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A note on a generalized form of the Laplacian and of sub-Laplacians

By

FRANCESCO UGUZZONI

Abstract. In this note we prove a recent conjecture of Hasson [11]: we show that, for a locally integrable function u, a sufficient condition to be harmonic is that $\lim_{r\to 0^+} r^{-2}(M_r u - u) = 0$ in the weak sense of distributions (M_r being the averaging operator on balls of radius r). We also extend this and other results to the setting of sub-Laplacians on Carnot groups.

1. Introduction. Let u be a smooth function on $\mathbb{R}^N$ and consider the averages $M_r u(x) = \displaystyle\fint_{|x-y|<r} u(y)\,\mathrm{d}y$. Then, at any point $x \in \mathbb{R}^N$, the following formula holds

$$\Delta u(x) = \lim_{r\to 0^+} \alpha_N \frac{M_r u(x) - u(x)}{r^2}, \tag{1}$$

being $\alpha_N > 0$ a dimensional constant and Δ the classical Laplace operator. This formula also suggests a definition of *generalized Laplacian* at x for less regular functions u, whenever the above limit exists. The result stated above is classical (see Pizzetti [14]) and related matters have been studied in several papers starting from the early 1900s up to the present days, see e.g. [14, 1, 15, 19, 18, 20, 8, 16, 13, 12, 11].

In the recent paper [11], M. Hasson shows that, for a function u in the Sobolev space $W^{2,1}$, a sufficient condition to be harmonic is that the limit in the right hand side of (1) is equal to zero for almost every x. In [11], Hasson also makes the conjecture that the same result holds for a function $u \in H^s$ with $s > \frac{3}{2}$, if the limit is assumed to be uniform in x whenever x is a Lebesgue point of u.

In this note we prove that the above conjecture is true. We also show that the assumptions of Hasson can be considerably weakened. Indeed we prove that a locally integrable function u is harmonic if the limit in (1) is equal to zero in the weak sense of distributions (see Corollary 6). Our proof is very easy and direct, as it only relies on the classical mean value formulas. Moreover such proof can be adapted to the more general setting of sub-Laplacians on Carnot groups allowing to obtain an analogous result for such operators (see Theorem 5).

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We also provide a simple proof of the $W^{2,1}$-result of Hasson, which can be easily extended to the setting of Carnot groups as well (see Theorem 8). Finally, in connection with the present work, we would like to quote the recent paper [2] where a Pizzetti-type formula and some related results are obtained in the setting of the Heisenberg groups.

2. Results and proofs. We start by giving the definition of a Carnot group. Let $\circ$ be an assigned Lie group law on $\mathbb{R}^N$. Suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(\nu)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^\nu x^{(\nu)}).$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, \nu$ and $N_1 + \cdots + N_\nu = N$. We denote by $\mathfrak{g}$ the Lie algebra of $(\mathbb{R}^N, \circ)$ i.e. the Lie algebra of left-invariant vector fields on $\mathbb{R}^N$. For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in $\mathfrak{g}$ that agrees at the origin with $\partial/\partial x_i$. We make the following assumption: the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$. With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *Carnot group*. We also say that $\mathbb{G}$ is of step ν and has $m := N_1$ generators. If $Y_1, \dots, Y_m$ is any basis for span$\{X_1, \dots, X_m\}$, the second order differential operator

$$\mathcal{L} = \sum_{i=1}^{m} Y_i^2$$

will be called a *sub-Laplacian* on $\mathbb{G}$. We shall also use the notation

$$\nabla_{\mathcal{L}} = (Y_1, \dots, Y_m)$$

for the $\mathcal{L}$-subelliptic gradient. The simplest example of Carnot group is $\mathbb{G} = (\mathbb{R}^N, +)$ (in this case the classical Laplace operator $\mathcal{L} = \Delta$ is a sub-Laplacian on $\mathbb{G}$). The most simple non-abelian example is the Heisenberg group $\mathbb{H}^n$ (with the Kohn-Laplace operator). In literature (see e.g. [7, 17, 10]) a Carnot group (or stratified group) $\mathbb{G}$ is usually defined as a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = V_1 \oplus \ldots \oplus V_\nu$ with $[V_1, V_i] = V_{i+1}$, $[V_1, V_\nu] = \{0\}$. The two definitions are indeed equivalent, up to isomorphisms (see e.g. [5]).

We next give a list of known results about Carnot groups. Since $X_1, \dots, X_m$ generate the whole $\mathfrak{g}$, which has rank N at any point, any sub-Laplacian $\mathcal{L}$ satisfies Hörmander's hypoellipticity condition. Moreover, the vector fields $X_1, \dots, X_m$ are homogeneous of degree 1 w.r.t. δ_λ and X_j^* (the adjoint operator of X_j) is $-X_j$. In particular, $\mathcal{L}$ is a self-adjoint operator in divergence form $\mathcal{L} = \operatorname{div}(M_{\mathcal{L}}(x)\,\nabla)$, $M_{\mathcal{L}}(x)$ being a suitable positive semi-definite symmetric matrix. We denote by $Q = \sum_{j=1}^{\nu} j\, N_j$ the *homogeneous dimension* of $\mathbb{G}$. If $Q \geqq 3$, then there exists a homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(x, y) = d(y^{-1} \circ x)^{2-Q}$$

is the fundamental solution for $\mathcal{L}$ (see [7, 9], see also [4]). We recall that a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{R}^N \to [0, \infty[$, smooth away from the origin, such that $d(\delta_\lambda(x)) = \lambda\, d(x)$, $d(x^{-1}) = d(x)$, and $d(x) = 0$ iff $x = 0$. Hereafter, we also denote $d(y^{-1} \circ x)$ by $d(x, y)$. Moreover we shall use the notation $B_d(x, r)$ for the d-ball of center x and radius r. The following quasi-triangle inequality holds $d(x, y) \leqq \beta\, (d(x, z) + d(z, y))$, for a suitable constant β.

In what follows, $\mathbb{G}$ will be a fixed Carnot group and $\mathcal{L}$ a fixed sub-Laplacian on $\mathbb{G}$. The following mean value formulas have been proved in [6] (see also [21, 3]). Let u be a C^2 function on an open subset Ω of $\mathbb{R}^N$ and suppose that $\overline{B_d(x, r)} \subseteqq \Omega$. Then

$$u(x) = M_r u(x) - N_r(\mathcal{L}u)(x), \tag{2}$$

where we have set

$$M_r u(x) = \mu\, r^{-Q} \int\limits_{d(x,y)<r} |\nabla_{\mathcal{L}} d|^2 (x^{-1} \circ y)\, u(y)\, \mathrm{d}y,$$

$$N_r(\mathcal{L}u)(x) = Q\, r^{-Q} \int\limits_0^r \rho^{Q-1} \int\limits_{d(x,y)<\rho} (\Gamma(x^{-1} \circ y) - \rho^{2-Q})\, \mathcal{L}u(y)\, \mathrm{d}y\, \mathrm{d}\rho,$$

being $\mu = (\int\limits_{B_d(0,1)} |\nabla_{\mathcal{L}} d|^2)^{-1}$. We remark that M_r, N_r are integral operators in the form

$$M_r u(x) = \int K_r(x^{-1} \circ y)\, u(y)\, \mathrm{d}y, \quad N_r f(x) = \int H_r(x^{-1} \circ y)\, f(y)\, \mathrm{d}y$$

with nonnegative kernels, supported in $B_d(0, r)$, $K_r \in L^\infty$ ($\nabla_{\mathcal{L}} d$ is bounded since it is δ_λ-homogeneous of degree zero) and $H_r \in L^{Q/(Q-2)}_{\text{weak}}$ (H_r has the same behavior of Γ at the origin and it is continuous away from the origin, see the proof of Lemma 1 below). The mean value operator M_r is an *approximation of the identity* in the sense that $K_r = r^{-Q} K_1 \circ \delta_{r^{-1}}$, $K_1 \geqq 0$, $\int K_1 = 1$ ($K_1 = \mu |\nabla_{\mathcal{L}} d|^2 \chi_{B_d(0,1)}$, where $\chi_{B_d(0,1)}$ denotes the characteristic function of the set $B_d(0, 1)$). The following lemma states that also

$$\widetilde{N}_r = \alpha\, r^{-2} N_r$$

is an approximation of the identity, for a suitable constant $\alpha > 0$.

Lemma 1. *There exists a positive constant α such that, setting $\widetilde{H}_r = \alpha\, r^{-2} H_r$, we have*

$$\widetilde{H}_r = r^{-Q} \widetilde{H}_1 \circ \delta_{r^{-1}}, \quad \widetilde{H}_1 \geqq 0, \quad \int \widetilde{H}_1 = 1. \tag{3}$$

In particular $\widetilde{N}_r(1)(x) = 1$ for every $x \in \mathbb{R}^N$ and $r > 0$.

P r o o f. It is easy to recognize that the explicit expression of H_r is

$$H_r(x) = Q\,r^{-Q}\chi_{B_d(0,r)}(x)\int_{d(x)}^{r}\rho^{Q-1}(d(x)^{2-Q}-\rho^{2-Q})\,\mathrm{d}\rho$$
$$= \left(\left(1-\left(\frac{d(x)}{r}\right)^{Q}\right)d(x)^{2-Q}-\frac{Q}{2}\left(1-\left(\frac{d(x)}{r}\right)^{2}\right)r^{2-Q}\right)\chi_{B_d(0,r)}(x).$$

Hence $\widetilde{H}_r = \alpha\,r^{-Q}((\frac{d}{r})^{2-Q}+\frac{Q-2}{2}(\frac{d}{r})^2-\frac{Q}{2})\chi_{B_d(0,r)}$. In order to prove (3), it is now sufficient to use the δ_λ-homogeneity of d. Finally, the last statement of the lemma immediately follows from the invariance of the Lebesgue measure w.r.t. the left translations of $\mathbb{G}$. Indeed we have

$$\widetilde{N}_r(1)(x) = \int \widetilde{H}_r(x^{-1}\circ y)\,\mathrm{d}y = \int \widetilde{H}_r(z)\,\mathrm{d}z$$
$$= r^{-Q}\int \widetilde{H}_1(\delta_{r^{-1}}z)\,\mathrm{d}z = \int \widetilde{H}_1(y)\,\mathrm{d}y = 1. \quad \square$$

We shall use the notation

$$\mathcal{L}_r u(x) = \alpha\,r^{-2}(M_r u(x) - u(x))$$

whenever $u \in L^1(B_d(x,r))$.

Lemma 2. *For every* $\varphi \in C_0^\infty(\mathbb{R}^N)$ *we have*

$$\mathcal{L}_r\varphi = \widetilde{N}_r(\mathcal{L}\varphi) \rightrightarrows \mathcal{L}\varphi, \quad \text{as } r \to 0^+, \text{ uniformly on } \mathbb{R}^N.$$

P r o o f. The identity $\mathcal{L}_r\varphi = \widetilde{N}_r(\mathcal{L}\varphi)$ is an immediate consequence of the mean value formula (2). Moreover, by Lemma 1,

$$|\mathcal{L}_r\varphi(x) - \mathcal{L}\varphi(x)| = |\widetilde{N}_r(\mathcal{L}\varphi - \mathcal{L}\varphi(x))(x)|$$
$$\leqq \int_{d(x,y)<r} \widetilde{H}_r(x^{-1}\circ y)|\mathcal{L}\varphi(y) - \mathcal{L}\varphi(x)|\,\mathrm{d}y.$$

We now only need to observe that $\mathcal{L}\varphi$ is uniformly continuous and to recall that $\int \widetilde{H}_r(x^{-1}\circ y)\,\mathrm{d}y = 1$. $\square$

Let us introduce the adjoint operators M_r^*, $\mathcal{L}_r^*$, defined by

$$M_r^* u(x) = \int K_r(y^{-1}\circ x)\,u(y)\,\mathrm{d}y, \quad \mathcal{L}_r^* u(x) = \alpha\,r^{-2}(M_r^* u(x) - u(x)).$$

Then, for every $u \in L_{\mathrm{loc}}(\Omega)$, $\varphi \in C_0^\infty(\Omega)$ and for every sufficiently small $r > 0$,

$$\int \varphi\,M_r u = \int u\,M_r^*\varphi, \quad \int \varphi\,\mathcal{L}_r u = \int u\,\mathcal{L}_r^*\varphi. \tag{4}$$

R e m a r k 3. If $|\nabla_{\mathcal{L}} d(x)| = |\nabla_{\mathcal{L}} d(x^{-1})|$ then $M_r^* = M_r$ and $\mathcal{L}_r^* = \mathcal{L}_r$. This is the case e.g. for $\mathcal{L} = \Delta$, the classical Laplace operator, and for $\mathcal{L} = \Delta_{\mathbb{H}^n}$, the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$.

Lemma 4. *Given $\varphi \in C_0^\infty(\mathbb{R}^N)$, the following assertions hold.*

(i) *If we set $\varphi_x^*(y) = \varphi(x \circ y^{-1})$, then $\varphi_x^* \in C_0^\infty(\mathbb{R}^N)$ and*

$$(M_r^*\varphi)(x) = (M_r\varphi_x^*)(0).$$

(ii) *If we set $\widetilde{\varphi}(x) = (\mathcal{L}\varphi_x^*)(0)$, then $\mathcal{L}_r^*\varphi \rightrightarrows \widetilde{\varphi}$, as $r \to 0^+$, uniformly on $\mathbb{R}^N$.*
(iii) *$\mathcal{L}_r^*\varphi \rightrightarrows \mathcal{L}\varphi$, as $r \to 0^+$, uniformly on $\mathbb{R}^N$.*

P r o o f. Recalling that the Jacobian determinant $|\det(\mathcal{J}_{(y \mapsto y^{-1} \circ x)}(y))| = 1$ for every x, y, (see e.g. [5, Theorem 5.13] for a complete proof of this assertion), the proof of (i) is straightforward:

$$M_r^*\varphi(x) = \int K_r(y^{-1} \circ x)\,\varphi(y)\,\mathrm{d}y = \int K_r(z)\,\varphi(x \circ z^{-1})\,\mathrm{d}z = M_r\varphi_x^*(0).$$

Moreover, from (i) and from Lemma 2 it follows that

$$\begin{aligned}
|\mathcal{L}_r^*\varphi(x) - \widetilde{\varphi}(x)| &= |\alpha\, r^{-2}(M_r\varphi_x^*(0) - \varphi(x)) - \widetilde{\varphi}(x)| \\
&= |\mathcal{L}_r\varphi_x^*(0) - \widetilde{\varphi}(x)| \\
&= |\widetilde{N}_r(\mathcal{L}\varphi_x^*)(0) - \widetilde{\varphi}(x)| = \left| \int \widetilde{H}_r(y)(\mathcal{L}\varphi_x^*(y) - \widetilde{\varphi}(x))\,\mathrm{d}y \right| \\
&\leqq \int\limits_{d(y)<r} \widetilde{H}_r(y)\ |\mathcal{L}_y((x, y) \mapsto \varphi(x \circ y^{-1}))(x, y) \\
&\quad - \mathcal{L}_y((x, y) \mapsto \varphi(x \circ y^{-1}))(x, 0)|\,\mathrm{d}y.
\end{aligned}$$

Recalling that $\int \widetilde{H}_r = 1$, that φ has compact support and that $\mathcal{L}_y((x, y) \mapsto \varphi(x \circ y^{-1}))$ is smooth, we obtain (ii). We now prove (iii). Using Lemma 2, (4) and (ii) and recalling that $\mathcal{L}$ is self-adjoint, we have

$$\int \psi\,\mathcal{L}\varphi = \int \varphi\,\mathcal{L}\psi = \lim_{r \to 0^+} \int \varphi\,\mathcal{L}_r\psi = \lim_{r \to 0^+} \int \psi\,\mathcal{L}_r^*\varphi = \int \psi\widetilde{\varphi},$$

for every test function $\psi \in C_0^\infty(\mathbb{R}^N)$. Thus $\widetilde{\varphi} = \mathcal{L}\varphi$ and (iii) follows from (ii). □

Theorem 5. *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in L_{\mathrm{loc}}(\Omega)$. Suppose that*

$$\int \varphi\,\mathcal{L}_r u \to 0, \quad \text{as } r \to 0^+, \tag{5}$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Then u is $\mathcal{L}$-harmonic in Ω (more precisely there exists $v \in C^\infty(\Omega)$ such that $\mathcal{L}v = 0$ in Ω, $u = v$ a.e. in Ω).

Proof. Collecting Lemma 4-(iii), (4) and (5), we obtain

$$\int u\,\mathcal{L}\varphi = \lim_{r\to 0^+}\int u\,\mathcal{L}_r^*\varphi = \lim_{r\to 0^+}\int \varphi\,\mathcal{L}_r u = 0$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Hence $\mathcal{L}u = 0$ in Ω in the sense of distributions. The thesis then follows from the hypoellipticity of $\mathcal{L}$. □

Let us restate more explicitly Theorem 5 when $\mathcal{L} = \Delta$ is the classical Laplace operator.

Corollary 6. *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in L_{\text{loc}}(\Omega)$. Suppose that*

$$\int \varphi(x)\left(\frac{\displaystyle\fint_{|x-y|<r} u(y)\,\mathrm{d}y - u(x)}{r^2}\right)\mathrm{d}x \to 0, \quad \text{as } r \to 0^+,$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Then u is harmonic in Ω.

Remark 7. The above corollary proves in particular [11, Conjecture 4.1]. The hypothesis in [11] is given in terms of surface averages. More precisely in the quoted conjecture it is supposed that there exists a function $w : \Omega \to \mathbb{R}$ such that the following condition holds:

$$\frac{1}{r^2}\left(\fint_{|x-y|=r} u(y)\,\mathrm{d}\sigma(y) - w(x)\right) \to 0, \quad \text{as } r \to 0^+,$$

uniformly in x, whenever x is a Lebesgue point of u. Integrating in r, it is easy to see that this condition implies an analogous condition involving solid averages, which in turns also implies that w must be equal to u a.e. Therefore the hypothesis in [11] is stronger than our hypothesis in Corollary 6.

Statement (8) of the following theorem extends to Carnot groups the $W^{2,1}$-result of Hasson [11, Theorem 1.2]. The hypothesis $u \in W^{2,1}$ is replaced in a natural way by the assumption $\mathcal{L}u \in L_{\text{loc}}$.

Theorem 8. *Let Ω be an open subset of $\mathbb{R}^N$. Suppose that $u \in L_{\text{loc}}(\Omega)$ has sub-Laplacian (in the sense of distributions) $\mathcal{L}u \in L_{\text{loc}}(\Omega)$. Then we have*

(6) $$u = M_r u - N_r(\mathcal{L}u) \qquad \text{a.e. in } \Omega_r = \{x \in \Omega \mid d(x, \mathbb{R}^N \setminus \Omega) > r\};$$

(7) $$\mathcal{L}_r u \to \mathcal{L}u \qquad \text{in } L_{\text{loc}}(\Omega), \text{ as } r \to 0^+;$$

(8) *if $\mathcal{L}_r u(x) \to 0$, as $r \to 0^+$, pointwise a.e. in Ω, then u is $\mathcal{L}$-harmonic in Ω.*

Before proving Theorem 8, let us briefly introduce mollifiers on Carnot groups. Let $\Phi \in C_0^\infty(B_d(0,1))$ be such that $\Phi \geqq 0$, $\int \Phi = 1$, $\Phi(x^{-1}) = \Phi(x)$ (in order to fulfil the last requirement it is sufficient to look for a $\Phi(x) = \widetilde{\Phi}(d(x)))$. Set $\Phi_\varepsilon = \varepsilon^{-Q}\Phi \circ \delta_{\varepsilon^{-1}}$ and define

$$u_\varepsilon(x) = \int \Phi_\varepsilon(y \circ x^{-1}) u(y)\, \mathrm{d}y$$

for any $u \in L_{\mathrm{loc}}(\mathbb{R}^N)$. Then it is a standard argument to recognize that u_ε is smooth and $u_\varepsilon \to u$ in $L_{\mathrm{loc}}(\mathbb{R}^N)$, as $\varepsilon \to 0^+$. Moreover the following result holds.

Remark 9. Let $u \in L_{\mathrm{loc}}(\mathbb{R}^N)$ and let $Z \in \mathfrak{g}$ (the Lie algebra of $\mathbb{G}$). Suppose that $Zu \in L_{\mathrm{loc}}(\mathbb{R}^N)$ (in the sense of distributions). Then we have $Z(u_\varepsilon) = (Zu)_\varepsilon$.

Proof. It is easy to see that $Z(\varphi_\varepsilon) = (Z\varphi)_\varepsilon$ holds for any test function $\varphi \in C_0^\infty(\mathbb{R}^N)$. Indeed it is sufficient to observe that $\varphi_\varepsilon(x) = \int \Phi(y)\, \varphi((\delta_\varepsilon y) \circ x)\, \mathrm{d}y$ and to use the left invariance of Z. Now, using the fact that $\Phi(x^{-1}) = \Phi(x)$ and recalling that $Z^* = -Z$ (being Z a linear combination of δ_λ-homogeneous vector fields, see e.g. [5, Remark 5.9] for more details), we obtain

$$\begin{aligned} \int \varphi\, Z(u_\varepsilon) &= -\int u_\varepsilon\, Z\varphi = -\int\int \Phi_\varepsilon(y \circ x^{-1})\, Z\varphi(x)\, u(y)\, \mathrm{d}x\, \mathrm{d}y \\ &= -\int u(y) \int \Phi_\varepsilon(x \circ y^{-1})\, Z\varphi(x)\, \mathrm{d}x\, \mathrm{d}y \\ &= -\int u\,(Z\varphi)_\varepsilon = -\int u\, Z(\varphi_\varepsilon) = \int \varphi_\varepsilon\, Zu = \int \varphi\, (Zu)_\varepsilon. \end{aligned}$$

This completes the proof. □

Proof of Theorem 8. Let $u \in L_{\mathrm{loc}}(\Omega)$. Then $u_\varepsilon \to u$ in $L_{\mathrm{loc}}(\Omega)$, as $\varepsilon \to 0^+$. It is easy to see that for every $K \subset\subset \Omega$ we also have $M_r u_\varepsilon \to M_r u$ as $\varepsilon \to 0^+$, uniformly on K, for sufficiently small $r > 0$. Moreover, if $f \in L_{\mathrm{loc}}(\Omega)$ and $K \subset\subset \Omega$, then (for sufficiently small $r > 0$) $N_r f(x)$ is well-posed for a.e. $x \in K$ and we have

$$\|N_r f\|_{L^1(K)} \leqq \|H_r\|_{L^1(\mathbb{R}^N)} \|f\|_{L^1(K_r)}$$

for a suitable $K_r \subset\subset \Omega$. As a consequence

$$N_r f_\varepsilon \to N_r f \qquad \text{in } L^1(K), \text{ as } \varepsilon \to 0^+.$$

Now, if $f = \mathcal{L}u \in L_{\mathrm{loc}}(\Omega)$, then $\mathcal{L}(u_\varepsilon) = (\mathcal{L}u)_\varepsilon$ (see Remark 9) and (2) becomes

$$u_\varepsilon = M_r u_\varepsilon - N_r f_\varepsilon.$$

We can thus obtain (6), letting ε go to zero.

In order to prove (7), we now use the properties of Lemma 1. Let $K \subset\subset \Omega$ and let $r > 0$ be sufficiently small. From (6) it follows that $\mathcal{L}_r u = \widetilde{N}_r(\mathcal{L}u)$ a.e. in K. Hence

$$\begin{aligned}\|\mathcal{L}_r u - \mathcal{L}u\|_{L^1(K)} &= \int_K \left| \int \widetilde{H}_r(x^{-1} \circ y)\mathcal{L}u(y)\,\mathrm{d}y - \mathcal{L}u(x) \right| \mathrm{d}x \\ &= \int_K \left| \int \widetilde{H}_1(z)(\mathcal{L}u(x \circ \delta_r z) - \mathcal{L}u(x))\,\mathrm{d}z \right| \mathrm{d}x \\ &\leqq \int_{B_d(0,1)} \widetilde{H}_1(z) \int_K |\mathcal{L}u(x \circ \delta_r z) - \mathcal{L}u(x)|\,\mathrm{d}x\,\mathrm{d}z\end{aligned}$$

and (7) easily follows being $\mathcal{L}u$ locally integrable. Finally (8) is an immediate consequence of (7) and of the hypoellipticity of $\mathcal{L}$. $\square$

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Francesco Uguzzoni
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
I-40126 Bologna
Italy
uguzzoni@dm.unibo.it