Università di Bologna

Solution of structured algebraic linear systems in PDE-constrained optimization problems

V. Simoncini

Dipartimento di Matematica, Università di Bologna

valeria.simoncini@unibo.it

The problem

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Computational Fluid Dynamics (Elman, Silvester, Wathen 2005)
- Elasticity problems
- Mixed (FE) formulations of II and IV order elliptic PDEs
- Linearly Constrained Programs
- Linear Regression in Statistics
- Image restoration
- ... Survey: Benzi, Golub and Liesen, Acta Num 2005

The problem. The setting

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Iterative solution by means of Krylov subspace methods

- Structural properties. Focus for this talk:
  - ⋆ A symmetric positive (semi)definite
  - ⋆ B^T tall (possibly rank deficient) or square nonsing.
  - ⋆ C symmetric positive (semi)definite

Spectral properties

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

$0 < \lambda_n \leq \cdots \leq \lambda_1$ eigs of A

$0 = \sigma_m \leq \cdots \leq \sigma_1$ sing. vals of B

$\lambda_{\max}(C) > 0, \quad BB^T + C$ full rank

$$\mathrm{spec}(\mathcal{M}) \subset [-a, -b] \cup [c, d], \quad a, b, c, d > 0$$

$\Rightarrow$ A large variety of results on spec($\mathcal{M}$), also for indefinite and singular A

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$\Rightarrow$ CG method not applicable (in general) $\Rightarrow$ MINRES

$\Rightarrow$ Search for good preconditioning strategies...

General preconditioning strategy

- Find $\mathcal{P}$ such that

$$\mathcal{M}\mathcal{P}^{-1}\hat{u} = b \qquad \hat{u} = \mathcal{P}u$$

  is easier (faster) to solve than $\mathcal{M}u = b$

- A look at efficiency:
  - Dealing with $\mathcal{P}$ should be cheap
  - Storage requirements for $\mathcal{P}$ should be low

  - Properties (algebraic/functional) should be exploited

  Mesh/parameter independence

Structure preserving preconditioners

Block diagonal Preconditioner

⋆ A nonsing., $C = 0$:

$$\mathcal{P}_0 = \begin{bmatrix} A & 0 \\ 0 & BA^{-1}B^T \end{bmatrix}$$

$$\Rightarrow \qquad \mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}} = \begin{bmatrix} I & A^{-\frac{1}{2}} B^T (BA^{-1}B^T)^{-\frac{1}{2}} \\ (BA^{-1}B^T)^{-\frac{1}{2}} BA^{-\frac{1}{2}} & 0 \end{bmatrix}$$

MINRES converges in at most 3 iterations. $\quad \mathrm{spec}(\mathcal{P}_0^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_0^{-\frac{1}{2}}) = \left\{1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}\right\}$

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A more practical choice:

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \qquad \text{spd.} \quad \widetilde{A} \approx A \qquad \widetilde{S} \approx BA^{-1}B^T$$

eigs of $\mathcal{M}\mathcal{P}^{-1}$ in $\qquad [-a, -b] \cup [c, d], \qquad a, b, c, d > 0$

Still an Indefinite Problem

Giving up symmetry ...

- Change the preconditioner: *Mimic the LU factors*

$$\mathcal{M} = \begin{bmatrix} I & O \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B^T \\ O & BA^{-1}B^T + C \end{bmatrix} \quad \Rightarrow \mathcal{P} \approx \begin{bmatrix} A & B^T \\ O & BA^{-1}B^T + C \end{bmatrix}$$

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- Change the preconditioner: *Mimic the Structure*

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \quad \Rightarrow \mathcal{P} \approx \mathcal{M}$$

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$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \quad \Rightarrow \mathcal{P} \approx \mathcal{M}$$

- Change the matrix: *Eliminate indef.* $\quad \mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}$

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- Change the matrix: *Eliminate indef.* $\mathcal{M}_- = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}$

- Change the matrix: *Regularize* $(C = 0)$

$$\mathcal{M} \;\Rightarrow\; \mathcal{M}_\gamma = \begin{bmatrix} A & B^T \\ B & -\gamma W \end{bmatrix} \text{ or } \mathcal{M}_\gamma = \begin{bmatrix} A + \frac{1}{\gamma} B^T W^{-1} B & B^T \\ B & O \end{bmatrix}$$

... But recovering symmetry in disguise

Nonstandard inner product:

Let $\mathcal{W}$ be any of $\mathcal{M}\mathcal{P}^{-1}, \mathcal{M}_-$

For $\mathrm{spec}(\mathcal{W})$ in $\mathbb{R}^+$, find symmetric matrix H such that

$$\mathcal{W}H = H\mathcal{W}^T$$

(that is, $\mathcal{W}$ is H-symmetric)

... But recovering symmetry in disguise

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If H is spd then

- $\mathcal{W}$ is diagonalizable
- Use PCG on $\mathcal{W}$ with H-inner product

Constraint (Indefinite) Preconditioner

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ B & -C \end{bmatrix} \quad \mathcal{M}\mathcal{P}^{-1} = \begin{bmatrix} A\widetilde{A}^{-1}(I-\Pi)+\Pi & \star \\ O & I \end{bmatrix}$$

with $\Pi = B(B\widetilde{A}^{-1}B^T + C)^{-1}B\widetilde{A}^{-1}$

- Constraint equation satisfied at each iteration
- If C nonsing $\Rightarrow$ all eigs real and positive
- If $B^TC = 0$ and $BB^T + C > 0 \Rightarrow$ all eigs real and positive

$\Rightarrow$ More general cases, $\widetilde{B} \approx B$, $\widetilde{C} \approx C$

The Stokes problem

Minimize

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f \cdot u \, dx$$

subject to $\nabla \cdot u = 0$ in Ω

Lagrangian: $\mathcal{L}(u,p) = J(u) + \int_\Omega p \, \nabla \cdot u \, dx$

Optimality condition on discretized Lagrangian leads to:

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

A second-order operator, B first-order operator, C zero-order operator

A standard choice: block diagonal preconditioning

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \qquad \text{spd.} \quad \widetilde{A} \approx A \qquad \widetilde{S} \approx BA^{-1}B^T$$

⇒ if $\widetilde{A}, A$ and $\widetilde{S}, BA^{-1}B^T$ spectrally equivalent, e.g.,

$$\exists\, \alpha_1, \alpha_2 > 0 \quad : \quad \forall x \neq 0,\ \alpha_1 x^T \widetilde{A} x \leq x^T A x \leq \alpha_2 x^T \widetilde{A} x,$$

then interval containing spec$(\mathcal{M}\mathcal{P}^{-1})$ is independent of mesh parameter

⇒ Krylov subspace solver MINRES will converge in a number of iterations bounded independently of mesh parameter

An example. Stokes problem

IFISS 3.1 (Elman, Ramage, Silvester): Lid driven cavity; Q2-Q1 approximation

$$\begin{bmatrix} -\Delta & -\mathrm{grad} \\ \mathrm{div} & \end{bmatrix} \approx \begin{bmatrix} -\widetilde{\Delta} & \\ & I \end{bmatrix}$$

In algebraic terms:

$I \to$ mass matrix
$-\widetilde{\Delta} \to$ Algebraic MG
(spectrally equivalent matrix)

(cf. K.-A. Mardal & R. Winther
JNLAA 2011)

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2D. Final residual norm $< 10^{-6}$

size($\mathcal{M}$)	its	Time (secs)
578	26	0.04
2178	26	0.14
8450	26	0.50
132098	26	11.17

The Stokes problem. Contraint preconditioning

$$\mathcal{P} = \begin{bmatrix} \widetilde{A} & B^T \\ B & B\widetilde{A}^{-1}B^T - S \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B\widetilde{A}^{-1} & I_m \end{bmatrix} \begin{bmatrix} \widetilde{A} & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} I_n & \widetilde{A}^{-1}B^T \\ 0 & I_m \end{bmatrix}$$

with $S \approx B\widetilde{A}^{-1}B^T + C \; spd$

The Stokes problem. Contraint preconditioning

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with $S \approx B\widetilde{A}^{-1}B^T + C$ *spd*

Selection of $\widetilde{A}$, S: $\widetilde{A} = \text{AMG}(A)$, $S = Q$ (pressure mass matrix)

IFISS 3.1 (Elman, Ramage, Silvester):

Flow over a backward facing step

Stable Q2-Q1 approximation

$(C = 0, B \in \mathbb{R}^{m\times n})$

stopping tolerance: 10^{-6}

non-symmetric solver

n	m	# it.
1538	209	18
5890	769	18
23042	2945	18
91138	11521	17
362498	45569	17

Distributed optimal control for time-periodic parabolic equations

Joint work with W. Zulehner and W. Krendl

Problem: *Find the state $y(x,t)$ and the control $u(x,t)$ that minimize the cost functional*

$$J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y(x,t) - y_d(x,t)|^2 \; dx \; dt + \frac{\nu}{2}\int_0^T \int_\Omega |u(x,t)|^2 \; dx \; dt$$

subject to the time-periodic parabolic problem

$$\begin{aligned}
\frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) &= u(x,t) && \text{in } \Omega \times (0,T),\\
y(x,t) &= 0 && \text{on } \partial\Omega \times (0,T),\\
y(x,0) &= y(x,T) && \text{in } \Omega,\\
u(x,0) &= u(x,T) && \text{in } \Omega.
\end{aligned}$$

Here $y_d(x,t)$ is a given target (or desired) state and $\nu > 0$ is a cost or regularization parameter.

Time-harmonic solution

Assume that y_d is time-harmonic: $\quad y_d(x,t) = y_d(x)e^{i\omega t},\ \omega = \frac{2\pi k}{T}$

Then there exists a time-periodic solution

$y(x,t) = y(x)e^{i\omega t}, u(x,t) = u(x)e^{i\omega t}$, where $y(x), u(x)$ solve:

Minimize

$$\frac{1}{2}\int_\Omega |y(x) - y_d(x)|^2 \ dx + \frac{\nu}{2}\int_\Omega |u(x)|^2 \ dx$$

subject to

$$\begin{aligned} i\omega\, y(x) - \Delta y(x) &= u(x) \quad \text{in } \Omega, \\ y(x) &= 0 \qquad\ \ \text{on } \partial\Omega \end{aligned}$$

Discrete version:

$$\frac{1}{2}(y - y_d)^* M (y - y_d) + \frac{\nu}{2} u^* M u, \quad \text{subject to} \quad i\omega M y + K y = M u$$

M, K real mass and stiffness matrices.

Solution of the discrete problem

Solution using Lagrange multipliers gives

$$\begin{bmatrix} M & 0 & K - i\omega M \\ 0 & \nu M & -M \\ K + i\omega M & -M & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ p \end{bmatrix} = \begin{bmatrix} M y_d \\ 0 \\ 0 \end{bmatrix}$$

Elimination of the control ($\nu M u = M p$) yields:

$$\begin{bmatrix} M & K - i\omega\, M \\ K + i\omega\, M & -\frac{1}{\nu} M \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix} = \begin{bmatrix} M y_d \\ 0 \end{bmatrix}$$

Zulehner, 2011 (for $\omega = 0$); Kolmbauer and Kollmann, tr2011

Solving the saddle point linear system

After simple scaling,

$$\begin{bmatrix} M & \sqrt{\nu}\,(K - i\omega\, M) \\ \sqrt{\nu}\,(K + i\omega\, M) & -M \end{bmatrix} \begin{bmatrix} y \\ \frac{1}{\sqrt{\nu}}\, p \end{bmatrix} = \begin{bmatrix} M y_d \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \mathcal{A}x = b$$

Ideal (Real) Block diagonal Preconditioner:

$$\mathcal{P} = \begin{bmatrix} M + \sqrt{\nu}\,(K + \omega\, M) & 0 \\ 0 & M + \sqrt{\nu}\,(K + \omega\, M) \end{bmatrix}$$

- Performance. Accurate estimates for the spectral intervals:

$$\mathrm{spec}(\mathcal{P}^{-1}\mathcal{A}) \subseteq \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$$

- Robustness. Convergence of MINRES bounded independently of the mesh, frequency and regularization parameters (h, ω, ν)

Distributed optimal control for the time-periodic Stokes equations. I

The problem.

Find the velocity $u(x,t)$, the pressure $p(x,t)$, and the force $f(x,t)$ that minimize the cost functional

$$J(u,f) = \frac{1}{2}\int_0^T \int_\Omega |u(x,t) - u_d(x,t)|^2 \; dx \; dt + \frac{\nu}{2}\int_0^T \int_\Omega |f(x,t)|^2 \; dx \; dt$$

subject to the time-periodic Stokes problem

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u}(x,t) - \Delta\mathbf{u}(x,t) + \nabla p(x,t) &= \mathbf{f}(x,t) && \textit{in } \Omega \times (0,T), \\
\nabla \cdot \mathbf{u}(x,t) &= 0 && \textit{in } \Omega \times (0,T), \\
\mathbf{u}(x,t) &= 0 && \textit{on } \partial\Omega \times (0,T), \\
\mathbf{u}(x,0) &= \mathbf{u}(x,T) && \textit{in } \Omega, \\
p(x,0) &= p(x,T) && \textit{in } \Omega, \\
\mathbf{f}(x,0) &= \mathbf{f}(x,T) && \textit{in } \Omega.
\end{aligned}$$

Distributed optimal control for the time-periodic Stokes equations. II

Similar solution strategy (time-harmonic solution, Lagrange multipliers, scaling) leads to a familiar structure:

$$\left[\begin{array}{cc|cc} \mathbf{M} & 0 & \sqrt{\nu}(\mathbf{K}-i\omega\,\mathbf{M}) & -\sqrt{\nu}\mathbf{D}^T \\ 0 & 0 & -\sqrt{\nu}\mathbf{D} & 0 \\ \hline \sqrt{\nu}(\mathbf{K}+i\omega\,\mathbf{M}) & -\sqrt{\nu}\mathbf{D}^T & -\mathbf{M} & 0 \\ -\sqrt{\nu}\mathbf{D} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{u} \\ p \\ \frac{1}{\sqrt{\nu}}\mathbf{w} \\ \frac{1}{\sqrt{\nu}} r \end{bmatrix} = \begin{bmatrix} \mathbf{M}\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(new setting for $\omega \neq 0$)

Optimal preconditioning technique

$$\left[\begin{array}{cc|cc} \mathbf{M} & 0 & \sqrt{\nu}(\mathbf{K} - i\omega\,\mathbf{M}) & -\sqrt{\nu}\mathbf{D}^T \\ 0 & 0 & -\sqrt{\nu}\mathbf{D} & 0 \\ \hline \sqrt{\nu}(\mathbf{K} + i\omega\,\mathbf{M}) & -\sqrt{\nu}\mathbf{D}^T & -\mathbf{M} & 0 \\ -\sqrt{\nu}\mathbf{D} & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{p} \\ \frac{1}{\sqrt{\nu}}\underline{\mathbf{w}} \\ \frac{1}{\sqrt{\nu}}\underline{r} \end{bmatrix} = \begin{bmatrix} \mathbf{M}\underline{\mathbf{u}}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Ideal real Block diagonal preconditioner:

$$\mathcal{P} = \begin{bmatrix} P & & & \\ & S & & \\ & & P & \\ & & & S \end{bmatrix}, \qquad \begin{aligned} P &= M + \sqrt{\nu}(K + \omega M), \\ S &= \nu D(M + \sqrt{\nu}(K + \omega M))^{-1} D^T \end{aligned}$$

- Performance. Accurate estimates for the spectral intervals:

$$\mathrm{spec}(\mathcal{P}^{-1}\mathcal{A}) \subseteq \left[-\frac{1}{2}(1+\sqrt{5}), -\phi\right] \cup \left[\phi, \frac{1}{2}(1+\sqrt{5})\right], \quad \phi = 0.306\ldots$$

- Robustness. Convergence of MINRES bounded independently of the mesh, frequency and regularization parameters (h, ω, ν)

An example for the time-periodic Stokes constraint

$\omega \backslash \nu$	10^{-8}	10^{-6}	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}
10^{-2}	58	58	61	48	32	22	20	20
10^{0}	58	58	61	48	36	32	32	32
10^{2}	58	57	66	62	62	62	62	62
10^{4}	48	56	60	60	60	60	60	60
10^{6}	30	30	30	30	30	30	30	30
10^{8}	16	16	16	16	16	16	16	16

(Taylor-Hood pair of FE spaces (P2-P1))

final tolerance: $\texttt{tol}=10^{-12}$

Practical block diagonal preconditioning

Ideal real Block diagonal preconditioner:

$$\mathcal{P} = \begin{bmatrix} P & & & \\ & S & & \\ & & P & \\ & & & S \end{bmatrix}, \qquad \begin{array}{l} P = M + \sqrt{\nu}(K + \omega M), \\ S = \nu D(M + \sqrt{\nu}(K + \omega M))^{-1} D^T \end{array}$$

Practical case:

$$S^{-1} \approx (1 + \omega\sqrt{\nu}) M_p^{-1} + \omega\sqrt{\nu} K_p^{-1}$$

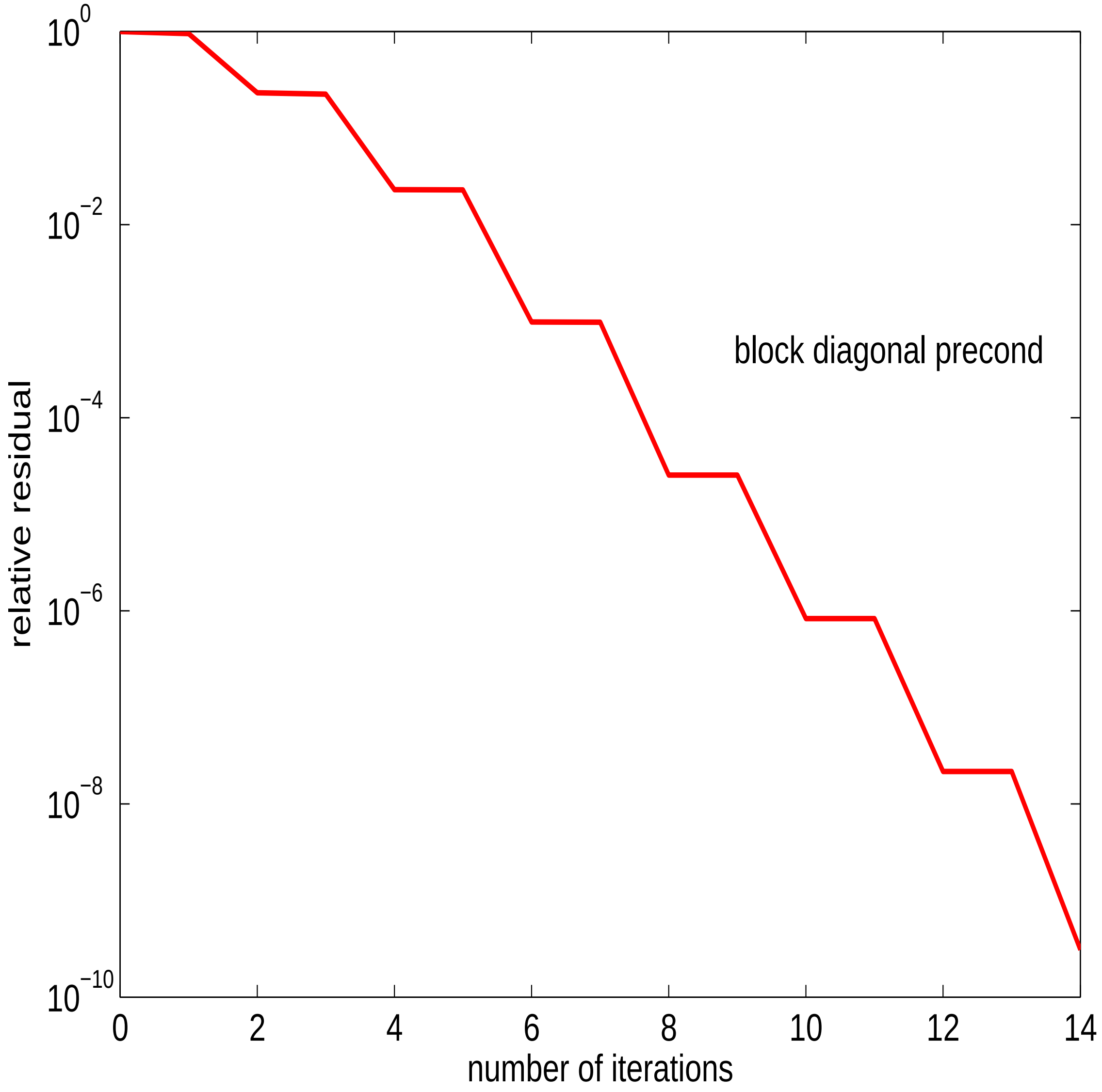
(Cahout-Charbard preconditioner)

with M_p, K_p the mass matrix and the discretized negative Laplacian in the finite element space for the pressure

$\Rightarrow$ M_p, K_p also replaced by, e.g., Multigrid versions

(Mardal, Winther, Bramble, Pasciak, Olshanskii, Peters, Reusken, ...)

Convergence history. Staircase behavior



Explanation of the Staircase behavior

Both matrices have the form:

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ B & -A \end{bmatrix} \in \mathbb{C}^{2n\times 2n},$$

with:

$A \in \mathbb{R}^{n\times n}$ symmetric and semidefinite

$B \in \mathbb{C}^{n\times n}$ complex symmetric (i.e., $B = B^T$)

THEOREM: Assume that B is nonsingular. Then the eigenvalues μ of $\mathcal{A}$ come in pairs, $(\mu, -\mu)$, with $\mu \in \mathbb{R}$.

(cf. Hamiltonian matrices)

Consequence: spec($\mathcal{A}$) is symmetric with respect to the origin, and $\text{spec}(\mathcal{A}) \subseteq [-b, -a] \cup [a, b]$

Symmetric spectrum. Consequences.

A classical result (e.g., Greenbaum 1997): Consider the linear system $\mathcal{A}x = r_0$

Let $\mathcal{A}$ be a Hermitian matrix, with spectrum in $[-a,-b] \cup [c,d]$, $a,b,c,d > 0$. Assume that $|b-a| = |d-c|$.

Then after m iterations, the MINRES residual r_m satisfies

$$\frac{\|r_m\|}{\|r_0\|} \le 2\left(\frac{\sqrt{|ad|}-\sqrt{|bc|}}{\sqrt{|ad|}+\sqrt{|bc|}}\right)^{[m/2]}$$

For equal intervals (our case):

$$\frac{\|r_m\|}{\|r_0\|} \le 2\left(\frac{d/c-1}{d/c+1}\right)^{[m/2]}$$

$\Rightarrow$ MINRES roughly behaves like CG on a matrix having only the squared (!) positive eigenvalues

Attempts to bypass quasi-stagnation. The time-periodic parabolic case

$$\mathcal{A} = \begin{bmatrix} M & \sqrt{\nu}(K - i\omega M) \\ \sqrt{\nu}(K + i\omega M) & -M \end{bmatrix}$$

An alternative (indefinite) preconditioner - work in progress:

$$\mathcal{P} = \begin{bmatrix} & M + \sqrt{\nu}(K - i\omega M) \\ M + \sqrt{\nu}(K + i\omega M) & -M \end{bmatrix}.$$

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Spectral independence wrto parameters: It holds that

$$\mathrm{spec}(\mathcal{A}\mathcal{P}^{-1}) \subset \ [\tfrac{1}{2},1) \times [-1, 1] \in \mathbb{C}^{+}$$

The actual rectangle may be much smaller, depending on ν, ω, h

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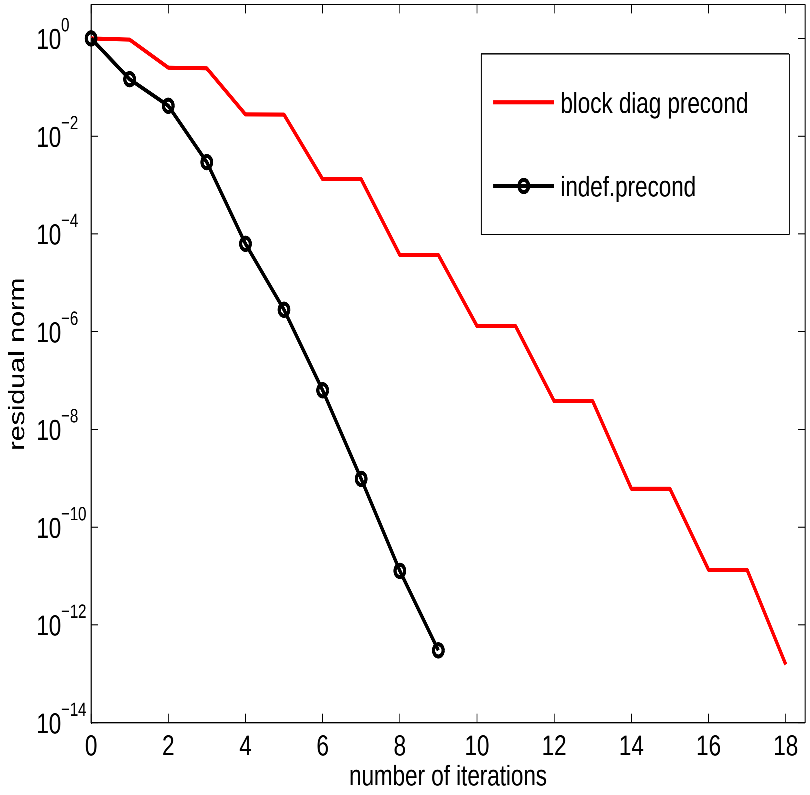
- Preconditioner not sensitive to $K \pm i\omega M$
- No results on eigenvectors

(Very) Preliminary numerical evidence. Time-periodic parabolic pb.

$\omega = 1, \nu = 10^{-2}$, $n = 1741$



MINRES vs GMRES

(Very) Preliminary numerical evidence. Time-periodic parabolic pb.

Block diagonal preconditioner: MINRES # its

$\omega \backslash \nu$	10^{-8}	10^{-6}	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}
10^{-2}	29	30	26	16	10	8	8	8
10^{-0}	29	30	26	18	14	14	14	14
10^{2}	29	38	34	30	30	30	30	30
10^{6}	26	30	30	30	30	30	30	30
10^{8}	10	10	10	10	10	10	10	10

(Very) Preliminary numerical evidence. Time-periodic parabolic pb.

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10^{-0}	29	30	26	18	14	14	14	14
10^{2}	29	38	34	30	30	30	30	30
10^{6}	26	30	30	30	30	30	30	30
10^{8}	10	10	10	10	10	10	10	10

Block indefinite preconditioner: GMRES # its

$\omega \backslash \nu$	10^{-8}	10^{-6}	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}
10^{-2}	42	32	15	8	5	4	3	3
10^{0}	42	32	15	8	5	4	3	3
10^{2}	42	29	11	6	4	3	3	2
10^{4}	11	5	4	3	2	2	2	2
10^{6}	3	3	2	2	2	1	1	1

Similar results with CGSTAB(ℓ)

A side consideration

Is the complex matrix formulation needed?

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} M & \sqrt{\nu}(K - i\omega M) \\ \sqrt{\nu}(K + i\omega M) & -M \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ i\omega\sqrt{\nu} I & I \end{bmatrix} \begin{bmatrix} M & \sqrt{\nu} K \\ \sqrt{\nu} K & -(1+\nu\omega^2) M \end{bmatrix} \begin{bmatrix} I & -i\omega\sqrt{\nu} I \\ 0 & I \end{bmatrix} \equiv R \mathcal{A}_r R^*
\end{aligned}
$$

(similar transformation in the Stokes case)

$$\mathcal{A} x = b \quad \Leftrightarrow \quad \mathcal{A}_r \hat{x} = \hat{b}$$

$\Rightarrow$ Convergence estimates (and expected performance) for real matrices

Final remarks

- Much is known about the behavior of structured preconditioners for well established problems and formulations
- New problems provide new challenges
- Understanding the underlying Linear Algebra may be key

References for this talk

W. Krendl, V. Simoncini and W. Zulehner, *Stability Estimates and Structural Spectral Properties of Saddle Point Problems*, submitted, 2012.

M. Benzi, G.H. Golub and J. Liesen, *Numerical Solution of Saddle Point Problems*, Acta Numerica, 14, 1-137 (2005)

K.A. Mardal and R. Winther, *Preconditioning discretizations of systems of partial differential equations*, Numer. Linear Algebra with Appl., 18, 1-40 (2011)