# Spectral properties of preconditioned saddle point linear systems

V. Simoncini

Dipartimento di Matematica, Università di Bologna

valeria@dm.unibo.it

## The problem

$$\begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Computational Fluid Dynamics (Elman, Silvester, Wathen 2005)
- Elasticity problems
- Mixed (FE) formulations of II and IV order elliptic PDEs
- Linearly Constrained Programs
- Linear Regression in Statistics
- Image restoration
- ... Survey: Benzi, Golub and Liesen, Acta Num 2005

## The problem

$$\begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Hypotheses:

- ⋆ $A \in \mathbb{R}^{n \times n}$ (non-)symmetric
- ⋆ $B^\top \in \mathbb{R}^{n \times m}$ tall, $m \leq n$
- ⋆ $C$ symmetric positive (semi)definite

More hypotheses later...

## Why are we interested in spectral bounds?

- To detect “sensitive” blocks in the coeff. matrix

  (guidelines for preconditioning strategies)

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- To detect "sensitive" blocks in the coeff. matrix
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- To "tune" the stabilization parameter (matrix $C$)

- To predict convergence behavior of the iterative solver

## Iterative solver. Convergence considerations.

$$\mathcal{M}x = b$$

$\mathcal{M}$ is symmetric and indefinite $\quad \longrightarrow \quad$ MINRES

$$x_k \in x_0 + K_k(\mathcal{M}, r_0), \quad \text{s.t.} \qquad \min \|b - \mathcal{M}x_k\|$$

$r_k = b - \mathcal{M}x_k$, $k = 0, 1, \ldots$, $x_0$ starting guess

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If $\mu(\mathcal{M}) \subset [-a, -b] \cup [c, d]$, with $|b - a| = |d - c|$, then

$$\|b - \mathcal{M}x_{2k}\| \leq 2 \left( \frac{\sqrt{ad} - \sqrt{bc}}{\sqrt{ad} + \sqrt{bc}} \right)^k \|b - \mathcal{M}x_0\|$$

Note: more general but less tractable bounds available

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Idea: write $r_{2k} = b - \mathcal{M}x_{2k} = q_{2k}(\mathcal{M})r_0, \qquad q_{2k} \in \mathbb{P}_{2k}^{(0)}$

$$\|r_{2k}\| = \min_{q_{2k} \in \mathbb{P}_{2k}^{(0)}} \|q_{2k}(\mathcal{M})r_0\| \leq \min_{q_k \in \mathbb{P}_k^{(0)}} \|q_k(\mathcal{M}^2)r_0\| \leq \ldots$$

Fischer etal. '98

## Spectral properties. $\mathcal{M}$ symmetric.

$$\mathcal{M} = \begin{bmatrix} A & B^\top \\ B & O \end{bmatrix} \qquad \begin{matrix} 0 < \lambda_n \leq \cdots \leq \lambda_1 & \text{eigs of } A \\ 0 < \sigma_m \leq \cdots \leq \sigma_1 & \text{sing. vals of } B \end{matrix}$$

$\mu(\mathcal{M})$ subset of (Rusten & Winther 1992)

$$\left[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}), \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2})\right] \quad \cup \quad \left[\lambda_n, \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 + 4\sigma_1^2})\right]$$

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e.g.

$$\mathcal{M} = \begin{bmatrix} I & U^\top \\ U & O \end{bmatrix}, \quad UU^\top = I$$

## Block diagonal Preconditioner

⋆ $A$ spd, $C = 0$:

$$\mathcal{P}_0 = \begin{bmatrix} A & 0 \\ 0 & BA^{-1}B^\top \end{bmatrix}$$

$$\Rightarrow \qquad \mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}} = \begin{bmatrix} I & A^{-\frac{1}{2}} B^\top (BA^{-1}B^\top)^{-\frac{1}{2}} \\ (BA^{-1}B^\top)^{-\frac{1}{2}} BA^{-\frac{1}{2}} & 0 \end{bmatrix}$$

MINRES converges in at most 3 iterations. $\mu(\mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}}) = \{1, 1/2 \pm \sqrt{5}/2\}$

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A more practical choice:

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \qquad \text{spd.} \quad \widetilde{A} \approx A \qquad \widetilde{S} \approx BA^{-1}B^\top$$

eigs in $\qquad [-a, -b] \cup [c, d], \qquad a, b, c, d > 0$

Still an Indefinite Problem, but possibly much easier to solve

## Symmetric and indefinite $A$

$$\mathcal{M} = \begin{bmatrix} A & B^\top \\ B & O \end{bmatrix}$$

$\lambda_n \leq \cdots \leq \lambda_1$ eigs of $A$

$0 < \sigma_m \leq \cdots \leq \sigma_1$ sing. vals of $B$

$A$ pos.def. on $\mathrm{Ker}(B)$

$\mu(\mathcal{M})$ subset of

$$\left[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}), \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2})\right] \quad \cup \quad \left[\Gamma, \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 + 4\sigma_1^2})\right]$$

If $m = n$, $\quad \Gamma = \frac{1}{2}(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_m^2})$

Gould & Simoncini, SIMAX to appear

## Indefinite $A$, $C = 0$. Cont'd

$$\left[\frac{1}{2}(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}), \frac{1}{2}(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2})\right] \quad \cup \quad \left[\Gamma, \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 + 4\sigma_1^2})\right]$$

Letting $\alpha_0 > 0$ be s.t. $\frac{u^\top Au}{u^\top u} > \alpha_0$, $u \in$ Ker$(B)$

$$\Gamma \geq \begin{cases} \dfrac{\alpha_0 \sigma_m^2}{|\alpha_0 \lambda_n - \|A\|^2 - \sigma_m^2|} & \text{if } \alpha_0 + \lambda_n \leq 0 \\ \dfrac{\alpha_0 \lambda_n - \|A\|^2 - \sigma_m^2}{2(\alpha_0 + \lambda_n)} + \sqrt{\left(\dfrac{\alpha_0 \lambda_n - \|A\|^2 - \sigma_m^2}{2(\alpha_0 + \lambda_n)}\right)^2 + \dfrac{\alpha_0 \sigma_m^2}{\alpha_0 + \lambda_n}} & \\ \text{otherwise.} & \end{cases}$$

## Sharpness of the bounds

Ex.1. $A = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$, $B^\top = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$ $\mu(\mathcal{M}) = \{-1.5441, 0.0014257, 4.5427\}$

Ex.2. $A = \begin{bmatrix} 0.01 & 3 \\ 3 & -0.01 \end{bmatrix}$, $B = [0,\ 3]$ $\mu(\mathcal{M}) = \{-4.2452, 5.0 \cdot 10^{-3}, 4.2402\}$

Ex.3. $A = \begin{bmatrix} 1 & -4 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B^\top = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mu(\mathcal{M}) = \{-4.3528, -0.22974, 0.22974, 2, 4.3528\}$

| case | $\lambda_n$ | $\lambda_1$ | $\alpha_0$ | $\sigma_m, \sigma_1$ | $\mathcal{I}^-$ | $\mathcal{I}^+$ |
|---|---|---|---|---|---|---|
| Ex.1 | -1.5414 | 4.5414 | 1.0 | 0.1 | [-1.5478, -0.0022] | [0.0004, 4.5436] |
| Ex.2 | -3.0000 | 3.0000 | 0.01 | 3 | [-4.8541, -1.8541] | [ $4.9917 \cdot 10^{-3}$, 4.8541] |
| Ex.3 | -4.1231 | 4.1231 | 2.0 | 1 | [-4.3528, -0.22974] | [0.0762, 4.3528] |

## Augmenting the (1,1) block

Equivalent formulation ($C = 0$):

$$\begin{bmatrix} A + \tau B^\top B & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f + \tau B^\top g \\ g \end{bmatrix}, \quad \tau \in \mathbb{R}$$

coefficient matrix: $\mathcal{M}(\tau)$

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Condition on $\tau$ for definiteness of $A + \tau B^\top B$:

$$\tau > \frac{1}{\sigma_m^2} \left( \frac{\|A\|^2}{\alpha_0} - \lambda_n \right)$$

Ex.2. $A = \begin{bmatrix} 0.01 & 3 \\ 3 & -0.01 \end{bmatrix}$, $\mu(\mathcal{M}) = \{-4.2452, 5.0 \cdot 10^{-3}, 4.2402\}$

$\frac{1}{\sigma_m^2} \left( \frac{\|A\|^2}{\alpha_0} - \lambda_n \right) = 100.33$

for $\tau = 100 \rightarrow A + \tau B^\top B$ is indefinite

## Augmenting the (1,1) block

Assume "good" $\tau$ is taken.

$$\begin{bmatrix} A + \tau B^\top B & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f + \tau B^\top g \\ g \end{bmatrix}, \quad \tau \in \mathbb{R}$$

Spectral intervals for (1,1) spd may be obtained

## “Regularized” problem

$$\begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad \mathcal{M}_C z = b$$

Warning: for $A$ indefinite, conditions on $C$ required:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \text{singular!}$$

Note: Perturbation results yield spectral bounds assuming $\lambda_{\max}^{C} < \Gamma$

## “Regularized” problem

More accurate result:

If $\lambda^C_{\max} < \frac{\alpha_0 \sigma_m^2}{\|A\|^2 - \lambda_n \alpha_0}$, then $\mu(\mathcal{M}_C) \subset \mathcal{I}^- \cup \mathcal{I}^+$ with

$$\mathcal{I}^- = \left[\frac{1}{2}\left(\lambda_n - \lambda^C_{\max} - \sqrt{(\lambda_n + \lambda^C_{\max})^2 + 4\sigma_1^2}\right), \frac{1}{2}\left(\lambda_1 - \sqrt{(\lambda_1)^2 + 4\sigma_m^2}\right)\right] \subset \mathbb{R}^-$$

$$\mathcal{I}^+ = \left[\Gamma_C, \frac{1}{2}\left(\lambda_1 + \sqrt{(\lambda_1)^2 + 4\sigma_1^2}\right)\right] \subset \mathbb{R}^+,$$

For $m = n$, $\quad \Gamma_C = \frac{1}{2}\left(\lambda_n - \lambda^C_{\max} + \sqrt{(\lambda_n + \lambda^C_{\max})^2 + 4\sigma_m^2}\right)$

more complicated (but explicit!) estimate for $m < n$

## “Regularized” problem

An example:

$$\mathcal{M}_C = \begin{bmatrix} \lambda_n & 0 & \sigma \\ 0 & \lambda_1 & 0 \\ \sigma & 0 & -\gamma^C \end{bmatrix},$$

with $\lambda_n < 0, \lambda_1 > 0, \sigma > 0$. If $\gamma^C = -\sigma^2/\lambda_n$ then $\mathcal{M}_C$ is singular.

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Our estimate requires (for $\|A\| = \alpha_0 = -\lambda_n$): $0 \le \gamma^C \le \frac{1}{2} \, \frac{-\sigma^2}{\lambda_n}$

(half the value from singularity!)

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Related results: Bai, Ng, Wang ('09) qualitatively similar bound based on $B^\top C^{-1} B$, $A + B^\top C^{-1} B$ (no full rank hyp. on $B$)
Bai (tech.rep.'09)

## Application to **practical** block diagonal preconditioners

Indefinite $A$ (and $C = 0$). Indefinite preconditioner:

Let $\mathcal{P}_\pm = \mathrm{blkdiag}(A, \pm\widetilde{S})$ with $A$, $\widetilde{S}$ nonsingular. Then

$$\mu(\mathcal{P}_\pm^{-1}\mathcal{M}) \subset \left\{1, \frac{1}{2}(1+\sqrt{1+4\xi}), \frac{1}{2}(1-\sqrt{1+4\xi})\right\} \subset \mathbb{C},$$

$\xi$ : (possibly complex) eigenvalues of $(BA^{-1}B^\top, \pm\widetilde{S})$

## Application to ideal block diagonal preconditioners

Indefinite $A$, $C \neq 0$. Indefinite preconditioner:

Let $\mathcal{P}_+ = \mathrm{blkdiag}(A, C + BA^{-1}B^\top)$. Then

$$\mu(\mathcal{P}_+^{-1}\mathcal{M}) \subset \left\{1, \frac{1}{2}(1 \pm \sqrt{5}), \frac{1}{2\theta}(\theta - 1 \pm \sqrt{(1-\theta)^2 + 4\theta^2})\right\} \subset \mathbb{R}.$$

$\theta$ finite eigs of $(C + BA^{-1}B^\top, C)$

Similar (but complex) results for $\mathcal{P}_- = \mathrm{blkdiag}(A, -C - BA^{-1}B^\top)$

## Application to ideal block diagonal preconditioners

Indefinite $A$, Definite preconditioner, $C = 0$:

$$\mathcal{P}(\tau) = \begin{bmatrix} P_A & \\ & P_C \end{bmatrix}, \qquad \begin{aligned} P_A &\approx P_A(\tau) = A + \tau B^\top B \\ P_C &\approx P_C(\tau) = B(A + \tau B^\top B)^{-1} B^\top \end{aligned}$$

- Definite preconditioner on definite problem:

$\mathcal{P}(\tau)^{-1}\mathcal{M}(\tau)$ has eigenvalues

$$1,\ \tfrac{1}{2}(1+\sqrt{5}),\ \tfrac{1}{2}(1-\sqrt{5})$$

with multiplicity $n-m$, $m$ and $m$, respectively.

## General nonsymmetric problem

$$\mathcal{M} = \begin{bmatrix} F & B^\top \\ B & -\beta C \end{bmatrix} \qquad F \quad \text{nonsymmetric}$$

Preconditioning strategies (other alternatives are possible):

$$\mathcal{P}_{tr} = \begin{bmatrix} \widetilde{F} & B \\ & \pm\widetilde{C} \end{bmatrix} \qquad \mathcal{P}_d = \begin{bmatrix} \widetilde{F} & \\ & \pm\widetilde{C} \end{bmatrix} \text{ with } \widetilde{C} > 0$$

- $\widetilde{F} \approx F$
- $\widetilde{F} \approx F + B^\top \widetilde{C}^{-1} B$ (augmentation block precond.)

For $+\widetilde{C}$: $\qquad \mathcal{M}\mathcal{P}^{-1}$ indefinite

## Augmentation block preconditioning

⋆ Appealing for $F$ singular

For $+\widetilde{C}$:

$$\mathcal{M}\mathcal{P}_d^{-1},\ \mathcal{M}\mathcal{P}_{tr}^{-1} \text{ have clusters in } \mathbb{C}^- \text{ and } \mathbb{C}^+$$

$\Rightarrow$ Indefinite matrix $\Rightarrow$ Elman's bound not applicable

Analysis of clusters:

- Schötzau & Greif '06 ($F$ sym)
- Cao '07

## Iterative solver. Convergence considerations.

$$\mathcal{M}x = b$$

$\mathcal{M}$ is nonsymmetric and indefinite $\quad\longrightarrow\quad$ GMRES

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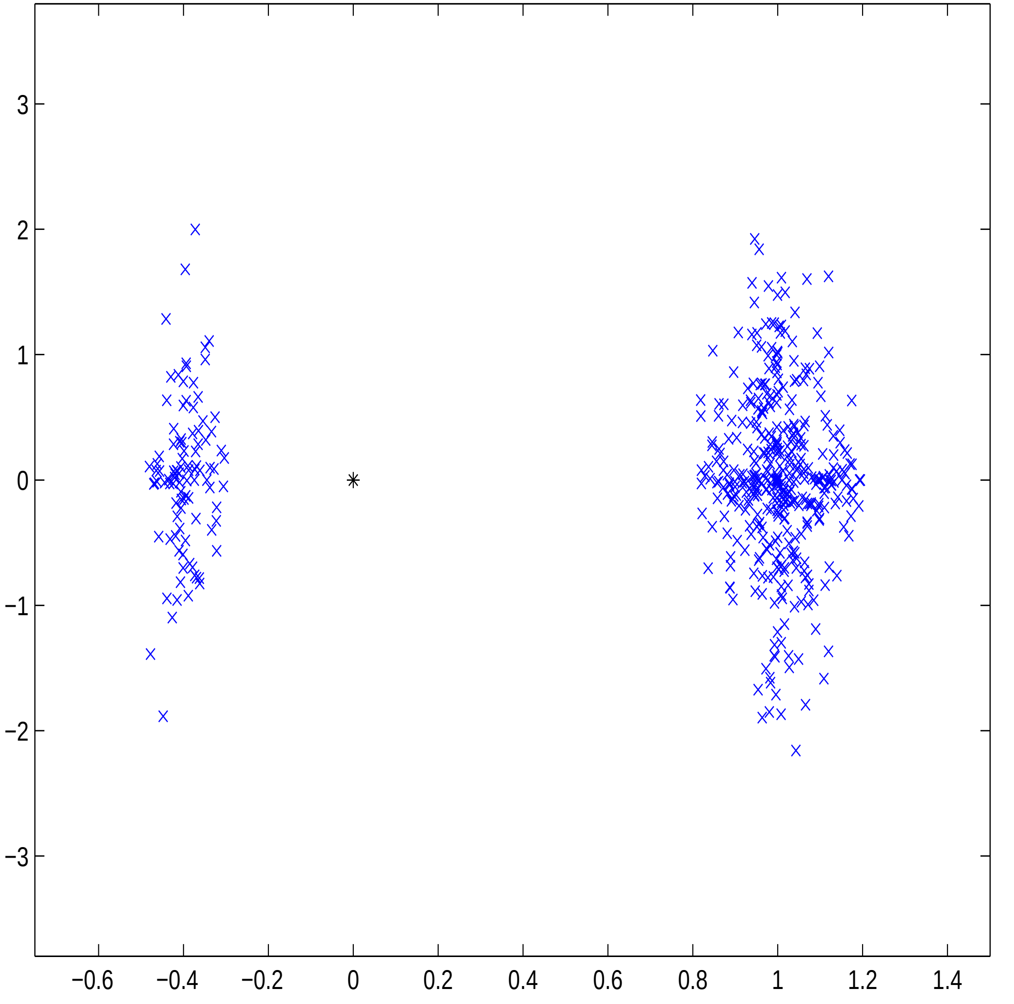
For $\mathcal{M}$ non-normal indefinite :

- In theory, complete stagnation is possible;
- Rule of thumb: tight spectral clusters help

# Rule of thumb: clustering helps



## GMRES: Nonstagnation condition (Simoncini & Szyld, '08)

Let $H = \frac{1}{2}(\mathcal{M} + \mathcal{M}^\top)$, $S = \frac{1}{2}(\mathcal{M} - \mathcal{M}^\top)$. If

$$H \text{ nonsingular and } \quad \|SH^{-1}\| < 1$$

Then

$$\|r_2\| \le \left(1 - \frac{\theta_{\min}^2}{\|\mathcal{M}^2\|^2}\right)^{\frac{1}{2}} \|r_0\| \qquad \theta_{\min} = \lambda_{\min}\left(\tfrac{1}{2}(\mathcal{M}^2 + (\mathcal{M}^2)^\top)\right) > 0$$

The same relation holds at every other iteration

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The same relation holds at every other iteration

Condition very suitable for preconditioned saddle point matrices

## Nonstagnation condition revisited. Grcar tech.rep'89

Let $\phi_k$ be polynomial with $\phi_k(0) = 0$. If $\frac{1}{2}(\phi_k(\mathcal{M}) + \phi_k(\mathcal{M})^\top) > 0$ then

$$\|r_k\| \leq \left(1 - \frac{\theta_{\min}^2}{\|\phi_k(\mathcal{M})\|^2}\right)^{\frac{1}{2}} \|r_0\| \qquad \theta_{\min} = \lambda_{\min}(\tfrac{1}{2}(\phi_k(\mathcal{M}) + \phi_k(\mathcal{M})^\top))$$

Elman's bound: $k = 1$

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The simplest case: $k = 2$

If $\phi_2(H) > 0$, then $\phi_2(\mathcal{M}) > 0$ iff $\|S\phi_2(H)^{-1/2}\| < 1$

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$\Rightarrow$ In Simoncini & Szyld '08: $\phi_2(\lambda) = \lambda^2$

$\Rightarrow$ Here: $\phi_2(\lambda) = \lambda(\lambda - \alpha)$, $\quad \alpha = \max\{0, \lambda_+(H) + \lambda_-(H)\}$

($\lambda_+(H), \lambda_-(H)$: pos/neg eigs closest to zero)

## Example. Navier-Stokes problem

IFISS Package (Elman, Ramage, Silvester)

"Flow over a step". Uniform grid, Q1-P0 elements

| Prec | blocks | $\lambda_{\min}(H)$ | $\lambda_{\max}(S^\top S, \phi_2(H))$ | $\alpha$ | # its |
|---|---|---|---|---|---|
| $P_{d,aug}$ | $\widetilde{C}(0)$ | -3.5512 | 0.9906 | 0.3951 | 16 |
| | $\widetilde{C}(10^{-1})$ | -2.7567 | 0.9724 | 0.4252 | 19 |
| | $Q$ | -4.2339 | 1.5620 | 0.3558 | 29 |
| $P_{tr,aug}$ | $\widetilde{C}(0)$ | -3.8091 | 0.9672 | 0 | 14 |
| | $\widetilde{C}(10^{-1})$ | -3.0814 | 1.1063 | 0.0216 | 21 |
| | $\widetilde{C}(10^{-2})$ | -3.7450 | 0.97097 | 0 | 16 |
| $P_{tr}$ | $\widehat{F}, W(1)$ | -7.3000 | 0.9923 | 0 | 11 |
| | $\widehat{F}, W(0)$ | -13.818 | 0.9924 | 0 | 17 |

$\widetilde{C}(\mathrm{tol}) = B\widetilde{F}^{-1}B^\top + \beta C \qquad \widetilde{F} = \mathrm{luinc}(F, \mathrm{tol})$

(2,2) block: $W(s_1) = B\widehat{F}^{-1}B^\top + s_1\beta C \qquad \widehat{F} = \mathrm{luinc}(F, 10^{-2})$

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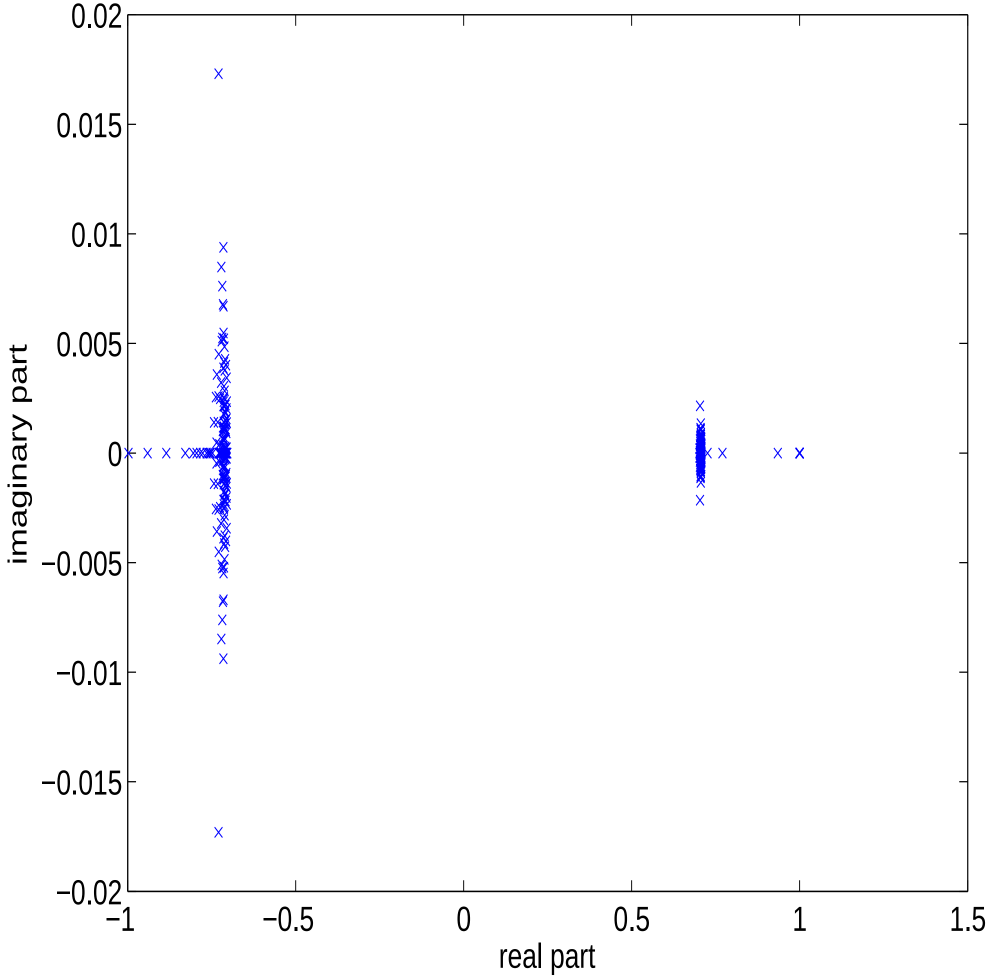
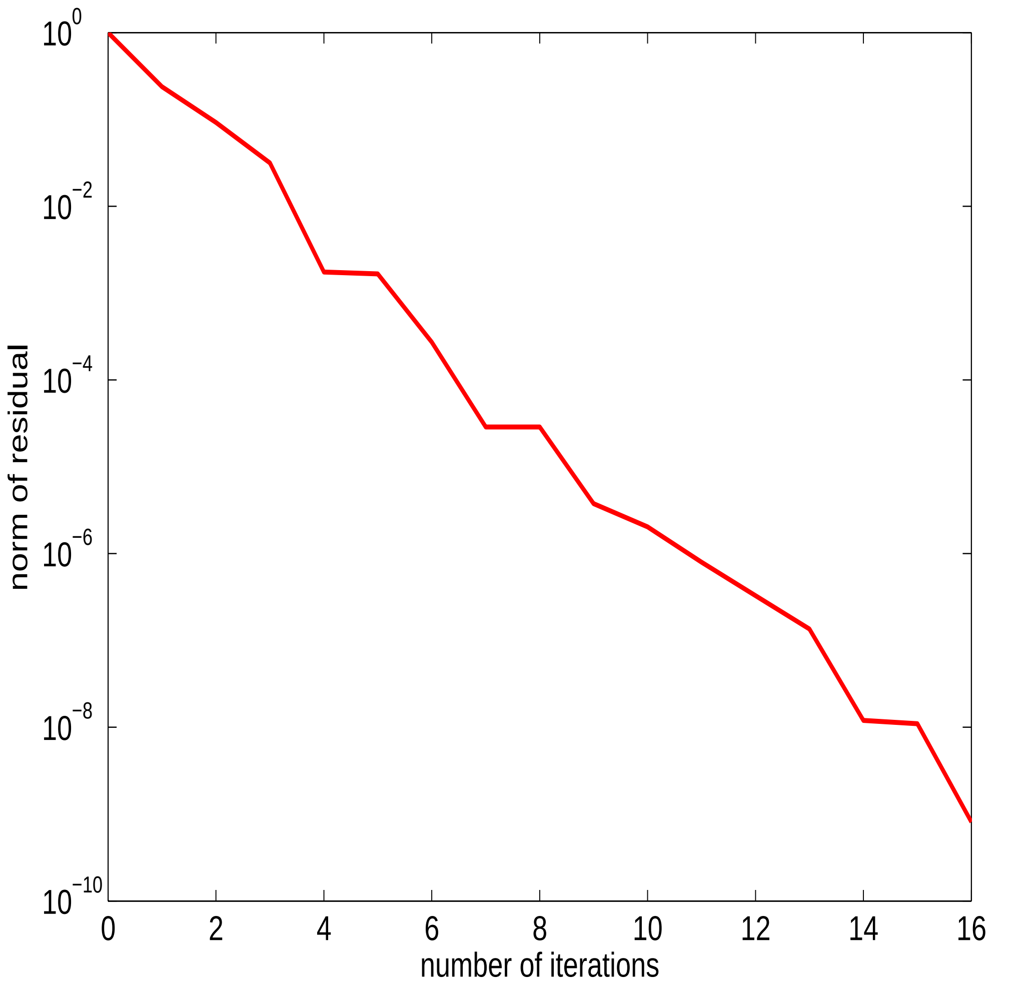
## Spectrum of $\mathcal{M}P_{tr,aug}^{-1}$



## GMRES Convergence history



## Mesh independence

$P_{tr,aug}$.  (2,2) block: $\widetilde{C} = \beta C + BF^{-1}B^\top$,

| $n$ | $m$ | $\lambda_{\min}(H)$ | $\lambda_{\max}(S^\top S, \phi_2(H))$ | $\alpha$ | # its |
|---|---|---|---|---|---|
| 418 | 176 | -3.8091 | 0.9672 | 0 | 14 |
| 1538 | 704 | -3.7057 | 0.9662 | 0 | 15 |
| 5890 | 2816 | -3.6710 | 0.9660 | 0 | 13 |

## Beyond non-stagnation. Indefinite vs. Definite

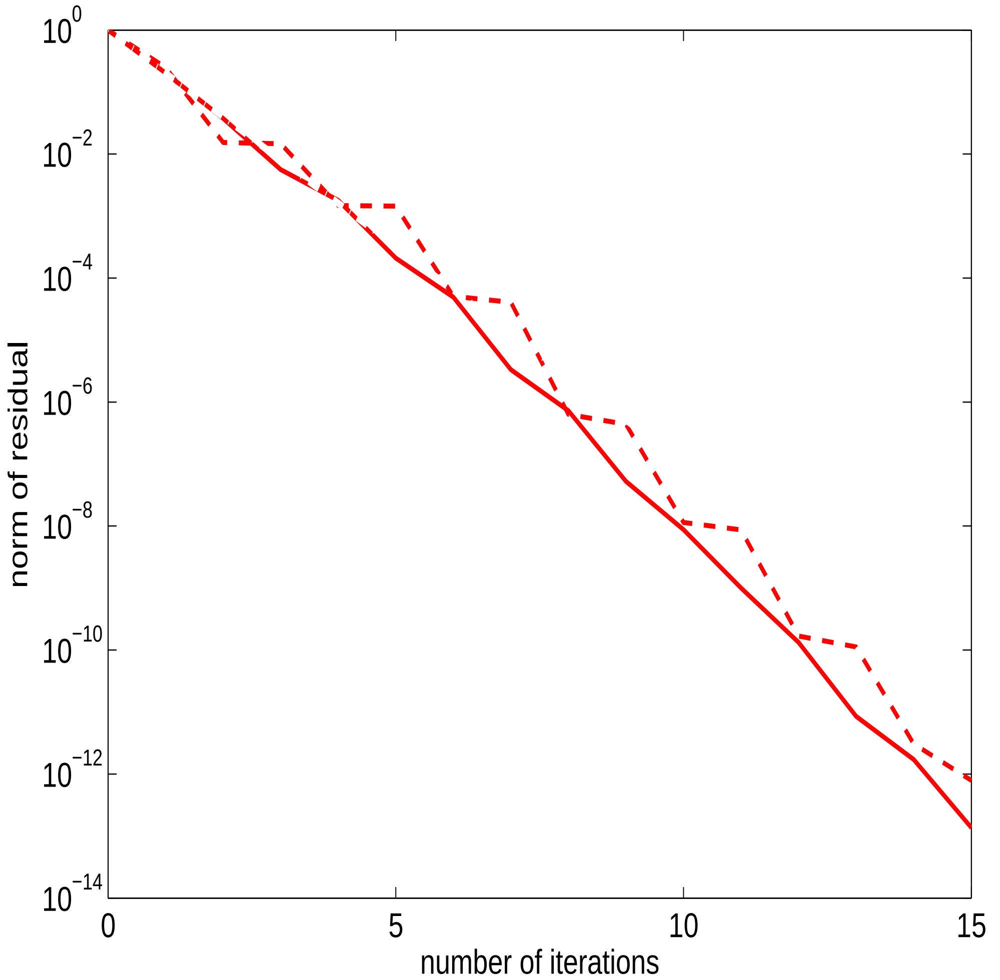
$$\mathcal{P}_{tr,+} = \begin{bmatrix} \widetilde{F} & B \\ & +\widetilde{C} \end{bmatrix} \qquad \mathcal{P}_{tr,-} = \begin{bmatrix} \widetilde{F} & B \\ & -\widetilde{C} \end{bmatrix}$$

$\mathcal{M}\mathcal{P}_{tr,+}^{-1}$ Indefinite

$\mathcal{M}\mathcal{P}_{tr,-}^{-1}$ Positive definite

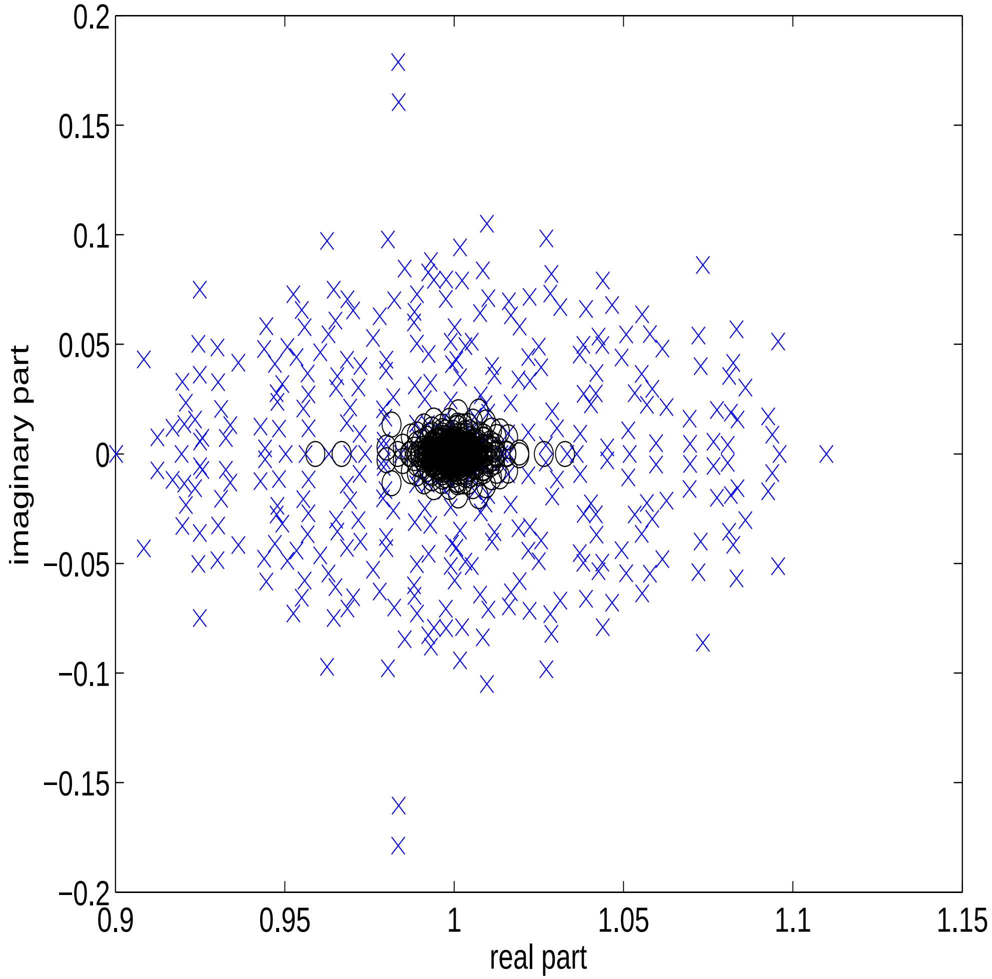
Should $\mathcal{M}\mathcal{P}_{tr,+}^{-1}$ be discarded?

## GMRES Convergence history: $\mathcal{P}_{tr,+}, \mathcal{P}_{tr,-}$



## Spectra and Preconditioned Problems: $\mathcal{MP}_{tr,+}^{-1}, \mathcal{MP}_{tr,-}^{-1}$

## Spectra and Preconditioned Problems: $\mathcal{M}\mathcal{P}_{tr,+}^{-1}, \mathcal{M}\mathcal{P}_{tr,-}^{-1}$




$\times$ : $\mathcal{M}\mathcal{P}_{tr,-}^{-1}$ $\circ$ : $(\mathcal{M}\mathcal{P}_{tr,+}^{-1})^2$

## Spectral properties

$E = (F - \widetilde{F})\widetilde{F}^{-1}$, $\qquad \widetilde{C} = B^\top \widetilde{F}^{-1} B + \beta C$:

$$MP_{tr,-}^{-1} = \begin{bmatrix} I & 0 \\ B\widetilde{F}^{-1} & I \end{bmatrix} + O(E)$$

So that $\lambda(MP_{tr,-}^{-1}) = 1 + O(\|E\|^{\frac{1}{2}})$

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$$(MP_{tr,+}^{-1})^2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + O(E)$$

So that $\lambda(MP_{tr,+}^{-1})^2 = 1 + O(\|E\|)$

## Asymptotic convergence rate

$X$ eigenvector matrix, $\Lambda$ eigenvalues matrix.

If $\Lambda$ definite then the following bound is useful:

$$\|r_{2k}\| \leq \kappa(X) \min_{q_{2k} \in \mathbb{P}_{2k}^{(0)}} \max_{\lambda \in \Lambda} |q_{2k}(\lambda))|,$$

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If $\phi_2(\Lambda)$ definite then the following bound is useful:

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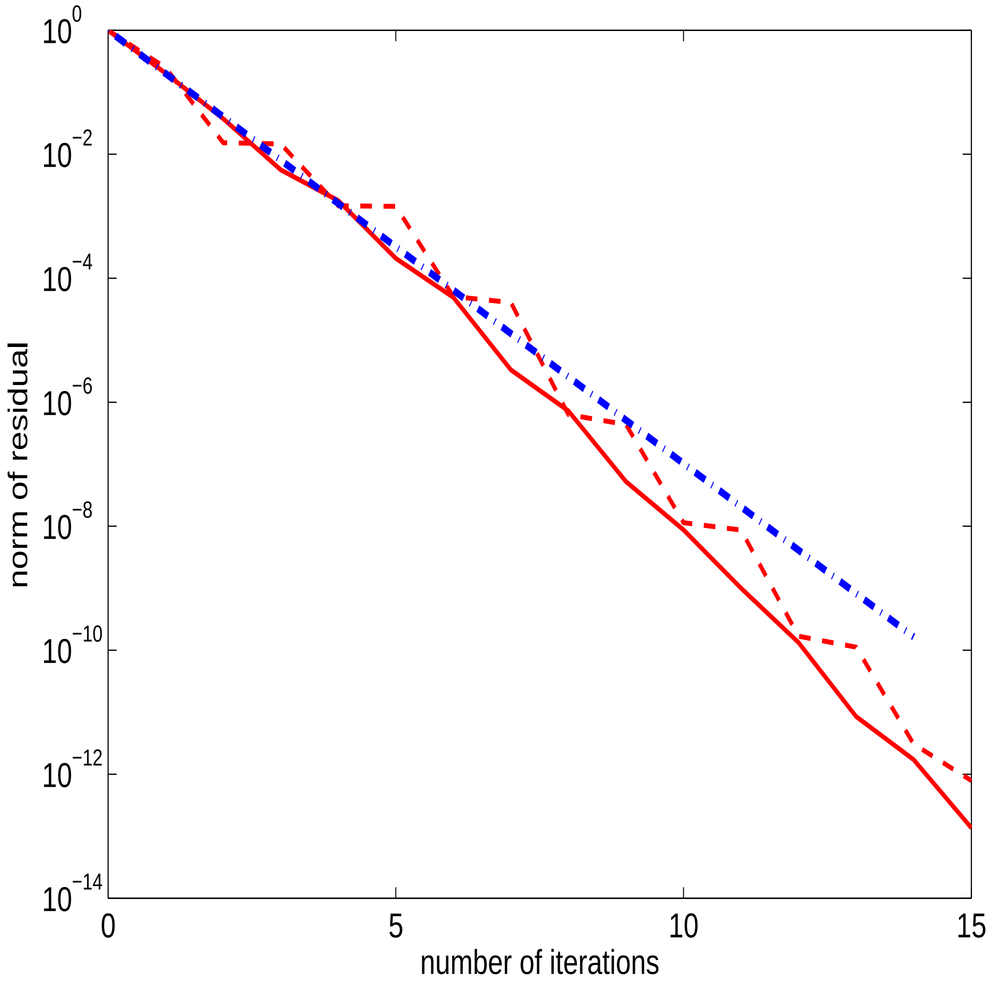
$\kappa(X)$ moderate

## Asymptotic rate

$MP_{tr,-}^{-1}$: $2k$-degree polyn but larger ($\sqrt{\text{radius}}$) eigenvalue set



## Final considerations and outlook

Symmetric case:

- Sharp bounds obtained for symmetric indefinite (1,1) block
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- Sharp bounds obtained for symmetric indefinite (1,1) block
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### Nonsymmetric case:

- First attempt to provide convergence information on indefinite problem
- Future work: devise more general non-stagnation conditions and expand on convergence analysis

## References


1. V. S. and Daniel B. Szyld , *New conditions for non-stagnation of minimal residual methods*. Numerische Mathematik, v. 109, n.3 (2008), pp. 477-487.

2. Nick Gould and V. S., *Spectral Analysis of saddle point matrices with indefinite leading blocks*. August 2008, To appear in SIAM J. Matrix Analysis Appl.

3. V.S., *On a non-stagnation condition for GMRES and application to saddle point matrices*, Sept 2009, submitted.