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# Computational methods for large-scale matrix equations and application to PDEs

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## Some matrix equations

- Sylvester matrix equation

$$A\mathbf{X} + \mathbf{X}B + D = 0$$

Eigenvalue pbs and tracking, Control, MOR, Assignment pbs, Riccati eqn

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Survey article: [V.Simoncini, SIAM Review 2016.](#)

## More matrix equations

- Systems of linear matrix equations:

$$A_2 \mathbf{X} + \mathbf{X} A_1 + B^T \mathbf{P} = F_1$$

$$A_1 \mathbf{Y} + \mathbf{Y} A_2 + \mathbf{P} B = F_2$$

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- Riccati equation: Find  $\mathbf{X} \in \mathbb{R}^{n \times n}$  such that

$$A \mathbf{X} + \mathbf{X} A^T - \mathbf{X} B B^T \mathbf{X} + C^T C = 0$$

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Focus: All or some of the matrices are large (and possibly sparse)

## Solving the Lyapunov equation. The problem

Approximate  $\mathbf{X}$  in:

$$A\mathbf{X} + \mathbf{X}A^{\top} + BB^{\top} = 0$$

$$A \in \mathbb{R}^{n \times n} \text{ neg.real} \quad B \in \mathbb{R}^{n \times p}, \quad 1 \leq p \ll n$$

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Time-invariant linear system:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = x_0$$

Closed form solution:

$$\mathbf{X} = \int_0^\infty e^{-tA} BB^\top e^{-tA^\top} dt$$

$\Rightarrow$   $\mathbf{X}$  symmetric semidef.

see, e.g., Antoulas '05, Benner '06

## Linear systems vs linear matrix equations

Large linear systems:

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}$$

- Krylov subspace methods (CG, MINRES, GMRES, BiCGSTAB, etc.)
- Preconditioners: find  $P$  such that

$$AP^{-1}\tilde{x} = b \quad x = P^{-1}\tilde{x}$$

is **easier** and **fast** to solve

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### Large linear matrix equations:

$$AX + XA^\top + BB^\top = 0$$

- No preconditioning - to preserve symmetry
- $X$  is a large, dense matrix  $\Rightarrow$  low rank approximation

$$X \approx \tilde{X} = ZZ^\top, \quad Z \text{ tall}$$

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### Large linear matrix equations:

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Kronecker formulation:

$$(A \otimes I + I \otimes A)x = b \quad x = \text{vec}(\mathbf{X})$$

## Projection-type methods

Given an low dimensional approximation space  $\mathcal{K}$ ,

$$\mathbf{X} \approx X_m \quad \text{col}(X_m) \in \mathcal{K}$$

**Galerkin condition:**  $R := AX_m + X_m A^\top + BB^\top \perp \mathcal{K}$

$$V_m^\top R V_m = 0 \quad \mathcal{K} = \text{Range}(V_m)$$

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**Projected Lyapunov equation:**

$$V_m^\top (AV_m Y_m V_m^\top + V_m Y_m V_m^\top A^\top + BB^\top) V_m = 0$$

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Early contributions: Saad '90, Jaimoukha & Kasenally '94, for

$$\mathcal{K} = \mathcal{K}_m(A, B) = \text{Range}([B, AB, \dots, A^{m-1}B])$$

## More recent options as approximation space

### Enrich space to decrease space dimension

- Extended Krylov subspace

$$\mathcal{K} = \mathbb{E}\mathcal{K} := \mathcal{K}_m(A, B) + \mathcal{K}_m(A^{-1}, A^{-1}B),$$

that is,  $\mathcal{K} = \text{Range}([B, A^{-1}B, AB, A^{-2}B, A^2B, A^{-3}B, \dots,])$

(Druskin & Knizhnerman '98, Simoncini '07)

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- Rational Krylov subspace

$$\mathcal{K} = \mathbb{K} := \text{Range}([B, (A - s_1I)^{-1}B, \dots, (A - s_mI)^{-1}B])$$

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In both cases, for  $\text{Range}(V_m) = \mathcal{K}$ , **projected Lyapunov equation:**

$$(V_m^\top A V_m) Y_m + Y_m (V_m^\top A^\top V_m) + V_m^\top B B^\top V_m = 0$$

$$X_m = V_m Y_m V_m^\top$$

## Multiterm linear matrix equation

$$A_1\mathbf{X}B_1 + A_2\mathbf{X}B_2 + \dots + A_\ell\mathbf{X}B_\ell = C$$

Applications:

- Control
- (Stochastic) PDEs
- Matrix least squares
- ...

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**Main device:** Kronecker formulation

$$(B_1^\top \otimes A_1 + \dots + B_\ell^\top \otimes A_\ell) x = c$$

Iterative methods: matrix-matrix multiplications and rank truncation

(Benner, Breiten, Bouhamidi, Chehab, Damm, Grasedyck, Jbilou, Kressner, Matthies, Onwunta, Raydan, Stoll, Tobler, Zander, and **many** others...)

## Multiterm linear matrix equation

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Applications:

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Alternative approaches:

- Projection onto rich approximation space
- Compression to two-term matrix equation
- Splitting strategy towards two-term matrix equation
- ...

## PDEs on uniform grids and separable coeffs

$$-\varepsilon \Delta u + \phi_1(x)\psi_1(y)u_x + \phi_2(x)\psi_2(y)u_y + \gamma_1(x)\gamma_2(y)u = f \quad (x, y) \in \Omega$$

$\phi_i, \psi_i, \gamma_i, i = 1, 2$  sufficiently regular functions + b.c.

Problem discretization by means of a tensor basis

Multiterm linear equation:

$$-\varepsilon T_1 \mathbf{U} - \varepsilon \mathbf{U} T_2 + \Phi_1 B_1 \mathbf{U} \Psi_1 + \Phi_2 \mathbf{U} B_2^\top \Psi_2 + \Gamma_1 \mathbf{U} \Gamma_2 = F$$

Finite Diff.:  $\mathbf{U}_{i,j} = \mathbf{U}(x_i, y_j)$  approximate solution at the nodes

## PDEs with random inputs

Stochastic steady-state diffusion eqn: Find  $u : D \times \Omega \rightarrow \mathbb{R}$  s.t.  $\mathbb{P}$ -a.s.,

$$\begin{cases} -\nabla \cdot (a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega)) = f(\mathbf{x}) & \text{in } D \\ u(\mathbf{x}, \omega) = 0 & \text{on } \partial D \end{cases}$$

$f$ : deterministic;

$a$ : random field, linear function of finite no. of real-valued random variables  $\xi_r : \Omega \rightarrow \Gamma_r \subset \mathbb{R}$

**Common choice:** truncated Karhunen–Loève (KL) expansion,

$$a(\mathbf{x}, \omega) = \mu(\mathbf{x}) + \sigma \sum_{r=1}^m \sqrt{\lambda_r} \phi_r(\mathbf{x}) \xi_r(\omega),$$

$\mu(\mathbf{x})$ : expected value of diffusion coef.                       $\sigma$ : std dev.

$(\lambda_r, \phi_r(\mathbf{x}))$  eigs of the integral operator  $\mathcal{V}$  wrto  $V(\mathbf{x}, \mathbf{x}') = \frac{1}{\sigma^2} C(\mathbf{x}, \mathbf{x}')$

$(\lambda_r \searrow \quad C : D \times D \rightarrow \mathbb{R} \text{ covariance fun. } )$

## Discretization by stochastic Galerkin

Approx with space in tensor product form<sup>a</sup>  $\mathcal{X}_h \times S_p$

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} = G_0 \otimes K_0 + \sum_{r=1}^m G_r \otimes K_r, \quad \mathbf{b} = \mathbf{g}_0 \otimes \mathbf{f}_0,$$

$\mathbf{x}$ : expansion coef. of approx to  $u$  in the tensor product basis  $\{\varphi_i \psi_k\}$

$K_r \in \mathbb{R}^{n_x \times n_x}$ , FE matrices (sym)

$G_r \in \mathbb{R}^{n_\xi \times n_\xi}$ ,  $r = 0, 1, \dots, m$  Galerkin matrices associated w/  $S_p$  (sym.)

$\mathbf{g}_0$ : first column of  $G_0$

$\mathbf{f}_0$ : FE rhs of deterministic PDE

$$n_\xi = \dim(S_p) = \frac{(m+p)!}{m!p!} \Rightarrow \boxed{n_x \cdot n_\xi} \text{ huge}$$

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<sup>a</sup> $S_p$  set of multivariate polyn of total degree  $\leq p$

## The matrix equation formulation

$$(G_0 \otimes K_0 + G_1 \otimes K_1 + \dots + G_m \otimes K_m) \mathbf{x} = \mathbf{g}_0 \otimes \mathbf{f}_0$$

transforms into

$$K_0 \mathbf{X} G_0 + K_1 \mathbf{X} G_1 + \dots + K_m \mathbf{X} G_m = F, \quad F = \mathbf{f}_0 \mathbf{g}_0^\top$$

$$(G_0 = I)$$

**Solution strategy.** Conjecture:

- $\{K_r\}$  from trunc'd Karhunen–Loève (KL) expansion

⇓

$$\mathbf{X} \approx \tilde{X} \text{ low rank, } \tilde{X} = X_1 X_2^T$$

(Possibly extending results of Grasedyck, 2004)

## Matrix Galerkin approximation of the deterministic part. 1

Approximation space  $\mathcal{K}_k$  and basis matrix  $V_k$ :  $\mathbf{X} \approx X_k = V_k Y$

$$V_k^\top R_k = 0, \quad R_k := K_0 X_k + K_1 X_k G_1 + \dots + K_m X_k G_m - \mathbf{f}_0 \mathbf{g}_0^\top$$

### Computational challenges:

- Generation of  $\mathcal{K}_k$  involved  $m + 1$  different matrices  $\{K_r\}$  !
- Matrices  $K_r$  have different spectral properties
- $n_x, n_\xi$  so large that  $X_k, R_k$  should not be formed !

(Powell & Silvester & Simoncini, SISC 2017)

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## Systems of linear matrix equations

Find  $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$  and  $\mathbf{P} \in \mathbb{R}^{m \times n_2}$  such that

$$\begin{aligned} A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} &= F_1 \\ B \mathbf{X} &= F_2 \end{aligned}$$

with  $A_i \in \mathbb{R}^{n_i \times n_i}$ ,  $B \in \mathbb{R}^{m \times n_1}$ ,  $F_1 \in \mathbb{R}^{n_1 \times n_2}$ ,  $F_2 \in \mathbb{R}^{m \times n_2}$ ,  $m \leq n_1$

### Emerging matrix formulation of different application problems

- Constraint control
- Mixed formulations of stochastic diffusion problems
- Discretized deterministic/stochastic (Navier-)Stokes equations
- ...

An example. Mixed FE formulation of stochastic Galerkin diffusion pb

$$\begin{aligned} c^{-1}\vec{u} - \nabla p &= 0, \\ -\nabla \cdot \vec{u} &= f, \end{aligned}$$

Assume that  $c^{-1} = c_0 + \sum_{r=1}^{\ell} \sqrt{\lambda_r} c_r(\vec{x}) \xi_r(\omega)$  and that an appropriate class of finite elements is used for the discretization of the problem (see, e.g., the derivation in Elman & Furnival & Powell, 2010)

After discretization the problem reads:

$$\begin{bmatrix} G_0 \otimes K_0 + \sum_{r=1}^{\ell} \sqrt{\lambda_r} G_r \otimes K_r & G_0^T \otimes B_0^T \\ G_0 \otimes B_0 & \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

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For  $\ell = 1$  we obtain

$$\begin{aligned} K_0 \mathbf{X} G_0 + K_1 \mathbf{X} G_1 + B_0^T \mathbf{P} G_0 &= 0, \\ B_0 \mathbf{X} G_0 &= F \end{aligned}$$

## Computational strategies for systems of matrix equations

$$\begin{aligned} A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} &= F_1 \\ B \mathbf{X} &= F_2 \end{aligned}$$

**Kronecker formulation (monolithic):**

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathcal{A} = I \otimes A_1 + A_2^T \otimes I, \quad \mathcal{B} = B \otimes I$$

with  $\mathbf{x} = \text{vec}(\mathbf{X})$ ,  $\mathbf{p} = \text{vec}(\mathbf{P})$ ,  $f_1 = \text{vec}(F_1)$  and  $f_2 = \text{vec}(F_2)$

Extremely rich literature from saddle point algebraic linear systems

**Problem:** Coefficient matrix has size  $(n_1 n_2 + m n_2) \times (n_1 n_2 + m n_2)$

## Computational strategies for systems of matrix equations. Cont'd

$$\begin{aligned} A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} &= F_1 \\ B \mathbf{X} &= F_2 \end{aligned}$$

★ Derive numerical strategies that directly work with the matrix equations:

- Small scale: Null space method
- Small and medium scale: Schur complement method  
(also directly applicable to trilinear case)
- Large scale: Iterative method for low rank  $F_i, i = 1, 2$

“Small and medium scale” actually means “Large scale” for the Kronecker form!

V.Simoncini, 2018, to appear in IMA J. Numer. Analysis

## Large scale problem. Iterative method. 1/3

$$\begin{aligned} A_1 \mathbf{X} + \mathbf{X} A_2 + B^T \mathbf{P} &= F_1 \\ B \mathbf{X} &= F_2 \end{aligned}$$

Rewrite as

$$\begin{bmatrix} A_1 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \Leftrightarrow \quad \mathcal{M} \mathbf{Z} + \mathcal{D}_0 \mathbf{Z} A_2 = F$$

with

$$\mathcal{M}, \mathcal{D}_0 \in \mathbb{R}^{(n_1+m) \times (n_1+m)}$$

$$A_2 \in \mathbb{R}^{n_2 \times n_2} \text{ nonsingular}$$

$\mathcal{D}_0$  highly singular

If  $F$  low rank, exploit projection-type strategies for Sylvester equations

## Large scale problem. Iterative method. 2/3

$$\begin{bmatrix} A_1 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} A_2 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \Leftrightarrow \quad \mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}A_2 = F$$

with  $F$  low rank. We rewrite the matrix equation as a Sylvester equation:

$$\mathbf{Z}A_2^{-1} + \mathcal{M}^{-1}\mathcal{D}_0\mathbf{Z} = \hat{F}$$

with  $\hat{F} = \mathcal{M}^{-1}FA_2^{-1}$  of low rank if  $F$  is of low rank,  $\hat{F} = \hat{F}_\ell \hat{F}_r^T$

$$\Rightarrow \quad \mathbf{Z} \approx \tilde{\mathbf{Z}}_k = V_k \mathbf{Z}_k W_k^T$$

with  $\text{Range}(V_k), \text{Range}(W_k)$  appropriate approximation spaces of small dimensions

## Large scale problem. Iterative method. 3/3

### Galerkin-Projection method

$$\mathbf{Z}A_2^{-1} + \mathcal{M}^{-1}\mathcal{D}_0\mathbf{Z} = \widehat{F}_l\widehat{F}_r^T \quad \Rightarrow \quad \mathbf{Z} \approx \widetilde{\mathbf{Z}}_k = V_k\mathcal{Z}_k W_k^T$$

Choice of  $V_k, W_k$ . A possible strategy:

- $W_k = \mathbb{E}\mathbb{K}_k(A_2^{-T}, \widehat{F}_r)$ , Extended Krylov subspace
- $V_k = K_k(\mathcal{M}^{-1}\mathcal{D}_0, \widehat{F}_l) \cup K_k((\mathcal{M}^{-1}\mathcal{D}_0 + \sigma I)^{-1}, \widehat{F}_l)$   
Augmented Krylov subspace,  $\sigma \in \mathbb{R}$   
(see, e.g., Shank & Simoncini, 2013)

**Note:**  $\mathcal{M}$  has size  $(n_1 + m) \times (n_1 + m)$

(Compare with  $(n_1n_2 + mn_2) \times (n_1n_2 + mn_2)$  of the Kronecker form)

## Numerical experiments

$$A_1 \mathbf{X} - \mathbf{X} A_2 + B^T \mathbf{P} = 0, \quad \text{vs} \quad \mathcal{A} \mathbf{z} = f$$

$$B \mathbf{X} = F_2$$

$$A_1 \rightarrow \mathcal{L}_1 = -u_{xx} - u_{yy}, \quad F_2 \text{ rank-1}$$

$$A_2 \rightarrow \mathcal{L}_1 = -(e^{-10xy} u_x)_x - (e^{10xy} u_y)_y + 10(x+y)u_x$$

$$B = \text{bidiag}(-1, \underline{1}) \in \mathbb{R}^{(n_2 - n_1) \times n_2}, \quad \text{params: tol} = 10^{-6}, \sigma = 10^{-2}$$

$n_1$	$n_2$	size( $\mathcal{A}$ )	Monolithic Elapsed Time	Matrix eqns Elapsed Time
400	100	79,000	6.9769e-02	3.1523e-02 (4)
900	225	401,625	3.4808e-01	5.0447e-02 (4)
1600	400	1272,000	1.1319e+00	7.8018e-02 (4)
2500	625	3109,375	3.1212e+00	1.5282e-01 (5)
3600	900	6453,000	1.0210e+01	2.8053e-01 (5)
4900	1225	11,962,125	3.7699e+01	1.4754e+00 (5)

Monolithic: direct solver (iterative not competitive)

Numerical experiments. 1D stochastic Stokes problem.

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \mathcal{H} = (\nu_0 G_0 + \nu_1 G_1) \otimes A_x, \quad \mathcal{B} = G_0 \otimes B_x$$

where  $\nu = \nu_0 + \nu_1 \xi(\omega)$  uncertain viscosity,  $\xi$  random variable

Then

$$A_x \mathbf{X} G_0 \nu_0 + A_x \mathbf{X} G_1 \nu_1 + B_x^T \mathbf{P} G_0 = F_1, \quad B_x \mathbf{X} = F_2$$

with  $G_0 = I$ . This corresponds to

$$\begin{bmatrix} \nu_0 A_x & B_x^T \\ B_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} \nu_1 A_x & \\ & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{P} \end{bmatrix} G_1 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

that is

$$\mathcal{M} \mathbf{Z} + \mathcal{D}_0 \mathbf{Z} G_1 = F$$

## Numerical experiments. 1D stochastic Stokes problem.

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \quad \text{vs} \quad \mathcal{A}\mathbf{z} = f$$

$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$       Powell & Silvester, 2012

### Elapsed Time

$n_1$	$n_2$	$n_B$	size( $\mathcal{A}$ )	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{E}\mathbb{K}(\sigma)$
1256	4	389	6,580	0.1852	0.146 (11)	0.19 (2)
3526	4	990	18,064	0.9063	0.275 (11)	0.52 (2)
9812	4	2615	49,708	4.6418	0.981 (10)	2.09 (2)
$n_1$	$n_2$	$n_B$	size( $\mathcal{A}$ )	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{E}\mathbb{K}(\sigma)$
1256	165	389	271,425	2.91	1.53 (11)	0.20 (2)
3526	165	990	745,140	12.16	7.43 (11)	0.45 (2)
9812	165	2615	2050,455	-	-	1.87 (2)

- $n_2$  could be much larger,  $n_2 = O(10^3)$
- Memory requirements are limited,  $\tilde{\mathbf{Z}} = Z_1 Z_2^T$  of very low rank

## Numerical experiments. 2D stochastic Stokes problem

$$\begin{bmatrix} \mathcal{H} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \begin{aligned} \mathcal{H} &= \text{blkdiag}((\nu_0 G_0 + \nu_1 G_1) \otimes A_x, (\nu_0 G_0 + \nu_1 G_1) \otimes A_y) \\ \mathcal{B} &= [G_0 \otimes B_x, G_0 \otimes B_y] \end{aligned}$$

$$\mathcal{M}\mathbf{Z} + \mathcal{D}_0\mathbf{Z}G_1 = F \quad \text{vs} \quad \mathcal{A}\mathbf{z} = f$$

$n_1$	$n_2$	size( $\mathcal{A}$ )	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{E}\mathbb{K}(\sigma)$
2512	4	11,604	0.55	0.12 (12)	0.28 (2)
7052	4	32,168	3.73	0.36 (12)	1.22 (2)
19624	4	88,956	11.93	1.51 (12)	4.37 (2)
$n_1$	$n_2$	size( $\mathcal{A}$ )	Monolithic direct	Monolithic MINRES	Iterative $\mathbb{E}\mathbb{K}(\sigma)$
2512	165	478 665	7.60	3.16 (17)	0.33 (2)
7052	165	1 326 930	34.08	15.52 (18)	1.32 (2)
19624	165	3 669 435	–	–	5.69 (3)

$$\nu_0 = 1/10, \nu_1 = 3\nu_0/10$$

Powell-Silvester (2012)

## Not discussed but in this category

- Sylvester-like linear matrix equations

$$AX + f(X)B = C$$

typically (but not only!):  $f(X) = \bar{X}$ ,  $f(X) = X^\top$ , or  $f(X) = X^*$   
(Bevis, Braden, Byers, Chiang, De Terán, Dopico, Duan, Feng, Gonzalez, Guillery, Hall, Hartwig, Ikramov, Kressner, Montealegre, Reyes, Schröder, Vorntsov, Watkins, Wu, ...)

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- Linear systems with complex tensor structure

$$\mathcal{A}\mathbf{x} = b \quad \text{with} \quad \mathcal{A} = \sum_{j=1}^k I_{n_1} \otimes \cdots \otimes I_{n_{j-1}} \otimes A_j \otimes I_{n_{j+1}} \cdots \otimes I_{n_k}.$$

Dolgov, Grasedyck, Khoromskij, Kressner, Oseledets, Tobler, Tyrtysnikov, and many more...

## Conclusions

Large-scale (Multiterm) linear equations are a new computational tool

- Great advances in solving really large linear matrix equations
- Second order (matrix) challenges rely on strength and maturity of linear system solvers
- Low-rank tensor formats is the new generation of approximations

**Reference for linear matrix equations:**

★ V. Simoncini,

*Computational methods for linear matrix equations,*

SIAM Review, Sept. 2016.