delta I_m \end{bmatrix}$$

$$P_{3,\mathcal{D}} = \begin{bmatrix} H + \rho I_n + X^{-1}Z + \delta^{-1}J^TJ & 0 & 0 \\ 0 & \delta I_m & 0 \\ 0 & 0 & X \end{bmatrix} = \begin{bmatrix} \multicolumn{2}{c}{P_{2,\mathcal{D}}} & 0 \\ & & 0 \\ 0 & 0 & X \end{bmatrix}$$

$P_{2,\mathcal{D}}$, $P_{3,\mathcal{D}}$ are positive definite.

Theorem

Upon elimination of Δz, the preconditioned 3×3 system reduces to the 2×2 preconditioned system.

$$K_{2,\mathrm{reg}} = \left[\begin{array}{c|c} H + \rho I_n + X^{-1}Z & J^T \\ \hline J & -\delta I_m \end{array}\right], \quad K_{3,\mathrm{reg}} = \left[\begin{array}{c|cc} H + \rho I_n & J^T & -I_n \\ \hline J & -\delta I_m & 0 \\ -Z & 0 & -X \end{array}\right]$$

Ideal preconditioner in terms of spectral distribution [Morini, Simoncini, Tani].

Numerical Results: condition number and direct solvers

QP problems from CUTEr solved with PDCO [Saunders].
$K_{3,\text{reg}}$ nonsingular at the solution.
$\delta = \rho = 10^{-6}$, accuracy on feasibility and complementarity: 10^{-6}.

Problem (n,m)	$\kappa_e(K_{3,\text{reg}})$ min-max	$\kappa_e(K_{2,\text{reg}})$ min-max	Backslash Time $K_{3,\text{reg}}$	Backslash Time $K_{2,\text{reg}}$
CVXQP1 (10000, 5000)	4-5	4-9	25.1	5.7
CVXQP2 (10000,7500)	3-5	3-9	13.0	4.4
CVXQP3 (10000, 7500)	4-5	4-9	34.4	6.2
STCQP1 (16385, 8190)	6-7	7-13	127.3	4.4
GOULDQP3 (19999, 9999)	7-10	7-13	2.9	0.6

$\kappa_e(\cdot)$: estimate of the 1-norm condition number (Matlab function `condest`), expressed in the form $10^{\text{min}-\text{max}}$.

Total execution time (secs) for solving the sequence of linear systems

Analogous results are valid without regularization though $\kappa_e(K_{2,\text{reg}})$ is higher than above.

Numerical Results: Iterative solvers

Control on inexactness

$$\|K_{3,\mathrm{reg}}\,\Delta_3 - f_3\| \le \eta\tau, \qquad \|K_{2,\mathrm{reg}}\Delta_2 - f_2\| \le \eta\tau$$

$\tau = x^T z/n$, $\eta = 10^{-2}$.

	$P_{\mathcal{C}}$-GMRES		$P_{\mathcal{D}}$-MINRES	
Problem	$K_{3,\mathrm{reg}}$ Time	$K_{2,\mathrm{reg}}$ Time	$K_{3,\mathrm{sym}}$ Time	$K_{2,\mathrm{reg}}$ Time
CVXQP1 (10000, 5000)	1.0	0.7	2.3	1.9
CVXQP2 (10000,7500)	0.8	0.5	1.6	1.0
CVXQP3 (10000, 7500)	2.1	1.7	3.7	3.2
STCQP1 (16385, 8190)	12.7	23.8	2.5	2.1
GOULDQP3 (19999, 9999)	1.6	0.9	1.8	1.9

Total execution time (secs) for solving the sequence of linear unreduced and reduced systems

Work in progress and open problems

- The use of unreduced systems may be appealing for stability however the effect of ill-conditioning is *benign* with direct solvers.

- As for the iterative solvers, the iteration counts of a Krylov method are similar for any considered formulation of the systems but the computational cost is higher in the 3x3 formulations.

- We are currently investigating when the effect of ill-conditioning is *benign* in Inexact IP methods.

  Morini and S., *Ill-conditioning in Inexact Interior-Point methods for convex quadratic programming*, in progress.

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