On the decay of the inverse of matrices that are sum of Kronecker products

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Joint work with

C. Canuto (Politecnico Torino) and M. Verani (Politecnico Milano)

The application. I

Adaptive Legendre-Galerkin discretizations for PDEs:

H_0^1 Tensorized Babuska-Shen basis in $\Omega = (0,1) \times (0,1)$:

$$\eta_{\mathbf{k}}(x_1, x_2) = \eta_{k_1}(x_1)\eta_{k_2}(x_2), \quad k_1, k_2 \geq 2, \quad \mathbf{k} = (k_1, k_2)$$

$\{\eta_{k_i}\}$: k_i-order Legendre polyn (1D BS basis)

Stiffness matrix:

$$(\eta_{\mathbf{k}}, \eta_{\mathbf{m}})_{H_0^1(\Omega)} = (\eta_{k_1}, \eta_{m_1})_{H_0^1(I)}(\eta_{k_2}, \eta_{m_2})_{L^2(I)} + (\eta_{k_1}, \eta_{m_1})_{L^2(I)}(\eta_{k_2}, \eta_{m_2})_{H_0^1(I)}$$

Kronecker structure: $S_\eta^p = M_p \otimes I_p + I_p \otimes M_p$ (max p polyn degree)

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Note: If higher order polynomial used, then S_η^p simply expands (augmented M_p)

The application. II

Adaptive Legendre-Galerkin discretizations for PDEs:

- Inner product:

$$v = \sum \hat{v}_{\mathbf{k}} \eta_{\mathbf{k}}, \qquad \|v\|^2_{H^1_0} = \hat{v}^T S_\eta \hat{v}$$

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with $G = L^{-1}$ where $S_\eta = LL^T$

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$$v = \sum \check{v}_{\mathbf{k}} \Psi_{\mathbf{k}}, \qquad \|v\|^2_{H^1_0} = \tilde{v}^T \check{G}^T S_\eta (\check{G}\tilde{v}) \approx \tilde{v}^T D \tilde{v}$$

  $\check{G}$ very sparse version of G, D diagonal

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  $\check{G}$ very sparse version of G, $\qquad D$ diagonal

Q: Does such a $\check{G}$ exist? ...Analyze sparsity of S_η^{-1}

The stiffness matrix

$$S := M \otimes I_n + I_n \otimes M,$$

with M symmetric and positive definite, banded with bandwidth b

- Finite differences: M is second order operator in one space dimension ($b = 1$)

  $\Rightarrow$ for instance, S: 2D Laplace operator

- Legendre Spectral methods: M spd, nonconstant ($b = 1$)

- ...

More generally,

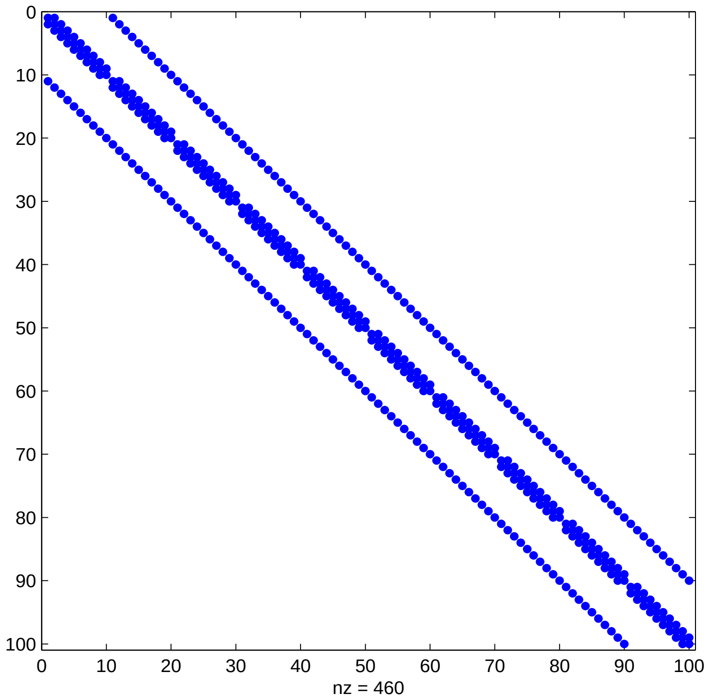
$$S_g := M_1 \otimes I_n + I_n \otimes M_2,$$

with $M_1 \neq M_2$, banded, with not necessarily the same dimensions

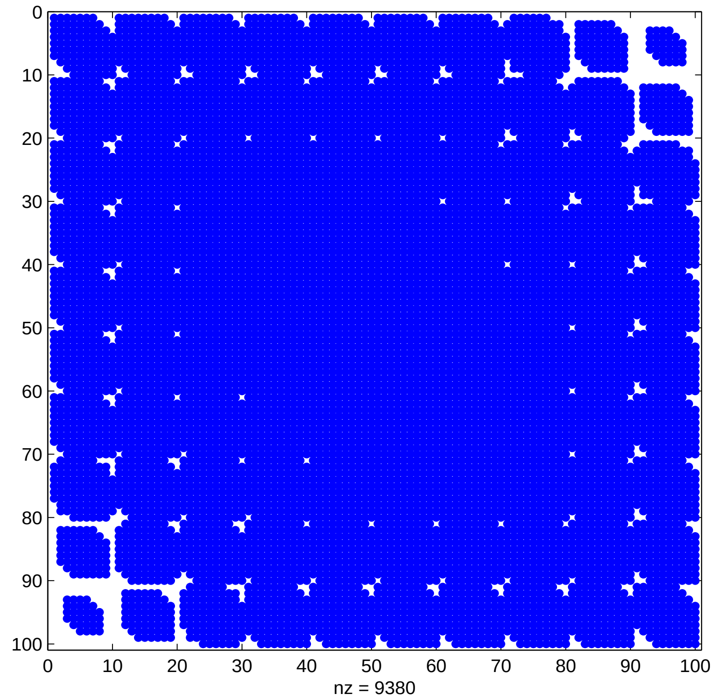
The inverse of the 2D Laplace matrix on the unit square

$$S := M \otimes I_n + I_n \otimes M, \qquad M = \text{tridiag}(-1, 2, -1)$$

Sparsity pattern:



Matrix S



S^{-1}

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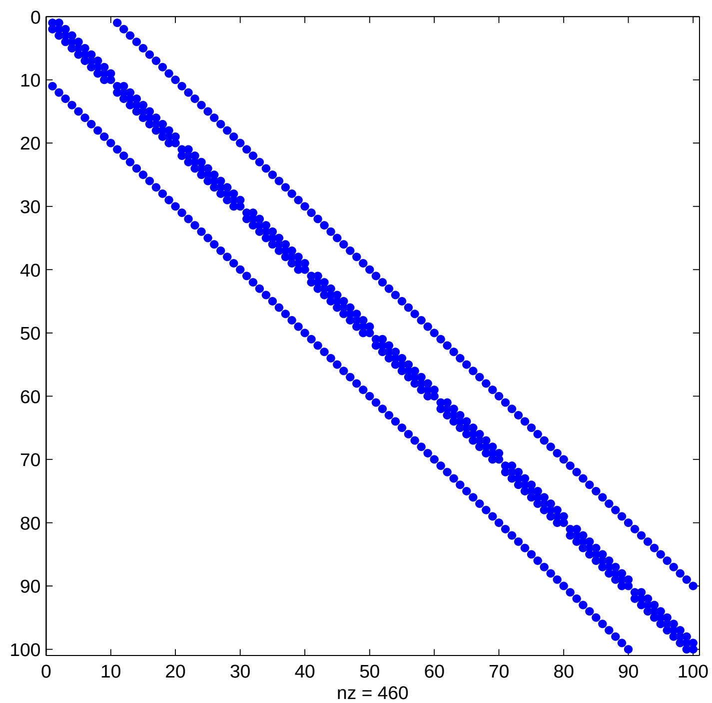
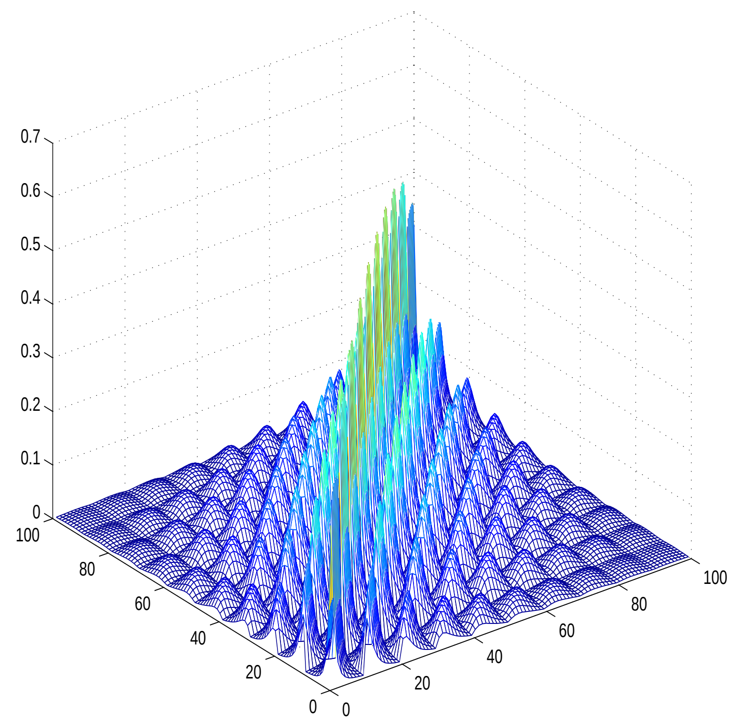




S $\qquad$ $|(S^{-1})_{ij}|$

The exponential decay of the entries of S^{-1}

The classical bound (Demko, Moss & Smith):

If S spd is banded with bandwidth b, then

$$|(S^{-1})_{ij}| \leq \gamma q^{\frac{|i-j|}{b}}$$

where

κ: condition number of S

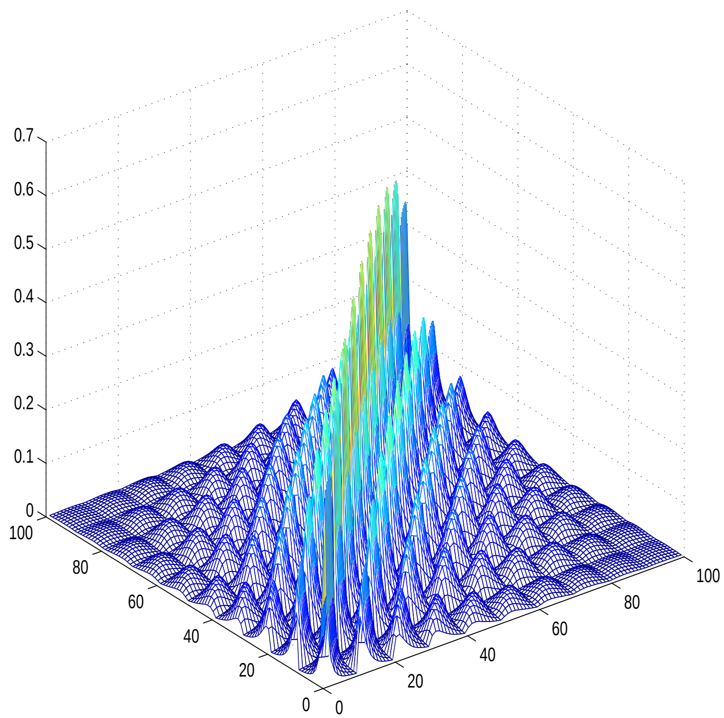
$q := \dfrac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} < 1$

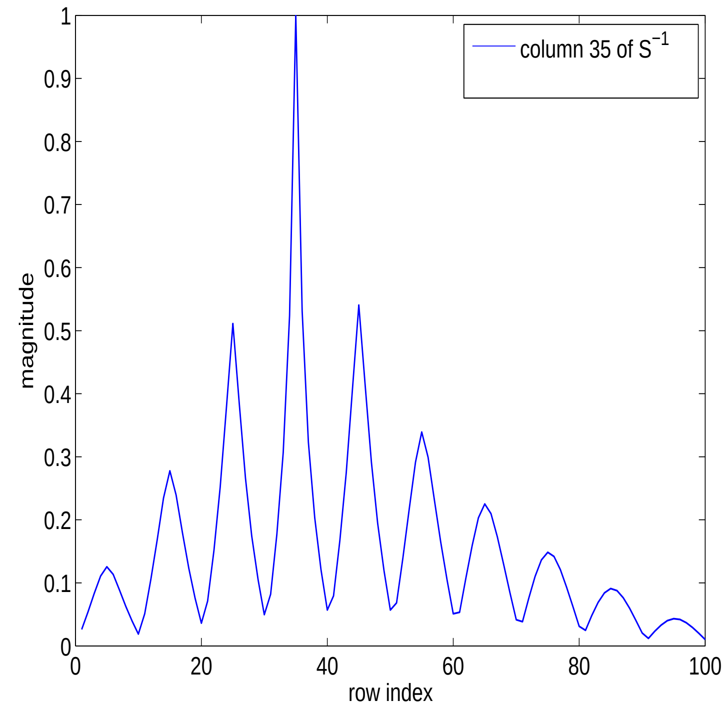
$\gamma := \max\{\lambda_{\min}(S)^{-1}, \hat{\gamma}\}$, and $\hat{\gamma} = \dfrac{(1+\sqrt{\kappa})^2}{2\lambda_{\max}(S)}$

($\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ smallest and largest eigenvalues of the given symmetric matrix)

Many contributions: Bebendorf, Hackbusch, Benzi, Boito, Razouk, Golub, Tuma, Concus, Meurant, Mastronardi, Ng, Tyrtyshnikov, Nabben, ...

The true decay





... a very peculiar pattern $\Rightarrow$ much higher sparsity

Where do the repeated peaks come from?

For $S = M \otimes I_n + I_n \otimes M \in \mathbb{R}^{n^2 \times n^2}$:

$$x_t := (S^{-1})_{:,t} = S^{-1} e_t \qquad \Leftrightarrow \qquad \text{Solve}: \; S x_t = e_t$$

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Let

$X_t \in \mathbb{R}^{n \times n}$ be such that $x_t = \text{vec}(X_t)$

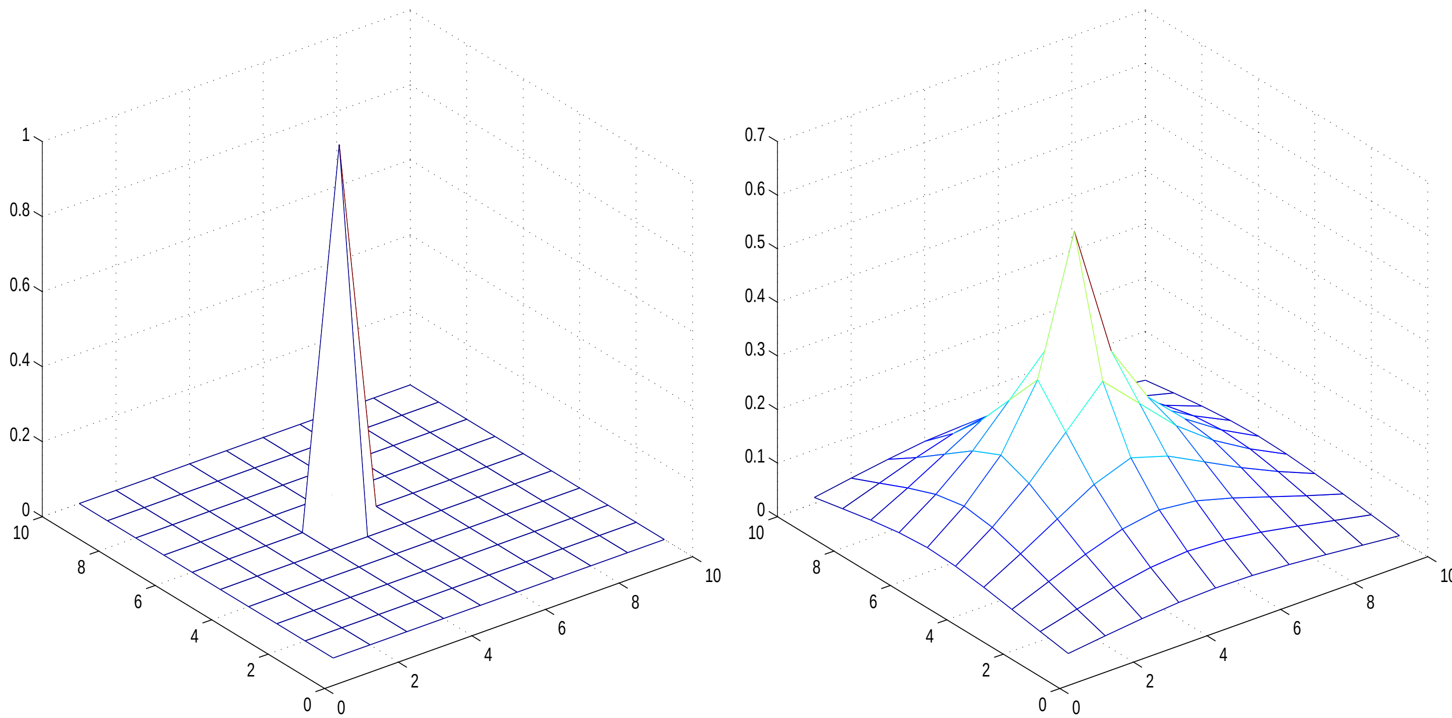
$E_t \in \mathbb{R}^{n \times n}$ be such that $e_t = \text{vec}(E_t)$

Then

$$S x_t = e_t \qquad \Leftrightarrow \qquad M X_t + X_t M = E_t$$

For S the 2D Laplace operator, $t = 1, \ldots, n^2$

$$t = 35, \qquad Sx_t = e_t \qquad \Leftrightarrow \qquad MX_t + X_tM = E_t$$



matrix E_t and matrix X_t

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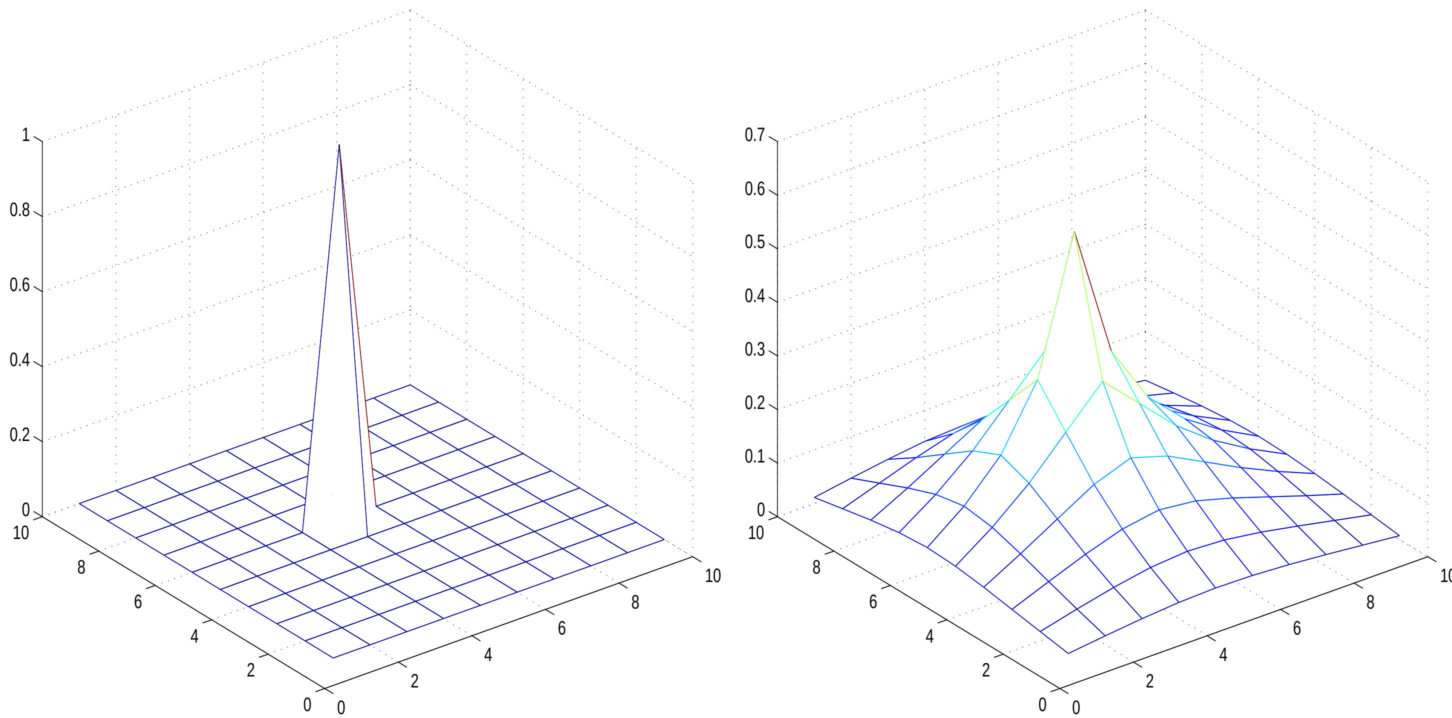
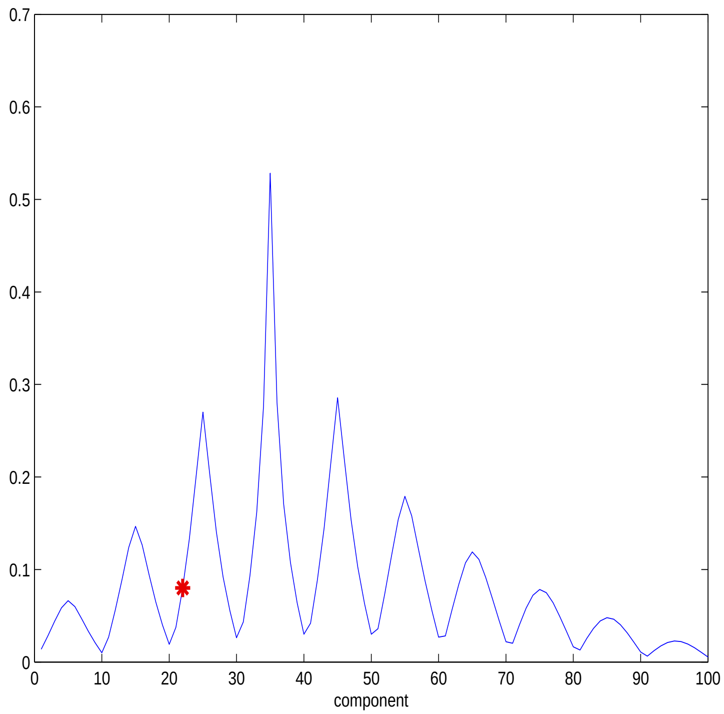
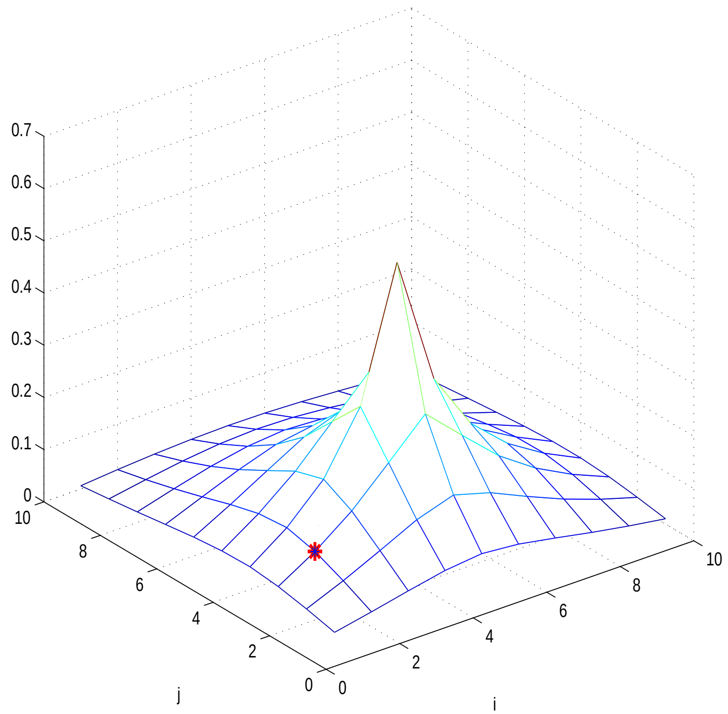


matrix E_t and matrix X_t

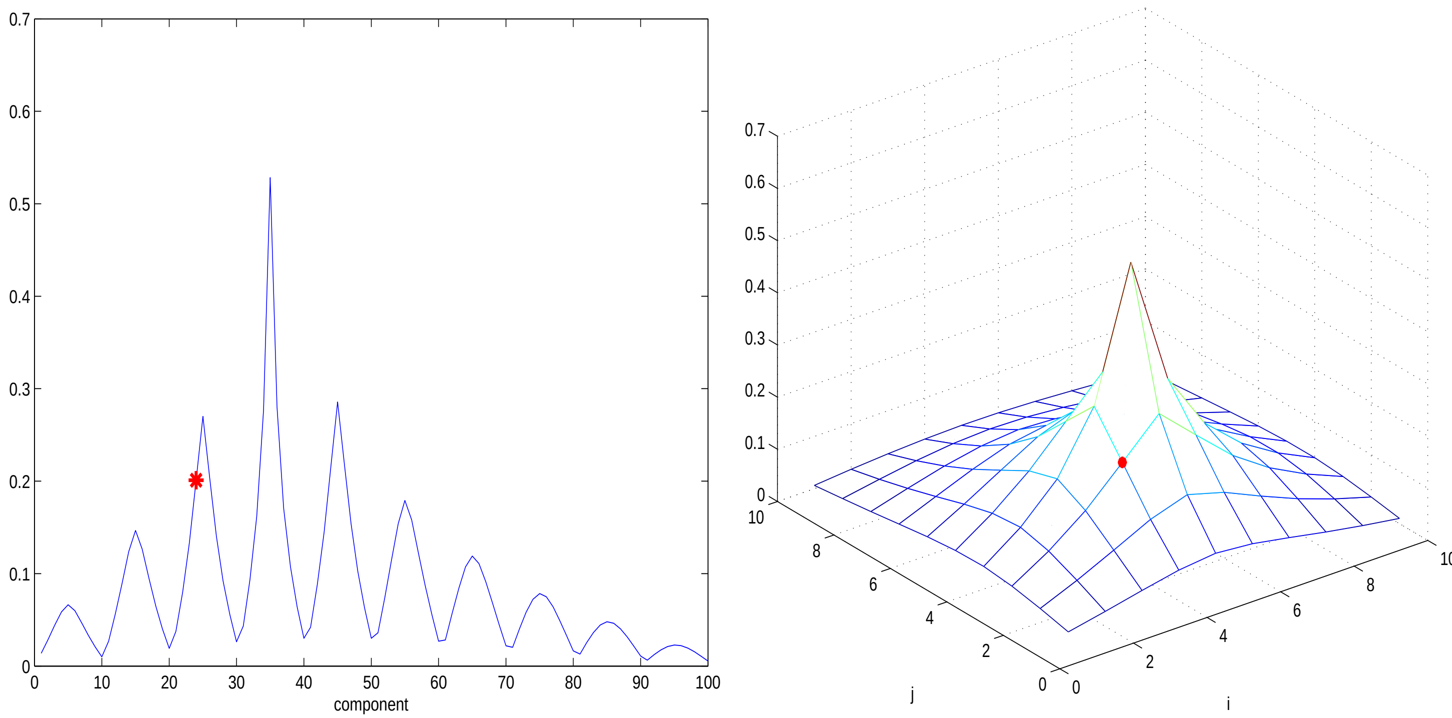
E_t has only one nonzero element

Lexicographic order: $(E_t)_{ij}$, $j = \lfloor (t-1)/n \rfloor + 1, \ i = tn\lfloor (t-1)/n \rfloor$





Left: Row of S^{-1} Right: same row on the grid



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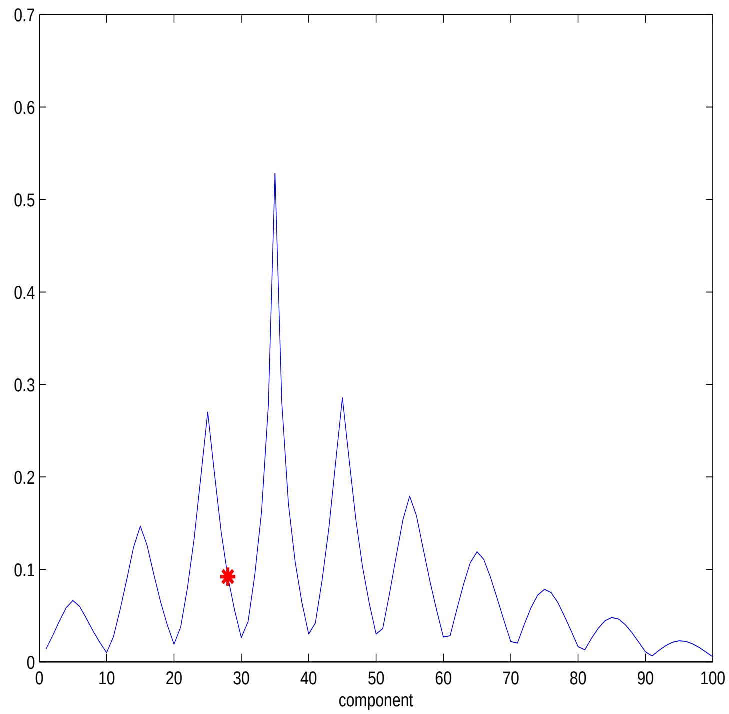

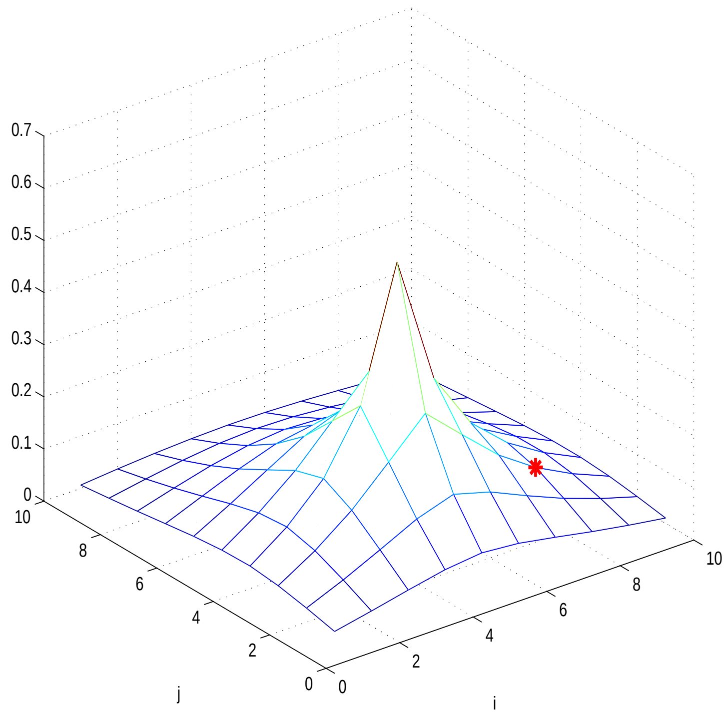

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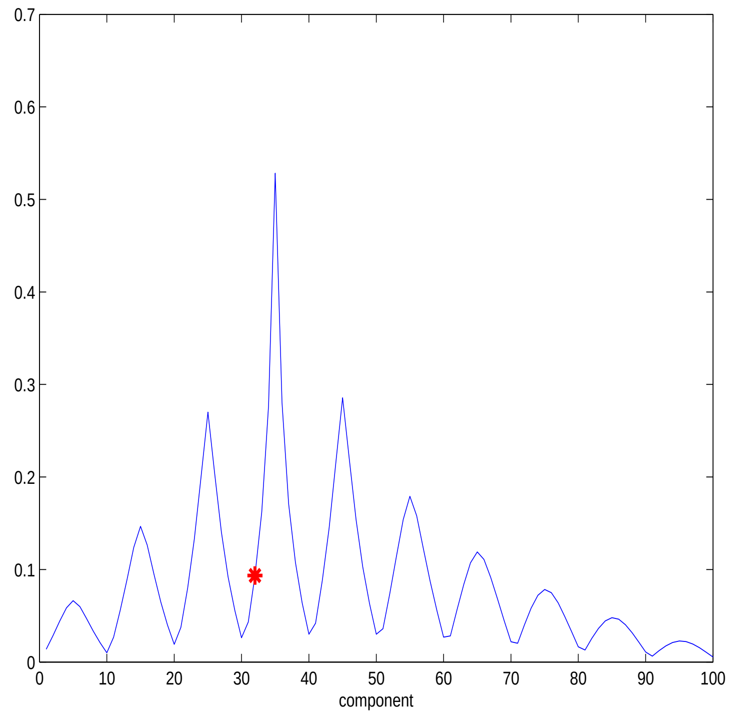

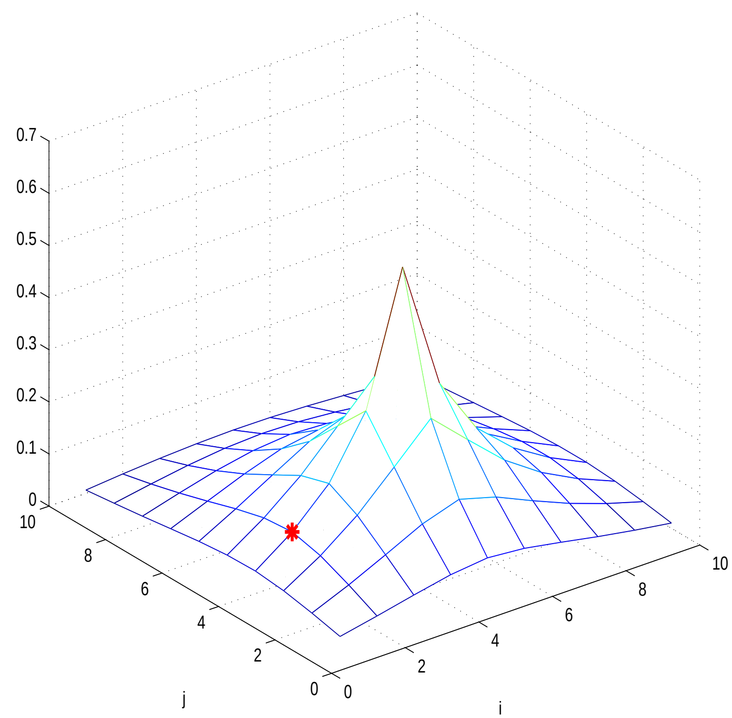

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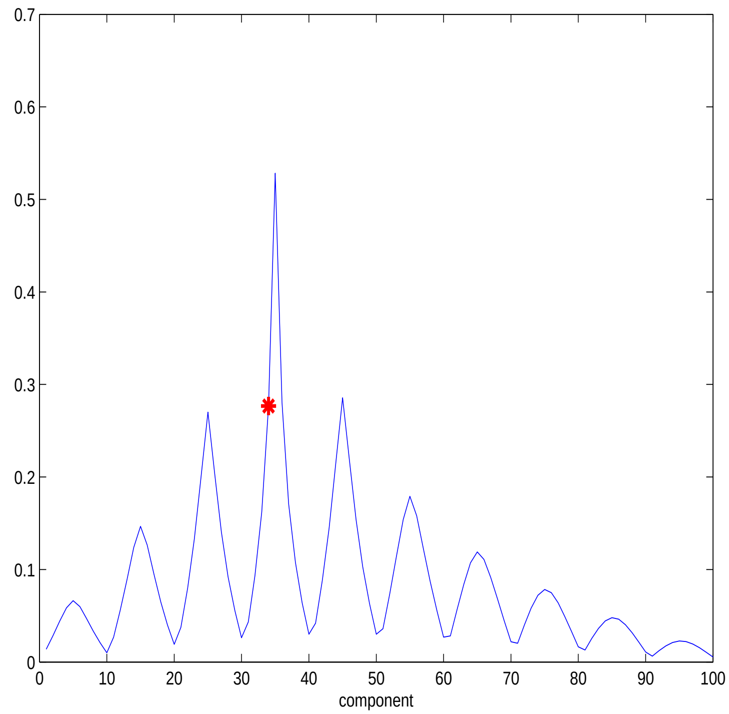

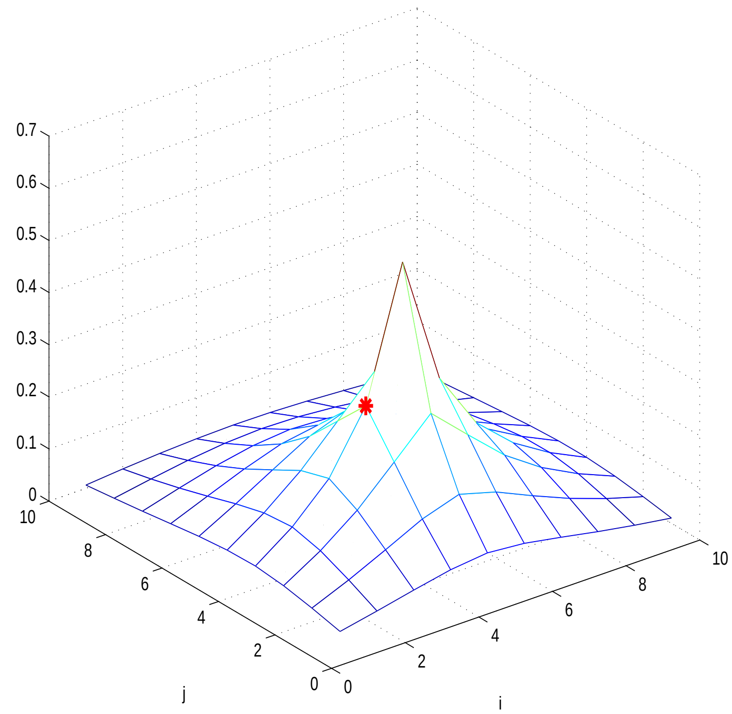

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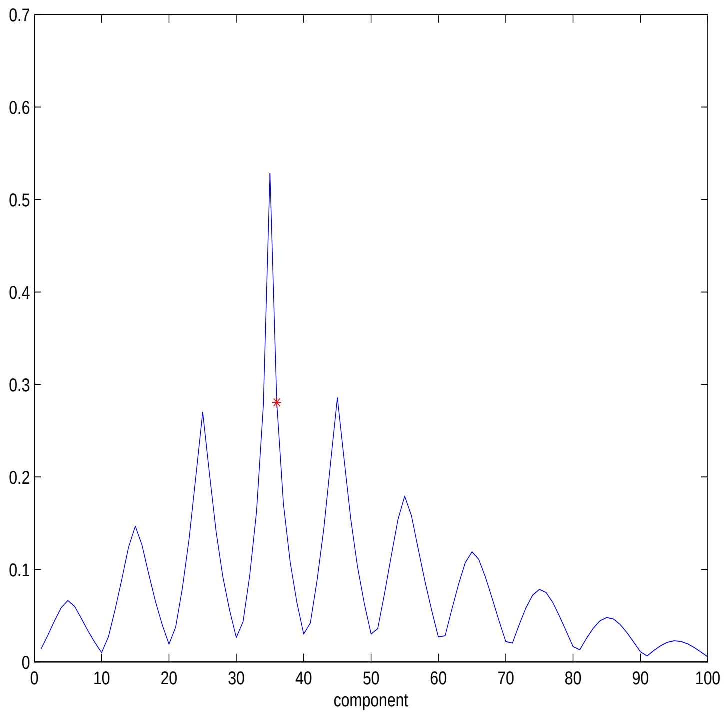

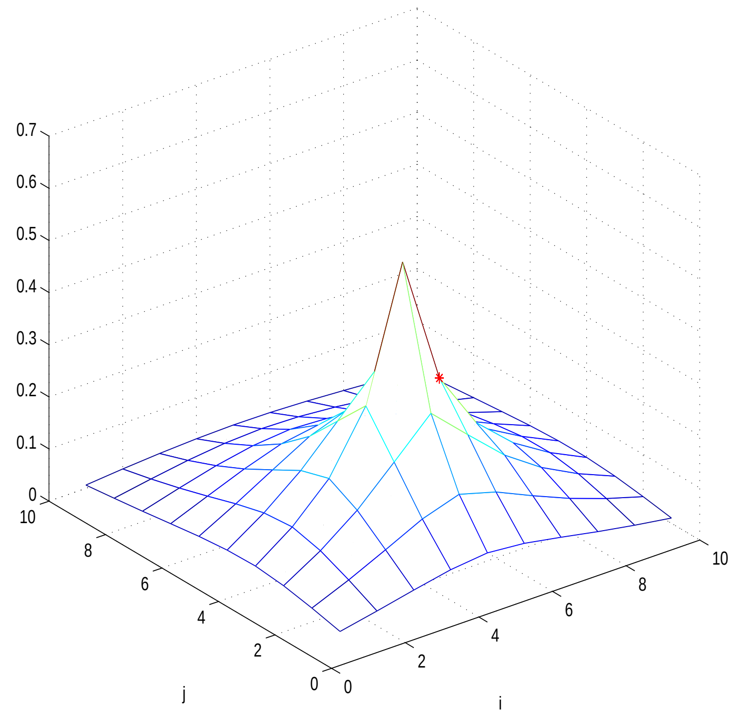

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Resolving the entry indexing using $MX_t + X_tM = E_t$

$$(S^{-1})_{k,t} = (S^{-1})_{\ell+n(m-1),t} = e_\ell^\top X_t e_m, \quad \ell, m \in \{1, \ldots, n\}$$

$\Rightarrow$ All the elements of the t-th column, $(S^{-1})_{:,t}$, are obtained by varying $m, \ell \in \{1, \ldots, n\}$

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From the Lyapunov equation theory,

$$X_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\imath\omega I + M)^{-1} E_t (\imath\omega I + M)^{-*} \mathrm{d}\omega$$

with $E_t = e_i e_j^\top$, $j = \lfloor (t-1)/n \rfloor + 1$, $i = t - n\lfloor (t-1)/n \rfloor$

Therefore,

$$e_\ell^\top \mathcal{X}_t e_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} e_\ell^\top (\imath\omega I + M)^{-1} e_i e_j^\top (\imath\omega I + M)^{-*} e_m \mathrm{d}\omega$$

Qualitative bounds

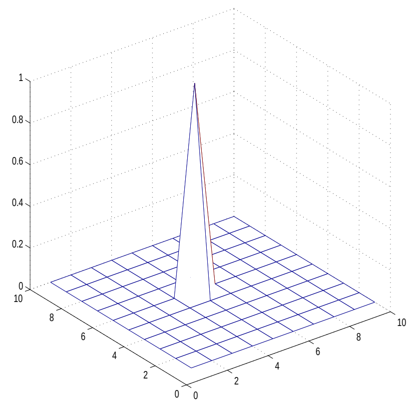
Let $\kappa = \lambda_{\max}/\lambda_{\min} = \mathrm{cond}(M)$

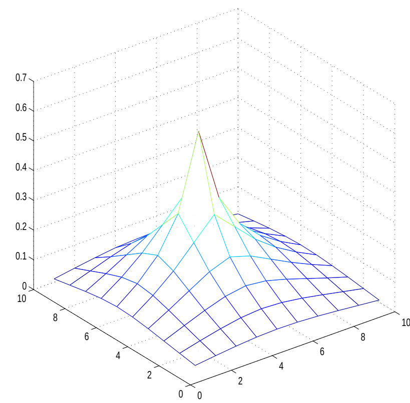
i)Assume $\ell, i, m, j \; : \; \ell \neq i, \; m \neq j$. $\mathfrak{n}_2 := |\ell - i| + |m - j| - 2 > 0$

$$|(S^{-1})_{k,t}| \leq \frac{\sqrt{\kappa^2+1}}{2\lambda_{\min}} \frac{1}{\sqrt{\mathfrak{n}_2}}.$$

ii)Assume $\ell, i, m, j \; : \; \ell = i$ or $m = j$. $\mathfrak{n}_1 := |\ell - i| + |m - j| - 1 > 0$

$$|(S^{-1})_{k,t}| \leq \frac{\kappa\sqrt{\kappa^2+1}}{2} \frac{1}{\sqrt{\mathfrak{n}_1}}.$$

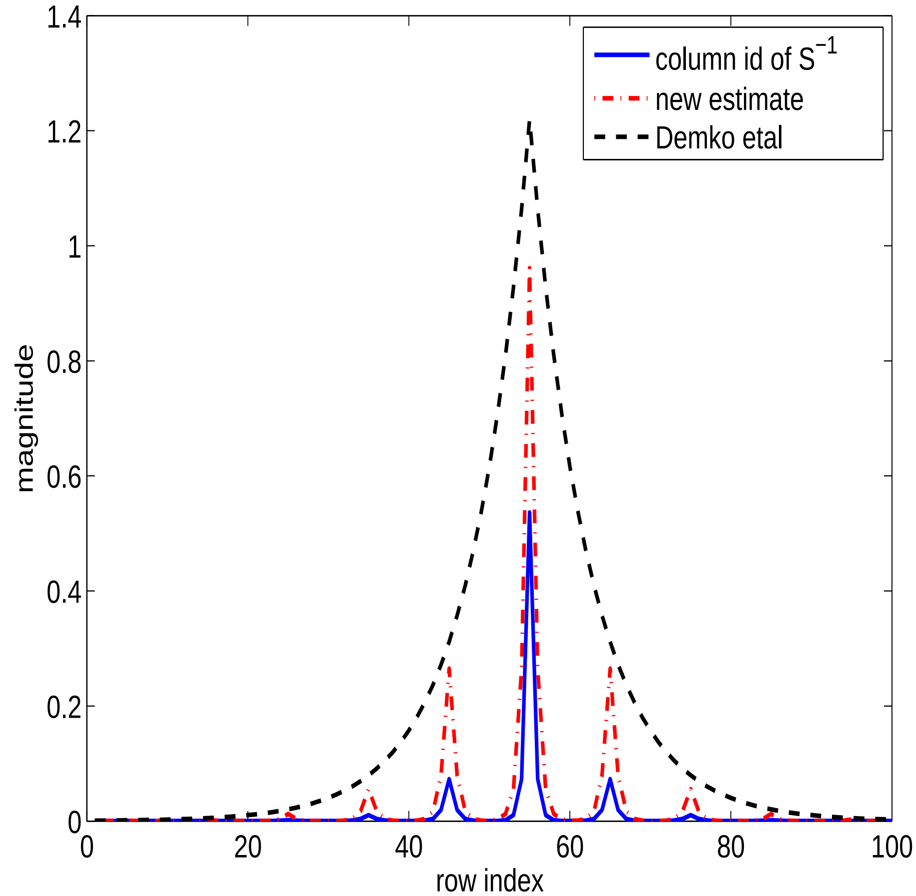




(i, j) and (ℓ, m)

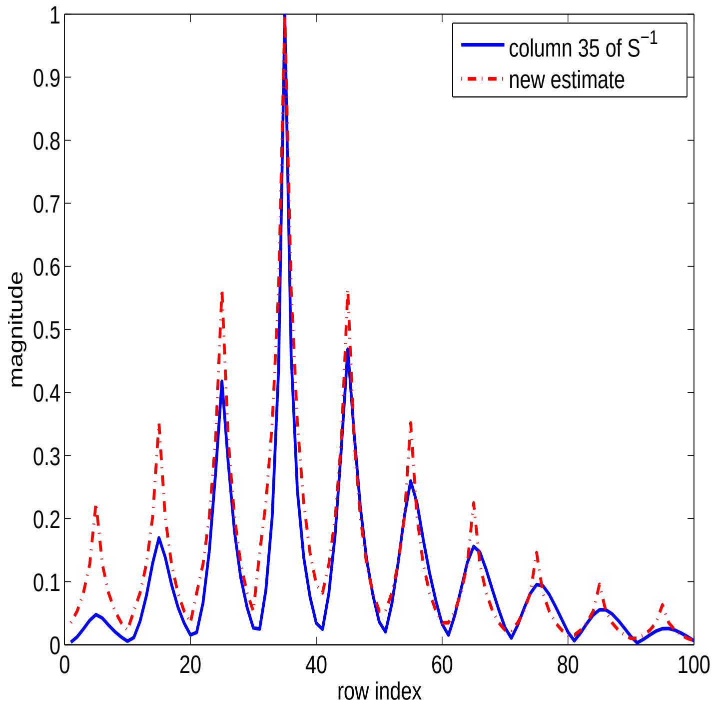
Examples. Symmetric positive definite matrix

$$M = \text{tridiag}(-0.5, \underline{2}, -0.5) \in \mathbb{R}^{10\times 10}$$



Examples. Legendre stiffness matrix (scaled to have peak equal to 1)

$$M = \text{tridiag}(\delta_k, \underline{\gamma_k}, \delta_k)$$



$$\gamma_k = \frac{2}{(4k-3)(4k+1)}$$

$$k = 1, \ldots, n, \quad \text{and}$$

$$\delta_k = \frac{-1}{(4k+1)\sqrt{(4k-1)(4k+3)}}$$

$$k = 1, \ldots, n-1$$

Connections to point-wise estimates for discrete Laplacian

For the discrete Green function G_h on the discrete d-dimensional grid R_h, there exist constants h_0 and C such that for $h \leq h_0$, $x, y \in R_h$,

$$G_h(x,y) \leq \begin{cases} C \log \frac{C}{|x-y|+h} & \text{if } d = 2 \\ \frac{C}{(|x-y|+h)^{d-2}} & \text{if } d \geq 3 \end{cases}$$

(Bramble & Thomee, '69)

Our estimate: entries depend on inverse square root of the distance!

Explored generalizations

- M spd of bandwidth $b > 1$
- $S = M_1 \otimes I + I \otimes M_2$, $M_1 \neq M_2$
- M_1, M_2 of different bandwidth
- $LL^T = S$, then L^{-1} (lower triang.) has same sparsity pattern

REFERENCES:

C. Canuto, V. Simoncini and M. Verani, LAA, v.452, 2014.

C. Canuto, V. Simoncini and M. Verani, Adaptive Legendre-Galerkin methods, in preparation, 2014.