Large-scale Lyapunov matrix equation with banded data

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Joint work with Davide Palitta, UniBO

The problem

$$A\mathbf{X} + \mathbf{X}A = D, \quad A, D \in \mathbb{R}^{n\times n}$$

A banded, sym pos.def. D banded sym. Large dimensions

- If D is low rank, then large body of literature/algorithms

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We address the case of D banded

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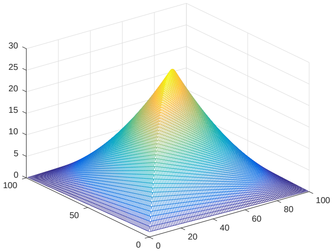
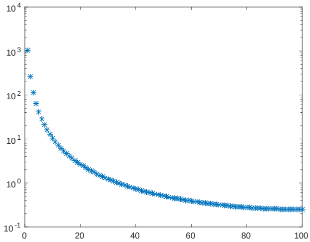
An example: $D = I, \; A = \mathrm{tridiag}(-1, \underline{2}, -1), \quad \Rightarrow \mathbf{X} = \frac{1}{2}A^{-1}$



mesh($\mathbf{X}$)



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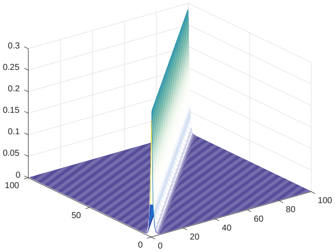

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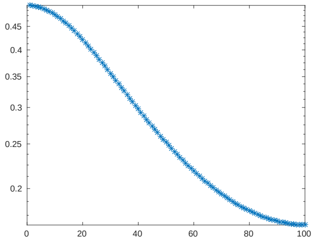


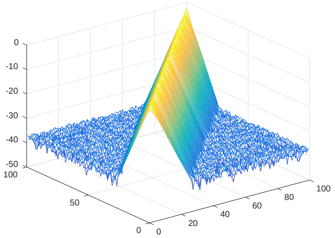
svd($\mathbf{X}$)

Relevance of conditioning for A banded

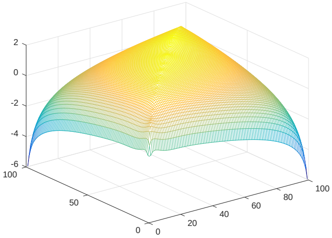
$$A\mathbf{X} + \mathbf{X}A = D, \quad A, D \in \mathbb{R}^{n\times n}$$

D diagonal with random entries.

Lyapunov Solution $\mathbf{X}$ (log-scale):



cond(A)=3



cond(A)=$5\,10^3$

Banded and well conditioned A

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Use formal equivalence with

$$(A \otimes I + I \otimes A)x = d \qquad (**)$$

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After k iterations the approximate matrix solution is banded
(w/bandwidth depending on k and bandwidth of A, D)

Banded and badly conditioned A

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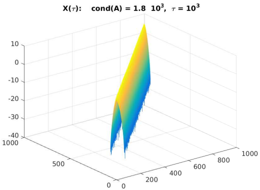
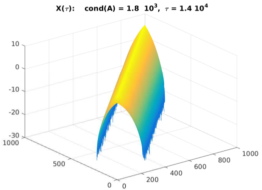
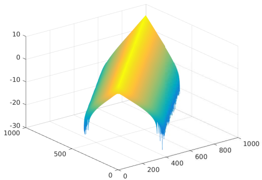


$\tau = \mathcal{O}(10^3)$



$\tau = \mathcal{O}(10^4)$



$\tau = \mathcal{O}(10^5)$

Splitting strategy for an approximate solution

For appropriate $\tau > 0$,

$$\mathbf{X} = \underbrace{X(\tau)}_{num.banded} + \underbrace{e^{-\tau A}\mathbf{X}e^{-\tau A}}_{low-rank}$$

(♣ $e^{-\tau A}$ low rank)

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$$\mathbf{X} \approx \begin{bmatrix} * & * & & & & \\ * & * & * & & & \\ & * & * & * & & \\ & & \ddots & \ddots & \ddots & \\ & & & * & * & * \\ & & & & * & * \end{bmatrix} + Z_k Z_k^T$$

with Z_k tall

Approximating the banded term $X(\tau)$

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- $(t_i A - \xi_j I)^{-1} \approx \mathrm{trunc}((t_i A - \xi_j I)^{-1})$ (banded truncation via sparse approx inverse)

Approximating the low-rank term $e^{-\tau A}\mathbf{X}e^{-\tau A}$

Let $\mathcal{V}_m = \mathrm{range}(V_m)$ be a space approximating the "smallest" invariant subspace of A so that

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  and Y_m solves the projected Lyapunov equation
  $H_m Y + Y H_m = V_m^T D V_m$

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Remark: X_m itself is not (necessarily) a good approximation to $\mathbf{X}$
(only the portion on relevant invariant subspace matters)

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  - τ estimated automatically by using a-priori decay of matrix exponential

Numerical experiments

$A \in \mathbb{R}^{n\times n}$: 3-point stencil discretization of $\mathcal{L}u = -\frac{1}{\gamma}\left(e^x u_x\right)_x + \gamma u$
$x \in (0,1)$, Dirichlet b.c. $\gamma > 0$, D sym tridiag. random

* Splitting τ: with $\beta_{\exp} = 500, \epsilon_{\exp} = 10^{-5}$ use theoretical bounds
* Parameters for banded portion:
  Deg. of Rational approx $\nu = 7$, truncation threshold $\epsilon_{trunc} = 10^{-8}$
* Parameter for low rank portion: stopping threshold 10^{-3}

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n	γ	$\kappa(A)$	τ	Time X_B (β_{X_B})	Time X_m (rank(X_m))	Time tot.
40000	1000	6.61e+3	2.72	1.57e+3 (489)	3.49e+0 (7)	1.57e+3
	500	2.68e+4	0.56	1.55e+3 (579)	2.16e+2 (374)	1.77e+3
	200	1.72e+5	0.08	1.63e+3 (595)	2.43e+2 (408)	1.87e+3
70000	1800	6.19e+3	2.97	2.81e+3 (475)	5.31e+0 (7)	2.82e+3
	800	3.17e+4	0.47	2.87e+3 (583)	1.07e+3 (654)	3.94e+3
	200	5.27e+5	0.02	2.92e+3 (597)	1.15e+3 (693)	4.07e+3
100000	2500	6.53e+3	2.77	4.08e+3 (487)	9.07e+0 (7)	4.08e+3
	1500	1.82e+4	0.84	4.17e+3 (571)	2.77e+3 (879)	*6.95e+3
	500	1.67e+5	0.08	3.99e+3 (595)	2.78e+3 (916)	*6.78e+3

Conclusions and outlook

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Further reading

⋆ V. Simoncini,
Computational methods for linear matrix equations,
SIAM Review, v.58, Sept. 2016.

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