Order reduction numerical methods for the algebraic Riccati equation

V. Simoncini

Dipartimento di Matematica

Alma Mater Studiorum - Università di Bologna

valeria.simoncini@unibo.it

The problem

Find $\mathbf{X} \in \mathbb{R}^{n\times n}$ such that

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{s\times n}$, $p, s = \mathcal{O}(1)$

Rich literature on analysis, applications and numerics:

Lancaster-Rodman 1995, Bini-Iannazzo-Meini 2012, Mehrmann etal 2003 ...

The problem

Find $\mathbf{X} \in \mathbb{R}^{n \times n}$ such that

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{s \times n}$, $p, s = \mathcal{O}(1)$

Rich literature on analysis, applications and numerics:

Lancaster-Rodman 1995, Bini-Iannazzo-Meini 2012, Mehrmann etal 2003 ...

We focus on the large scale case: $n \gg 1000$

Different strategies

- (Inexact) Kleinman iteration (Newton-type method)
- Projection methods
- Invariant subspace iteration
- (Sparse) multilevel methods
- ....

Newton-Kleinman iteration

Assume A stable. Compute sequence $\{\mathbf{X}_k\}$ with $\mathbf{X}_k \to_{k\to\infty} \mathbf{X}$

$$(A - \mathbf{X}_k BB^\top)\mathbf{X}_{k+1} + \mathbf{X}_{k+1}(A^\top - BB^\top\mathbf{X}_k) + C^\top C + \mathbf{X}_k BB^\top \mathbf{X}_k = 0$$

Newton-Kleinman iteration

Assume A stable. Compute sequence $\{\mathbf{X}_k\}$ with $\mathbf{X}_k \to_{k\to\infty} \mathbf{X}$

$$(A - \mathbf{X}_k BB^\top)\mathbf{X}_{k+1} + \mathbf{X}_{k+1}(A^\top - BB^\top \mathbf{X}_k) + C^\top C + \mathbf{X}_k BB^\top \mathbf{X}_k = 0$$

1: Given $\mathbf{X}_0 \in \mathbb{R}^{n\times n}$ such that $\mathbf{X}_0 = \mathbf{X}_0^\top$, $A^\top - BB^\top \mathbf{X}_0$ is stable
2: **For** $k = 0, 1, \ldots,$ until convergence
3: **Set** $\mathcal{A}_k^\top = A^\top - BB^\top \mathbf{X}_k$
4: **Set** $\mathcal{C}_k^\top = [\mathbf{X}_k B,\ C^\top]$
5: **Solve** $\mathcal{A}_k \mathbf{X}_{k+1} + \mathbf{X}_{k+1}\mathcal{A}_k^\top + \mathcal{C}_k^\top \mathcal{C}_k = 0$

Newton-Kleinman iteration

Assume A stable. Compute sequence $\{\mathbf{X}_k\}$ with $\mathbf{X}_k \to_{k\to\infty} \mathbf{X}$

$$(A - \mathbf{X}_k BB^\top)\mathbf{X}_{k+1} + \mathbf{X}_{k+1}(A^\top - BB^\top \mathbf{X}_k) + C^\top C + \mathbf{X}_k BB^\top \mathbf{X}_k = 0$$

1: Given $\mathbf{X}_0 \in \mathbb{R}^{n\times n}$ such that $\mathbf{X}_0 = \mathbf{X}_0^\top$, $A^\top - BB^\top \mathbf{X}_0$ is stable.

2: **For** $k = 0, 1, \ldots,$ until convergence

3: **Set** $\mathcal{A}_k^\top = A^\top - BB^\top \mathbf{X}_k$

4: **Set** $\mathcal{C}_k^\top = [\mathbf{X}_k B, \ \ C^\top]$

5: **Solve** $\mathcal{A}_k \mathbf{X}_{k+1} + \mathbf{X}_{k+1}\mathcal{A}_k^\top + \mathcal{C}_k^\top \mathcal{C}_k = 0$

Critical issues:

- The full matrix $\mathbf{X}_k$ cannot be stored (sparse or low-rank approx)
- Need a computable stopping criterion
- Each iteration k requires the solution of the Lyapunov equation

(Benner, Feitzinger, Hylla, Saak, Sachs, ...)

Galerkin projection method for the Riccati equation

Given the orth basis V_k for an approximation space, determine

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

to the Riccati solution matrix by orthogonal projection:

Galerkin condition:	Residual orthogonal to approximation space

Galerkin projection method for the Riccati equation

Given the orth basis V_k for an approximation space, determine

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

to the Riccati solution matrix by orthogonal projection:

Galerkin condition:	Residual orthogonal to approximation space

$$V_k^\top (A\mathbf{X}_k + \mathbf{X}_k A^\top - \mathbf{X}_k B B^\top \mathbf{X}_k + C^\top C) V_k = 0$$

Galerkin projection method for the Riccati equation

Given the orth basis V_k for an approximation space, determine

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

to the Riccati solution matrix by orthogonal projection:

Galerkin condition:	Residual orthogonal to approximation space

$$V_k^\top (A\mathbf{X}_k + \mathbf{X}_k A^\top - \mathbf{X}_k BB^\top \mathbf{X}_k + C^\top C) V_k = 0$$

giving the reduced Riccati equation

$$(V_k^\top A V_k)\mathbf{Y} + \mathbf{Y}(V_k^\top A^\top V_k) - \mathbf{Y}(V_k^\top BB^\top V_k)\mathbf{Y} + (V_k^\top C^\top)(CV_k) = 0$$

$\mathbf{Y}_k$ is the stabilizing solution (Heyouni-Jbilou 2009)

Galerkin projection method for the Riccati equation

Given the orth basis V_k for an approximation space, determine

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

to the Riccati solution matrix by orthogonal projection:

Galerkin condition:	Residual orthogonal to approximation space

$$V_k^\top (A\mathbf{X}_k + \mathbf{X}_k A^\top - \mathbf{X}_k BB^\top \mathbf{X}_k + C^\top C) V_k = 0$$

giving the reduced Riccati equation

$$(V_k^\top A V_k)\mathbf{Y} + \mathbf{Y}(V_k^\top A^\top V_k) - \mathbf{Y}(V_k^\top BB^\top V_k)\mathbf{Y} + (V_k^\top C^\top)(CV_k) = 0$$

$\mathbf{Y}_k$ is the stabilizing solution (Heyouni-Jbilou 2009)

Key questions:

- Which approximation space?
- Is this meaningful from a Control Theory perspective?

Dynamical systems and the Riccati equation

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

Time-invariant linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) = Cx(t), \end{cases}$$

$u(t)$: control (input) vector; $\qquad$ $y(t)$: output vector

$x(t)$: state vector; $\qquad$ x_0 : initial state

Dynamical systems and the Riccati equation

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

Time-invariant linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \qquad x(0) = x_0 \\ y(t) = Cx(t), & \end{cases}$$

$u(t)$: control (input) vector; $\qquad y(t)$: output vector

$x(t)$: state vector; $\qquad x_0$: initial state

Minimization problem for a Cost functional: (simplified form)

$$\inf_u \mathcal{J}(u, x_0) \qquad \mathcal{J}(u, x_0) := \int_0^\infty \left(x(t)^\top C^\top C x(t) + u(t)^\top u(t) \right) dt$$

Dynamical systems and the Riccati equation

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

$$\inf_u \mathcal{J}(u, x_0) \qquad \mathcal{J}(u, x_0) := \int_0^\infty \left(x(t)^\top C^\top C x(t) + u(t)^\top u(t)\right) dt$$

THEOREM Let the pair (A, B) be stabilizable and (C, A) observable. Then there is a unique solution $\mathbf{X} \geq 0$ of the Riccati equation. Moreover,

i) For each x_0 there is a unique optimal control, and it is given by

$$u_*(t) = -B^\top\mathbf{X}\exp((A - BB^\top\mathbf{X})t)x_0 \quad \text{for} \quad t \geq 0$$

ii) $\mathcal{J}(u_*, x_0) = x_0^\top\mathbf{X}x_0$ for all $x_0 \in \mathbb{R}^n$

see, e.g., Lancaster & Rodman, 1995

Order reduction of dynamical systems by projection

Let $V_k \in \mathbb{R}^{n \times d_k}$ have orthonormal columns, $d_k \ll n$

Let $\quad T_k = V_k^\top A V_k, \quad B_k = V_k^\top B, \quad C_k^\top = V_k^\top C^\top$

Reduced order dynamical system:

$$\begin{cases} \dot{\widehat{x}}(t) = T_k \widehat{x}(t) + B_k \widehat{u}(t), & \widehat{x}(0) = \widehat{x}_0 := V_k^\top x_0 \\ \widehat{y}(t) = C_k \widehat{x}(t) \end{cases}$$

$x_k(t) = V_k \widehat{x}(t) \approx x(t)$

Typical frameworks:

- Transfer function approximation
- Model reduction

The role of the projected Riccati equation

Consider again the reduced Riccati equation:

$$(V_k^\top A V_k)\mathbf{Y} + \mathbf{Y}(V_k^\top A^\top V_k) - \mathbf{Y}(V_k^\top B B^\top V_k)\mathbf{Y} + (V_k^\top C^\top)(C V_k) = 0$$

that is

$$T_k \mathbf{Y} + \mathbf{Y} T_k^\top - \mathbf{Y} B_k B_k^\top \mathbf{Y} + C_k^\top C_k = 0 \qquad (*)$$

The role of the projected Riccati equation

Consider again the reduced Riccati equation:

$$(V_k^\top A V_k)\mathbf{Y} + \mathbf{Y}(V_k^\top A^\top V_k) - \mathbf{Y}(V_k^\top B B^\top V_k)\mathbf{Y} + (V_k^\top C^\top)(C V_k) = 0$$

that is

$$T_k \mathbf{Y} + \mathbf{Y} T_k^\top - \mathbf{Y} B_k B_k^\top \mathbf{Y} + C_k^\top C_k = 0 \qquad (*)$$

THEOREM. Let the pair (T_k, B_k) be stabilizable and (C_k, T_k) observable. Then there is a unique solution $\mathbf{Y}_k \geq 0$ of (*) that for each $\widehat{x}_0$ gives the feedback optimal control

$$\widehat{u}_*(t) = -B_k^* \mathbf{Y}_k \exp((T_k - B_k B_k^* \mathbf{Y}_k)t)\widehat{x}_0, \quad t \geq 0$$

for the reduced system.

The role of the projected Riccati equation

Consider again the reduced Riccati equation:

$$(V_k^\top A V_k)\mathbf{Y} + \mathbf{Y}(V_k^\top A^\top V_k) - \mathbf{Y}(V_k^\top B B^\top V_k)\mathbf{Y} + (V_k^\top C^\top)(C V_k) = 0$$

that is

$$T_k \mathbf{Y} + \mathbf{Y} T_k^\top - \mathbf{Y} B_k B_k^\top \mathbf{Y} + C_k^\top C_k = 0 \qquad (*)$$

THEOREM. Let the pair (T_k, B_k) be stabilizable and (C_k, T_k) observable. Then there is a unique solution $\mathbf{Y}_k \geq 0$ of (*) that for each $\widehat{x}_0$ gives the feedback optimal control

$$\widehat{u}_*(t) = -B_k^* \mathbf{Y}_k \exp((T_k - B_k B_k^* \mathbf{Y}_k)t)\widehat{x}_0, \quad t \geq 0$$

for the reduced system.

♣ If there exists a matrix K such that $A - BK$ is passive, then the pair (T_k, B_k) is stabilizable.

Projected optimal control vs approximate control

⋆ Our projected optimal control function:

$$\widehat{u}_*(t) = -B_k^\top \mathbf{Y}_k \exp((T_k - B_k B_k^\top \mathbf{Y}_k)t)\widehat{x}_0, \quad t \geq 0$$

with $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$

⋆ Commonly used approximate control function:

If $\widetilde{\mathbf{X}}$ is some approximation to $\mathbf{X}$, then

$$\widetilde{u}(t) := -B^\top \widetilde{\mathbf{X}} \widetilde{x}(t)$$

where $\widetilde{x}(t) := \exp((A - BB^\top \widetilde{\mathbf{X}})t)x_0$

$$\widehat{u}_* \neq \widetilde{u}$$

They induce different actions on the functional $\mathcal{J}$, even for $\widetilde{\mathbf{X}} = \mathbf{X}_k$

Projected optimal control vs approximate control

$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$

Residual matrix: $\quad R_k := A\mathbf{X}_k + \mathbf{X}_k A - \mathbf{X}_k BB^\top \mathbf{X}_k + C^\top C$

$\star$ Projected optimal control function:

$\widehat{u}_*(t) = -B_k^\top \mathbf{Y}_k \exp((T_k - B_k B_k^\top \mathbf{Y}_k)t)$

THEOREM. Assume that $A - BB^\top \mathbf{X}_k$ is stable and that $\widetilde{u}(t) := -B^\top \mathbf{X}_k x(t)$ approx control. Then

$$|\mathcal{J}(\widetilde{u}, x_0) - \widehat{\mathcal{J}}_k(\widehat{u}_*, \widehat{x}_0)| = \mathcal{E}_k, \quad \text{with} \quad \mathcal{E}_k \leq \frac{\|R_k\|}{2\alpha} x_0^\top x_0,$$

where $\alpha > 0$ is such that $\|e^{(A-BB^\top \mathbf{X}_k)t}\| \leq e^{-\alpha t}$ for all $t \geq 0$.

Note: $|\mathcal{J}(\widetilde{u}, x_0) - \widehat{\mathcal{J}}_k(\widehat{u}_*, \widehat{x}_0)|$ is nonzero for $R_k \neq 0$

On the choice of approximation space

Approximate solution $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$, with

$$(V_k^\top A V_k)\mathbf{Y}_k + \mathbf{Y}_k(V_k^\top A^\top V_k) - \mathbf{Y}_k(V_k^\top BB^\top V_k)\mathbf{Y}_k + (V_k^\top C^\top)(CV_k) = 0$$

Krylov-type subspaces: (extensively used in the linear case)

- $\mathcal{K}_k(A, C^\top) := \mathrm{Range}([C^\top, AC^\top, \ldots, A^{k-1}C^\top])$ (Polynomial)
- $\mathcal{EK}_k(A, C^\top) := \mathcal{K}_k(A, C^\top) + \mathcal{K}_k(A^{-1}, A^{-1}C^\top)$ (EKS, Rational)
- $\mathcal{RK}_k(A, C^\top, \mathbf{s}) :=$

  $$\mathrm{Range}([C^\top, (A - s_2 I)^{-1}C^\top, \ldots, \prod_{j=1}^{k-1}(A - s_{j+1}I)^{-1}C^\top])$$

  (RKS, Rational)

On the choice of approximation space

Approximate solution $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$, with

$$(V_k^\top A V_k)\mathbf{Y}_k + \mathbf{Y}_k(V_k^\top A^\top V_k) - \mathbf{Y}_k(V_k^\top BB^\top V_k)\mathbf{Y}_k + (V_k^\top C^\top)(CV_k) = 0$$

Krylov-type subspaces: (extensively used in the linear case)

- $\mathcal{K}_k(A, C^\top) := \mathrm{Range}([C^\top, AC^\top, \ldots, A^{k-1}C^\top])$ (Polynomial)
- $\mathcal{EK}_k(A, C^\top) := \mathcal{K}_k(A, C^\top) + \mathcal{K}_k(A^{-1}, A^{-1}C^\top)$ (EKS, Rational)
- $\mathcal{RK}_k(A, C^\top, \mathbf{s}) :=$

  $$\mathrm{Range}([C^\top, (A - s_2 I)^{-1}C^\top, \ldots, \prod_{j=1}^{k-1}(A - s_{j+1}I)^{-1}C^\top])$$

  (RKS, Rational)

$\star$ Matrix $BB^\top$ **not** involved (nonlinear term!)

On the choice of approximation space

Approximate solution $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$, with

$$(V_k^\top A V_k)\mathbf{Y}_k + \mathbf{Y}_k(V_k^\top A^\top V_k) - \mathbf{Y}_k(V_k^\top BB^\top V_k)\mathbf{Y}_k + (V_k^\top C^\top)(CV_k) = 0$$

Krylov-type subspaces: (extensively used in the linear case)

- $\mathcal{K}_k(A, C^\top) := \mathrm{Range}([C^\top, AC^\top, \ldots, A^{k-1}C^\top])$ (Polynomial)
- $\mathcal{EK}_k(A, C^\top) := \mathcal{K}_k(A, C^\top) + \mathcal{K}_k(A^{-1}, A^{-1}C^\top)$ (EKS, Rational)
- $\mathcal{RK}_k(A, C^\top, \mathbf{s}) :=$
  $$\mathrm{Range}([C^\top, (A - s_2 I)^{-1}C^\top, \ldots, \prod_{j=1}^{k-1}(A - s_{j+1}I)^{-1}C^\top])$$
  (RKS, Rational)

$\star$ Matrix $BB^\top$ **not** involved (nonlinear term!)

$\star$ Parameters s_j (adaptively) chosen in field of values of $-A$

Performance of solvers

Problem: A: 3D Laplace operator, B, C `randn` matrices, tol$=10^{-8}$

$(n, p, s) = (125000, 5, 5)$

	its	inner its	time	space dim	rank(X_f)
Newton $X_0 = 0$	15	5, . . . , 5	808	100	95
GP-EKS	20		531	200	105
GP-RKS	25		524	125	105

$(n, p, s) = (125000, 20, 20)$

	its	inner its	time	space dim	rank(X_f)
Newton $X_0 = 0$	19	5, . . . , 5	2332	400	346
GP-EKS	15		622	600	364
GP-RKS	20		720	400	358

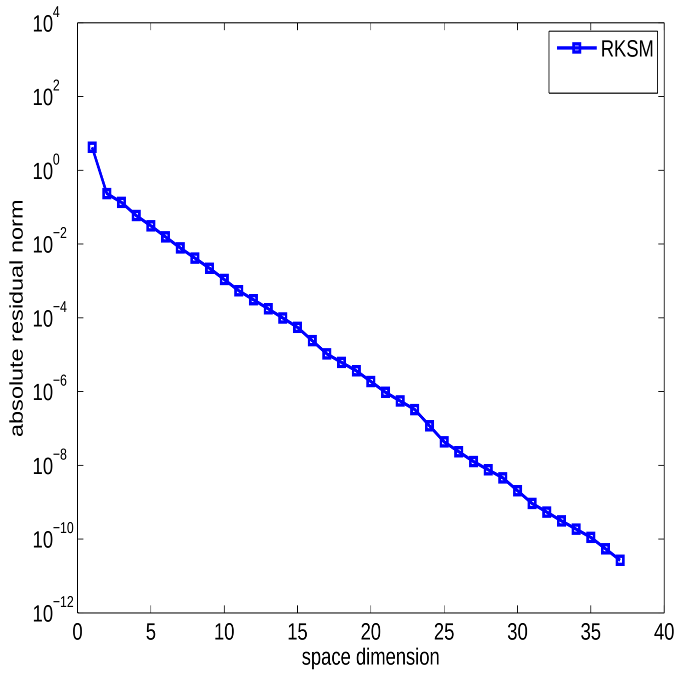
GP=Galerkin projection

(V.Simoncini & D.Szyld & M.Monsalve, 2014)

A numerical example on the role of $BB^\top$

Consider the 500×500 Toeplitz matrix

$$A = \text{toeplitz}(-1, \underline{2.5}, 1, 1, 1), \quad C = [1, -2, 1, -2, \ldots], B = \mathbf{1}$$



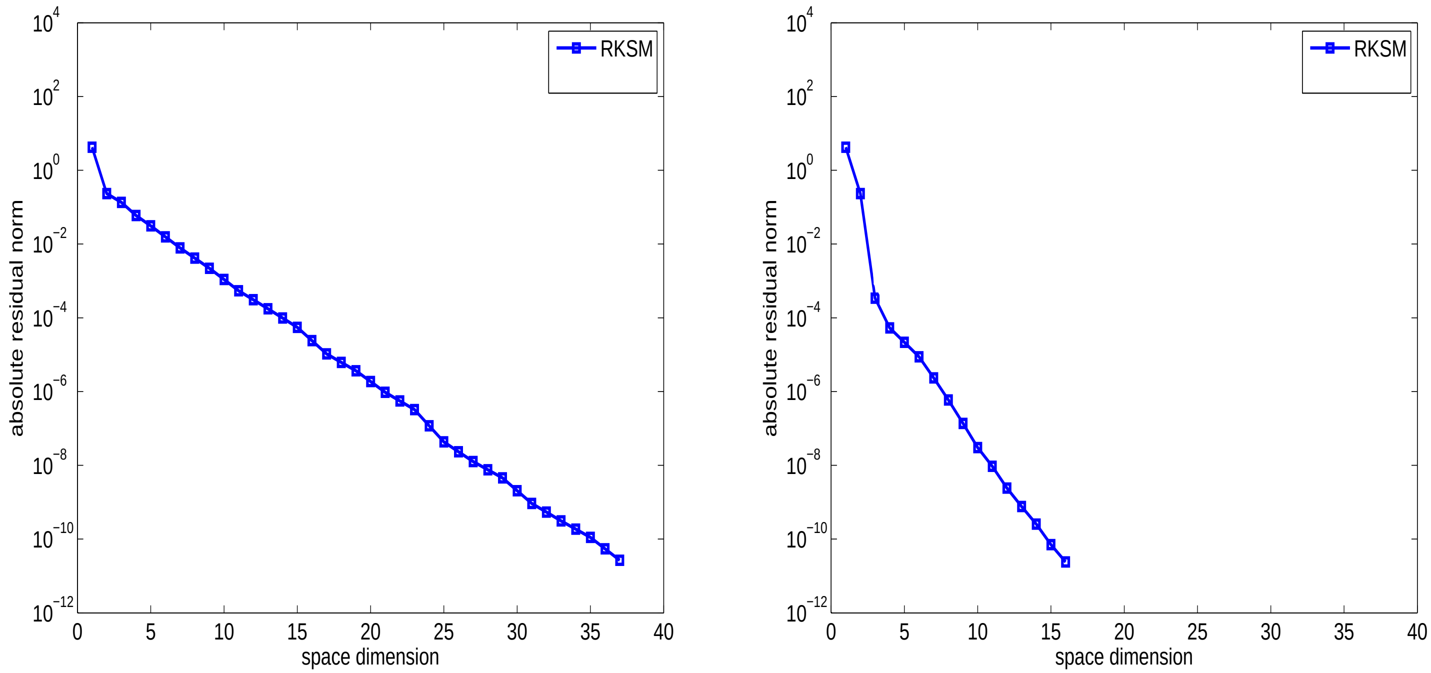
Parameter computation:

Left: adaptive RKS on A

A numerical example on the role of $BB^\top$

Consider the 500×500 Toeplitz matrix

$$A = \mathrm{toeplitz}(-1, \underline{2.5}, 1, 1, 1), \quad C = [1, -2, 1, -2, \ldots], B = \mathbf{1}$$



Parameter computation:
Left: adaptive RKS on A **Right**: adaptive RKS on $A - BB^\top \mathbf{X}_k$

(Lin & Simoncini 2015)

On the residual matrix and adaptive RKS

$$R_k := A\mathbf{X}_k + \mathbf{X}_k A - \mathbf{X}_k BB^\top \mathbf{X}_k + C^\top C$$

THEOREM. Let $\mathcal{T}_k = T_k - B_k B_k^\top \mathbf{Y}_k$. Then

$$R_k = \widehat{R}_k V_k^\top + V_k \widehat{R}_k^\top, \qquad \text{with} \quad \widehat{R}_k = AV_k\mathbf{Y}_k + V_k\mathbf{Y}_k\mathcal{T}_k^\top + C^\top(CV_k)$$

so that $\|R_k\|_F = \sqrt{2}\|\widehat{R}_k\|_F$

At least formally:

$\Rightarrow V_k\mathbf{Y}_k V_k^\top$ is a solution to the Riccati equation ($R_k = 0$) if and only if $Z_k = V_k\mathbf{Y}_k$ is the solution to the Sylvester equation ($\widehat{R}_k = 0$)

On the residual matrix and adaptive RKS

$$R_k = \widehat{R}_k V_k^\top + V_k \widehat{R}_k^\top$$

Expression for the semi-residual $\widehat{R}_k$:

THEOREM. Assume $C^\top \in \mathbb{R}^n$, Range(V_k)= $\mathcal{RK}_k(A, C^\top, \mathbf{s})$. Assume that $\mathcal{T}_k = T_k - B_k B_k^\top \mathbf{Y}_k$ is diagonalizable. Then

$$\widehat{R}_k = \psi_{k,T_k}(A) C^\top C V_k (\psi_{k,T_k}(-\mathcal{T}_k^\top))^{-1}.$$

where

$$\psi_{k,T_k}(z) = \frac{\det(zI - T_k)}{\prod_{j=1}^k (z - s_j)}$$

(see also Beckermann 2011 for the linear case)

On the choice of the next parameters s_{k+1}

$$\widehat{R}_k = \psi_{k,T_k}(A)C^\top C V_k(\psi_{k,T_k}(-\mathcal{T}_k^\top))^{-1}.$$

with $\psi_{k,T_k}(z) = \frac{\det(zI - T_k)}{\prod_{j=1}^k (z - s_j)}$

$\star$ Greedy strategy: Next shift should make $(\psi_{k,T_k}(-\mathcal{T}_k^\top))^{-1}$ smaller

$$\Downarrow$$

Determine for which s in the spectral region of $\mathcal{T}_k$ the quantity $(\psi_{k,T_k}(-s))^{-1}$ is large, and add a root there

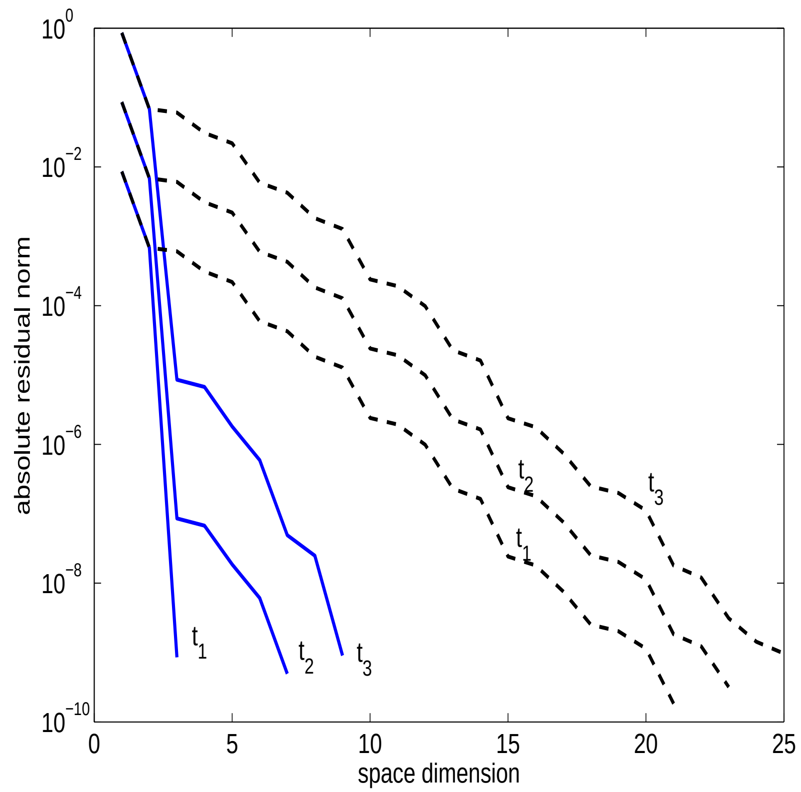
$$s_{k+1} = \arg \max_{s \in \partial \mathbb{S}_k} \left| \frac{1}{\psi_{k,T_k}(s)} \right|$$

$\mathbb{S}_k$ region enclosing the eigenvalues of $-\mathcal{T}_k = -(T_k - B_k B_k^\top \mathbf{Y}_k)$

(This argument is new also for linear equations)

Selection of s_{k+1} in RKS. An example

A: 900×900 2D Laplacian, $B = t\,\mathbf{1}$ with $t_j = 5 \cdot 10^{-j}$, $C = [1, -2, 1, -2, 1, -2, ...]$



RKS convergence with and without modified shift selection as t varies

Solid curves: use of $\mathcal{T}_k$ **Dashed curves:** use of T_k

Further results not presented but relevant

- Stabilization properties of the approx solution $\mathbf{X}_k$
- Accuracy tracking as the approximation space grows
- Interpretation via invariant subspace approximation

(V.Simoncini, 2016)

Further results not presented but relevant

- Stabilization properties of the approx solution $\mathbf{X}_k$
- Accuracy tracking as the approximation space grows
- Interpretation via invariant subspace approximation

(V.Simoncini, 2016)

Wrap-up

♡ Projection-type methods fill the gap between MOR and Riccati equation

♡ Clearer role of the non-linear term during the projection

Outlook

- ♠ Petrov-Galerkin projection (à la Balanced truncation)
  (work in progress, with A. Alla, Florida State Univ.)

- ♠ Projected Differential Riccati equations
  (see, e.g., Koskela & Mena, tr 2017)

- ♠ Parameterized Algebraic Riccati equations
  (see, e.g., Schmidt & Haasdonk, tr 2017)

Outlook

- ♠ Petrov-Galerkin projection (à la Balanced truncation)
  (work in progress, with A. Alla, Florida State Univ.)

- ♠ Projected Differential Riccati equations
  (see, e.g., Koskela & Mena, tr 2017)

- ♠ Parameterized Algebraic Riccati equations
  (see, e.g., Schmidt & Haasdonk, tr 2017)

REFERENCES

- V. Simoncini, Daniel B. Szyld and Marlliny Monsalve, *On two numerical methods for the solution of large-scale algebraic Riccati equations* IMA Journal of Numerical Analysis, (2014)
- Yiding Lin and V. Simoncini, *A new subspace iteration method for the algebraic Riccati equation* Numerical Linear Algebra w/Appl., (2015)
- V.Simoncini, *Analysis of the rational Krylov subspace projection method for large-scale algebraic Riccati equations* SIAM J. Matrix Anal.Appl, (2016)
- V. Simoncini, *Computational methods for linear matrix equations* SIAM Review, (2016)