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## Some existence results for the Webster scalar curvature problem in presence of symmetry

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**Abstract.** We prove some existence results for the Webster scalar curvature problem on the Heisenberg group and on the unit sphere of  $\mathbb{C}^{n+1}$ , under the assumption of some natural symmetries of the prescribed curvatures. We use variational and perturbation techniques.

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**Key words.** Webster curvature – Heisenberg group – subelliptic equations

### 1. Introduction

In this paper we prove some existence results for the equation

$$-\Delta_{\mathbb{H}^n} u(\xi) = K(\xi)u(\xi)^{\frac{Q+2}{Q-2}}, \quad \xi \in \mathbb{H}^n, \quad (1.1)$$

where  $\Delta_{\mathbb{H}^n}$  is the sublaplacian on the Heisenberg group  $\mathbb{H}^n$  and  $Q = 2n + 2$  is the homogeneous dimension of  $\mathbb{H}^n$ . Our results provide existence of solutions for the Webster scalar curvature problem on  $\mathbb{H}^n$  and on the unit sphere  $\mathbb{S}^{2n+1}$  of  $\mathbb{C}^{n+1}$ , under suitable assumption on the prescribed curvatures. This problem is the CR counterpart of the classical Nirenberg problem. In this paper we shall mainly assume that the prescribed curvature  $K$  has a natural symmetry, namely a cylindrical-type symmetry. Our main results are contained in Theorems 2.1, 2.3, 2.5, and 2.8 below.

We remark that one of the main features of the above equation (1.1) is a lack of compactness due both to the criticality of the exponent  $(Q + 2)/(Q - 2)$  and to the unboundedness of the domain. Non-existence results for (1.1) can be obtained using the Pohozaev-type identities of [12] under certain conditions on  $K$ . In particular it turns out that a positive solution  $u$  to (1.1) in the Sobolev space  $S_0^1(\mathbb{H}^n)$  (with the notation of Section 2) satisfies the following identity

$$\int_{\mathbb{H}^n} \langle (z, 2t), \nabla K(z, t) \rangle u(z, t)^{\frac{2Q}{Q-2}} dz dt = 0$$

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provided the integral is convergent and  $K$  is bounded and smooth enough. This implies that there are no such solutions if  $\langle (z, 2t), \nabla K(z, t) \rangle$  does not change sign in  $\mathbb{H}^n$  and  $K$  is not constant.

The link between equation (1.1) and the Webster scalar curvature problem on the sphere is briefly discussed below. Let us denote by  $\theta_0$  the standard contact form of the CR manifold  $\mathbb{S}^{2n+1}$ . Given a smooth function  $\bar{K}$  on  $\mathbb{S}^{2n+1}$ , the Webster scalar curvature problem on  $\mathbb{S}^{2n+1}$  consists in finding a contact form  $\theta$  conformal to  $\theta_0$  such that the corresponding Webster scalar curvature is  $\bar{K}$  (for the definition of the Webster scalar curvature, see [22]). This problem is equivalent to solve the semilinear equation

$$b_n \Delta_{\theta_0} v(\zeta) + \bar{K}_0 v(\zeta) = \bar{K}(\zeta) v(\zeta)^{b_n-1}, \quad \zeta \in \mathbb{S}^{2n+1}, \tag{1.2}$$

where  $b_n = 2 + \frac{2}{n}$ ,  $\Delta_{\theta_0}$  is the sublaplacian on  $(\mathbb{S}^{2n+1}, \theta_0)$  and  $\bar{K}_0 = \frac{n(n+1)}{2}$  is the Webster scalar curvature of  $(\mathbb{S}^{2n+1}, \theta_0)$ . If  $v$  is a positive solution to (1.2), then  $(\mathbb{S}^{2n+1}, \theta = v^{\frac{2}{n}} \theta_0)$  has Webster scalar curvature  $\bar{K}$ . Using the CR equivalence  $F$  (given by the Cayley transform, see definition (2.6) below) between  $\mathbb{S}^{2n+1}$  minus a point and  $\mathbb{H}^n$ , equation (1.2) is equivalent to (1.1) with  $K = \bar{K} \circ F^{-1}$ , up to an uninfuent constant. We refer to [14] for a more detailed presentation of the problem.

Indeed in the papers [14–16], Jerison and Lee extensively studied the Yamabe problem on CR manifolds (see also the recent papers [10, 11]). On the contrary, to the authors’ knowledge, very few results have been established on the Webster scalar curvature problem. In a recent paper [20] by Malchiodi and one of the authors, a new result is obtained in the perturbative case, i.e., when  $K$  is assumed to be a small perturbation of a constant (see the papers [3, 8], for analogous results concerning the Riemannian context).

The aim of this paper is to begin to study a case analogous to the radial one in the Riemannian setting. The natural counterpart in our context seems to be that of cylindrical curvatures

$$K(z, t) = K(|z|, t)$$

(see Section 2 for all the notation) and not that of “radial” ones  $K = K(\rho)$ . Indeed cylindrical curvatures  $K$  on  $\mathbb{H}^n$  correspond on  $\mathbb{S}^{2n+1}$  to curvatures  $\bar{K}$  depending only on the last complex variable of  $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ ,

$$\bar{K}(\zeta_1, \dots, \zeta_{n+1}) = \bar{K}(\zeta_{n+1}),$$

in analogy with the Riemannian case where radial curvatures  $R$  on  $\mathbb{R}^N$  correspond to curvatures  $\bar{R}(x_1, \dots, x_{N+1}) = \bar{R}(x_{N+1})$  in  $\mathbb{S}^N \subseteq \mathbb{R}^{N+1}$ , via the stereographic projection.

However, the cylindrical case presents higher difficulties with respect to the radial Riemannian case. Indeed, when  $K$  is cylindrical, one can reduce equation (1.1) to a two variables PDE, but not to an ODE as in the radial case. Nevertheless we are able to adapt a technique by Bianchi and Egnell [6] in order to obtain our first existence result Theorem 2.1. This technique consists in a minimization on a space of cylindrically symmetric functions and is based on a concentration-compactness

lemma that can be proved by just adapting the classical result by P.L. Lions [17, 18] holding in the euclidean context.

We then deal with the perturbative case, obtaining some results via the abstract Ambrosetti–Badiale finite dimensional reduction method [1, 2] (see Theorems 2.3, 2.5, and 2.8). Using this method we are able to prove, in the CR setting, some results analogous to those found in [3] in the Riemannian setting. More precisely in [3], Section 4, the radial symmetry allows to reduce the perturbation problem to the study of critical points of a one variable function, thus obtaining more precise and neat results than for the nonradial case. Similarly in our setting the cylindrical symmetry leads us to treat a two variables problem and to find results like Theorems 2.3 and 2.8, which have no counterpart in [20], where the Webster curvature problem is treated without requiring any symmetry, and like Theorem 2.5, which requires assumptions of the type of [20] only on the function  $K$  restricted to the axis  $\{z = 0\}$ . For other results related to the Riemannian case in presence of symmetry we refer to [4–6, 13].

We finally remark that some other results for equation (1.1) on the Heisenberg group have been obtained in the papers [7, 19, 21]. However, our hypotheses on  $K$  are very different from the ones in such papers where  $K$  is assumed to satisfy suitable decaying conditions at infinity. In particular, the results of [7] require an estimate of the type  $K_1(\rho)\Delta_{\mathbb{H}^n}\rho \leq K \leq K_2(\rho)\Delta_{\mathbb{H}^n}\rho$  ( $\rho$  is the homogeneous norm on  $\mathbb{H}^n$  defined in (2.1) below) involving the degenerate term  $\Delta_{\mathbb{H}^n}\rho$ , which allows to “radialize” the problem and to apply ODE methods.

## 2. Notation and main results

Denoting by  $\xi = (z, t) = (x + iy, t) \equiv (x, y, t)$  the points of  $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ , let us recall that the group law on the Heisenberg group is

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2x' \cdot y - 2x \cdot y')$$

where  $\cdot$  denotes the usual inner product in  $\mathbb{R}^n$ . Let us denote by  $\tau_\xi(\xi') = \xi \circ \xi'$  the left translations, by  $\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)$ ,  $\lambda > 0$  the natural dilations, by  $Q = 2n + 2$  the homogeneous dimension, and by

$$\rho(\xi) = (|z|^4 + t^2)^{1/4} \tag{2.1}$$

the homogeneous norm on  $\mathbb{H}^n$ . The Lie algebra of the left-invariant vector fields on  $\mathbb{H}^n$  is generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \dots, n.$$

The sub-elliptic gradient on  $\mathbb{H}^n$  is given by  $\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$  and the Kohn Laplacian on  $\mathbb{H}^n$  is the degenerate-elliptic PDO

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^n (X_j^2 + Y_j^2).$$

We will say that a function  $f : \mathbb{H}^n \rightarrow \mathbb{R}$  is continuous on  $\overline{\mathbb{H}^n}$  if it is continuous and there exists  $\lim_{(z,t) \rightarrow \infty} f(z, t) \in \mathbb{R}$ . In this case we will denote by  $f(\infty)$  such a limit.

Let  $K$  be a continuous function on  $\overline{\mathbb{H}^n}$ . We shall always suppose that  $K$  has cylindrical symmetry, i.e.  $K(z, t) = \tilde{K}(|z|, t)$ , and that  $\tilde{K}$  is locally Hölder continuous in  $]0, \infty[ \times \mathbb{R}$ . Let us consider the following equation on  $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}^n} u = K u^{Q^*-1}, \quad u > 0 \quad \text{in } \mathbb{H}^n, \tag{P}$$

where  $Q^* = \frac{2Q}{Q-2}$ . We will work in the space of cylindrically symmetric functions of the Folland–Stein Sobolev space  $S_0^1(\mathbb{H}^n)$ , namely in

$$S_{\text{cyl}}^1(\mathbb{H}^n) = \{u \in S_0^1(\mathbb{H}^n) : u(z, t) = u(|z|, t)\},$$

where  $S_0^1(\mathbb{H}^n)$  is defined as the completion of  $C_0^\infty(\mathbb{H}^n)$  with respect to the norm

$$\|u\|_{S_0^1(\mathbb{H}^n)}^2 = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dz dt.$$

Let us remark that  $Q^*$  is the critical exponent for the embedding  $S_0^1(\mathbb{H}^n) \hookrightarrow L^{Q^*}(\mathbb{H}^n)$ . Choosing suitable regularization functions, it is not difficult to recognize that we have

$$S_0^1(\mathbb{H}^n) = \left\{ u \in L^{Q^*}(\mathbb{H}^n) : \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dz dt < \infty \right\}$$

and that  $S_{\text{cyl}}^1(\mathbb{H}^n)$  is equal to the closure in  $S_0^1(\mathbb{H}^n)$  of the set of cylindrically symmetric  $C_0^\infty$  functions. Let us also observe that  $S_{\text{cyl}}^1(\mathbb{H}^n)$  is a Hilbert space endowed with the scalar product  $(u, v) = \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v dz dt$ . Let us denote by  $S$  the best constant in the Sobolev-type inequality (see [15])

$$S \|v\|_{Q^*}^2 \leq \|v\|_{S_0^1(\mathbb{H}^n)}^2, \quad \forall v \in S_0^1(\mathbb{H}^n). \tag{2.2}$$

It is known (see [15]) that all the positive cylindrically symmetric solutions in  $S_0^1(\mathbb{H}^n)$  to the problem

$$-\Delta_{\mathbb{H}^n} U = S \frac{Q}{Q-2} U^{Q^*-1}$$

are of the form

$$U_{\mu,s}(z, t) = c_n \mu^{-\frac{Q-2}{2}} U_0\left(\frac{r}{\mu}, \frac{t-s}{\mu^2}\right) \tag{2.3}$$

(for  $\mu > 0$  and  $s \in \mathbb{R}$ ) where  $r = |z|$ ,

$$U_0(r, t) = \left( \frac{1}{t^2 + (1+r^2)^2} \right)^{\frac{Q-2}{4}}$$

and  $c_n$  is a positive constant to be chosen in such a way that  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} U_{\mu,s}|^2 = 1$ . By solutions to problem (P) we mean weak solutions in the sense of  $S_0^1(\mathbb{H}^n)$ .

On the other hand, under our hypotheses solutions in the  $S^1_{\text{cyl}}(\mathbb{H}^n)$  sense are also solutions in the  $S^1_0(\mathbb{H}^n)$  sense, as is shown in Lemma A.2 in the Appendix.

Our first result is the counterpart in the Heisenberg context of a result of Bianchi and Egnell [6] about radial solutions of the corresponding problem for the Laplacian on the euclidean space.

**Theorem 2.1.** *Assume that  $K$  is a continuous cylindrically symmetric function on  $\mathbb{H}^n$  such that there exists  $K(\infty) = \lim_{(z,t) \rightarrow \infty} K(z, t) \in \mathbb{R}$ ,  $K$  is positive somewhere and*

$$\sup_{t \in \mathbb{R}} K(0, t) \leq K(\infty). \tag{2.4}$$

*If either  $K(\infty) \leq 0$  or there exist  $\mu > 0$  and  $s \in \mathbb{R}$  such that*

$$\int_{\mathbb{H}^n} (K(z, t) - K(\infty)) U_{\mu,s}^{Q*} dz dt \geq 0, \tag{2.5}$$

*then problem  $(\mathcal{P})$  has a cylindrically symmetric solution.*

*Remark 2.2.* The solution  $u$  found in Theorem 2.1 above, satisfies the decay condition  $u = O(\rho^{2-Q})$  at infinity (see e.g. [20, Proposition 2]). Moreover, by means of standard regularization techniques based on the results of Folland and Stein [9], one can prove that  $u$  is smooth if  $K$  is smooth. We finally remark that Theorem 2.1 gives also an existence result for the Webster scalar curvature problem on the sphere  $\mathbb{S}^{2n+1}$ , by means of the CR equivalence  $F : \mathbb{S}^{2n+1} \setminus \{(0, \dots, 0, -1)\} \rightarrow \mathbb{H}^n$ ,

$$F(\zeta_1, \dots, \zeta_{n+1}) = \left( \frac{\zeta_1}{1 + \zeta_{n+1}}, \dots, \frac{\zeta_n}{1 + \zeta_{n+1}}, \operatorname{Re} \left( i \frac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}} \right) \right). \tag{2.6}$$

We also remark that the set  $\{(0, t) | t \in \mathbb{R}\} \subseteq \mathbb{H}^n$  (i.e. the center of the Heisenberg group) corresponds via  $F$  to the circle  $\{(0, w) \in \mathbb{C}^n \times \mathbb{C} | w \in \mathbb{S}^1\} \subseteq \mathbb{S}^{2n+1}$ . Hence condition (2.4) is equivalent to

$$\max_{w \in \mathbb{S}^1} \bar{K}(0, w) \leq \bar{K}(0, -1)$$

(where  $\bar{K} = K \circ F$  is the prescribed curvature on  $\mathbb{S}^{2n+1}$ ).

Theorem 2.1 is proved by a minimization technique that makes use of a concentration compactness argument. Since the solution is found as a constrained minimum, it is not possible to obtain in the same way an analogous result with inequalities (2.4) and (2.5) in the opposite sense.

In the second part of the paper, we shall deal with the case in which  $K$  is close to a constant, namely  $K(z, t) = 1 + \varepsilon k(z, t)$ . We will consider the perturbation problem

$$-\Delta_{\mathbb{H}^n} u = (1 + \varepsilon k) u^{Q^*-1}, \quad u > 0 \quad \text{in } \mathbb{H}^n, \tag{\mathcal{P}_\varepsilon}$$

where  $\varepsilon$  is a small parameter and  $k$  is a bounded cylindrically symmetric function on  $\mathbb{H}^n$ . Following the Ambrosetti and Badiale [1,2] finite dimensional reduction method we are able to prove some perturbative existence results, which in most cases require weaker assumptions than Theorem 2.1. Our first perturbation result is the counterpart of [3], Theorem 4.5.

**Theorem 2.3.** Assume that  $k$  is a continuous cylindrically symmetric function on  $\mathbb{H}^n$  with  $k(\infty) := \lim_{(z,t) \rightarrow \infty} k(z, t) \in \mathbb{R}$  and that there exist  $\mu > 0$  and  $s \in \mathbb{R}$  such that either

$$\int_{\mathbb{H}^n} \left( k(z, t) - \sup_{\sigma \in \mathbb{R}} k(0, \sigma) \right) U_{\mu,s}^{Q^*} dz dt > 0 \tag{2.7}$$

or

$$\int_{\mathbb{H}^n} \left( k(z, t) - \inf_{\sigma \in \mathbb{R}} k(0, \sigma) \right) U_{\mu,s}^{Q^*} dz dt < 0. \tag{2.8}$$

Then problem  $(\mathcal{P}_\varepsilon)$  has a cylindrically symmetric solution for  $|\varepsilon|$  sufficiently small.

*Remark 2.4.* 1. In fact, from the proof it will be clear that we do not need to assume that the limit of  $k$  at  $\infty$  exists, if we assume instead that either

$$\int_{\mathbb{H}^n} \left( k(z, t) - \limsup_{(\mu,s) \rightarrow \infty} k(\mu, s) \right) U_{\mu,s}^{Q^*} dz dt > 0$$

or

$$\int_{\mathbb{H}^n} \left( k(z, t) - \liminf_{(\mu,s) \rightarrow \infty} k(\mu, s) \right) U_{\mu,s}^{Q^*} dz dt < 0.$$

2. Note the presence of the strict inequality in (2.7) and (2.8) which is due to technical reasons.
3. Assumption (2.7) is weaker than assumptions (2.4) and (2.5), with strict inequality in at least one of them. The case covered by (2.8) has no counterpart in Theorem 2.1. Actually in such a case the solutions we will find are not constrained minima like the ones found in Theorem 2.1.

Using the perturbation method, it is also possible to find some other results, requiring assumptions on the behavior of  $k$  on the axis  $\{z = 0\}$  instead of integral assumptions of the type (2.7) and (2.8). The following result is analogous for the Heisenberg group of [3], Theorem 4.4.

**Theorem 2.5.** Assume that  $k$  is a cylindrically symmetric function such that  $\bar{k} = k \circ F$  is a smooth function on  $\mathbb{S}^{2n+1}$ , and that there exists a point  $(0, \bar{s}) \in \mathbb{H}^n$  such that  $k(0, \bar{s}) = \max_{\sigma} k(0, \sigma)$  and

$$\Delta_{x,y} k(0, \bar{s}) > 0$$

where  $\Delta_{x,y} k = \sum_{i=1}^n \left[ \frac{\partial^2 k}{\partial x_i^2} + \frac{\partial^2 k}{\partial y_i^2} \right]$ . Then problem  $(\mathcal{P}_\varepsilon)$  has a cylindrically symmetric solution for  $|\varepsilon|$  sufficiently small.

*Remark 2.6.* It is also possible to find solutions under the assumption that there exists a point  $(0, \bar{s}) \in \mathbb{H}^n$  such that  $k(0, \bar{s}) = \min_{\sigma} k(0, \sigma)$  and  $\Delta_{x,y} k(0, \bar{s}) < 0$ .

*Remark 2.7.* Let us remark that the assumption that  $k$  comes from a regular function on the sphere through the Cayley transform implies that  $k$  has finite limit at  $\infty$  and that  $|k|, |\nabla k|, |\nabla^2 k|$ , and  $|t^2 \partial_t^2 k|$  are bounded.

Our last result is inspired by [4], Theorem 5.1(a).

**Theorem 2.8.** Assume that  $k$  is a cylindrically symmetric continuous function such that there exists  $\lim_{(z,t) \rightarrow \infty} k(z, t)$  and  $k(0, t) \equiv k(\infty)$  for any  $t \in \mathbb{R}$ . Then problem  $(\mathcal{P}_\varepsilon)$  has a cylindrically symmetric solution for  $|\varepsilon|$  sufficiently small.

### 3. Proof of Theorem 2.1

In this section we shall prove Theorem 2.1 by finding a solution of  $(\mathcal{P})$  as a minimizer on the constraint

$$M = \left\{ f \in S_{\text{cyl}}^1(\mathbb{H}^n) : \|f\|_{S_0^1(\mathbb{H}^n)} = 1, \int_{\mathbb{H}^n} K |f|^{Q^*} > 0 \right\}.$$

Note that the assumption that  $K$  is positive somewhere ensures that  $M$  is nonempty. Let us consider the minimum problem

$$\gamma = \inf_{u \in M} \mathcal{F}_K(u) \tag{I_K}$$

where  $\mathcal{F}_K : M \rightarrow \mathbb{R}$  is defined by

$$\mathcal{F}_K(u) = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-\frac{Q}{Q^*}} = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-\frac{Q-2}{2}}.$$

Suppose that  $u \in M$  attains the infimum in  $(I_K)$ . Then  $u$  is a critical point of  $\mathcal{F}_K$  constrained on  $M$ . Then there exists a Lagrange multiplier  $\lambda \in \mathbb{R}$  such that

$$(\mathcal{F}'_K(u), v) = \lambda(\mathcal{G}'(u), v) \quad \forall v \in S_{\text{cyl}}^1(\mathbb{H}^n)$$

where  $\mathcal{G}(u) = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2$ , namely for any  $v \in S_{\text{cyl}}^1(\mathbb{H}^n)$

$$-Q \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-\frac{Q}{2}} \int_{\mathbb{H}^n} K |u|^{Q^*-2} u v = 2\lambda \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v. \tag{3.1}$$

Testing (3.1) with  $v = u$ , we can easily compute the value of  $\lambda$ , thus finding  $\lambda = -\frac{Q}{2} \mathcal{F}_K(u)$ . Hence (3.1) can be written in the form

$$\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-1} \int_{\mathbb{H}^n} K |u|^{Q^*-2} u v \quad \forall v \in S_{\text{cyl}}^1(\mathbb{H}^n)$$

which is equivalent to the fact that  $u$  is a weak solution in the  $S_{\text{cyl}}^1(\mathbb{H}^n)$  sense of the equation

$$-\Delta_{\mathbb{H}^n} u = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-1} K |u|^{Q^*-2} u.$$

Without loss of generality, we can assume  $u \geq 0$ ; otherwise one takes  $|u|$  after noticing that if  $u \in M$ , then  $|u| \in M$  and  $\mathcal{F}_K(|u|) = \mathcal{F}_K(u)$ . From Lemma A.2, [20, Proposition 2] and the Harnack inequality proved in [14, Proposition 5.12] it follows that  $u$  is strictly positive and hence  $u$  satisfies

$$-\Delta_{\mathbb{H}^n} u = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{-1} K u^{Q^*-1}.$$

Let us consider the rescaled function

$$\bar{u} = \left( \int_{\mathbb{H}^n} K |u|^{Q^*} \right)^{\frac{1}{2-Q^*}} u.$$

It is easy to check that  $\bar{u} > 0$  satisfies

$$-\Delta_{\mathbb{H}^n} \bar{u} = K \bar{u}^{Q^*-1},$$

hence  $\bar{u}$  is a solution to problem  $(\mathcal{P})$ . Therefore, the above argument shows that the existence of minimizers of  $(\mathcal{I}_K)$  provides a weak solution in  $S^1_{\text{cyl}}(\mathbb{H}^n)$  to  $(\mathcal{P})$  (and hence in  $S^1_0(\mathbb{H}^n)$ , thanks to Lemma A.2).

A sufficient condition for the existence of minimizers of  $(\mathcal{I}_K)$  is given in the following lemma.

**Lemma 3.1.** *Let  $\delta = \sup\{K^+(0, t) : t \in \mathbb{R}\}$  and assume that there exists  $V \in M$  such that*

$$\mathcal{F}_K(V) \leq \inf_{v \in M} \mathcal{F}_\delta(v) = \mathcal{F}_\delta(U_{\mu,s}) = \delta^{-\frac{Q-2}{2}} S^{\frac{Q}{2}}.$$

*If  $\delta = 0$  it is enough to assume  $M \neq \emptyset$  (i.e.,  $K$  positive somewhere). Then the infimum in  $(\mathcal{I}_K)$  is attained.*

*Proof.* Let  $(u_m)_m$  be a minimizing sequence for  $(\mathcal{I}_K)$ , i.e.,

$$\left( \int_{\mathbb{H}^n} K |u_m|^{Q^*} \right)^{\frac{2-Q}{2}} \xrightarrow{m \rightarrow +\infty} \gamma, \quad \int_{\mathbb{H}^n} K |u_m|^{Q^*} > 0, \quad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u_m|^2 = 1.$$

Up to a subsequence, we can assume that  $u_m \rightharpoonup u$  in  $S^1_{\text{cyl}}(\mathbb{H}^n)$  (and in  $S^1_0(\mathbb{H}^n)$  and  $L^{Q^*}(\mathbb{H}^n)$ ). In view of Lemma A.3 we have that  $u_m$  converges to  $u$  in  $L^q(C)$  for any set  $C$  of the type  $\{z : 0 < c_1 \leq |z| \leq c_2\} \times [-c_3, c_3]$  (and hence on any compact set away from the axis  $\{z = 0\}$ ) and for any  $1 \leq q < +\infty$ . From a suitable  $\mathbb{H}^n$  version of the concentration-compactness principle of P.L. Lions (see Theorem A.4) there exist some nonnegative finite regular Borel measures  $\nu, \nu_K, \mu$  on  $\overline{\mathbb{H}^n}$ , an at most countable index set  $J$ , a sequence  $(z_j, t_j) \in \mathbb{H}^n, \nu^j, \nu^\infty \in (0, \infty)$  such that (passing to a subsequence) the following convergences in the weak sense of measures hold

$$|u_m|^{Q^*} dx \xrightarrow{\mathcal{M}} \nu = |u|^{Q^*} + \sum_{j \in J} \nu^j \delta_{(z^j, t^j)} + \nu^\infty \delta_\infty \tag{3.2}$$

$$K |u_m|^{Q^*} dx \xrightarrow{\mathcal{M}} \nu_K = K |u|^{Q^*} + \sum_{j \in J} K(z^j, t^j) \nu^j \delta_{(z^j, t^j)} + K(\infty) \nu^\infty \delta_\infty \tag{3.3}$$

$$|\nabla_{\mathbb{H}^n} u_m|^2 dx \xrightarrow{\mathcal{M}} \mu \geq |\nabla_{\mathbb{H}^n} u|^2 + \sum_j S(\nu^j)^{\frac{2}{Q^*}} \delta_{(z^j, t^j)} + S(\nu^\infty)^{\frac{2}{Q^*}} \delta_\infty. \tag{3.4}$$

Moreover if  $(z^j, t^j)$  does not belong to the axis  $\{z = 0\}$ , i.e., if  $z^j \neq 0$ , then  $(z^j, t^j)$  is in some compact set of the type  $\{z : 0 < c_1 \leq |z| \leq c_2\} \times [-c_3, c_3]$ . Hence,

in view of Lemma A.3 in the Appendix,  $\int_C |u_m - u|^{Q^*} \rightarrow 0$  and consequently  $\int_C |u_m|^{Q^*} \rightarrow \int_C |u|^{Q^*}$ . Take some continuous nonnegative function  $\varphi$  with compact support and satisfying  $\varphi(z^j, t^j) \neq 0$ . For such a  $\varphi$  we have that  $\int_C \varphi |u_m|^{Q^*} \rightarrow \int_C \varphi |u|^{Q^*}$  and hence

$$0 = v^j \varphi(z^j, t^j).$$

Therefore it must be  $v^j = 0$ , so that we can assume, without loss of generality, that  $z^j = 0$  for any  $j \in J$ . We claim that  $u \in M$ . Let us distinguish two cases.

*Case  $\delta > 0$ .* We can assume  $\gamma < \delta^{-\frac{Q-2}{2}} S^{\frac{Q}{2}}$ . Otherwise if  $\gamma \geq \delta^{-\frac{Q-2}{2}} S^{\frac{Q}{2}}$  we have  $\mathcal{F}_k(V) \leq \delta^{-\frac{Q-2}{2}} S^{\frac{Q}{2}} \leq \gamma$  and hence  $V$  is a minimizer. Let  $\gamma < \delta^{-\frac{Q-2}{2}} S^{\frac{Q}{2}}$ . Since  $u_m$  is a minimizing sequence, from (3.3) and (3.4) we have

$$\delta S^{-\frac{Q^*}{2}} < \gamma^{-\frac{2}{Q-2}} = v_K(\overline{\mathbb{H}^n}) = \int_{\mathbb{H}^n} K |u|^{Q^*} + \sum_j K(0, t^j) v^j + K(\infty) v^\infty, \quad (3.5)$$

$$1 \geq \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 + \sum_j S(v^j)^{\frac{2}{Q^*}} + S(v^\infty)^{\frac{2}{Q^*}}. \quad (3.6)$$

We claim that  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 = 1$ . By a way of contradiction, assume that  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 = \rho$  with  $\rho \in [0, 1[$ . From (3.6) we have

$$S^{-\frac{Q^*}{2}} \geq (1 - \rho)^{-\frac{Q^*}{2}} \left[ \sum_j (v^j)^{\frac{2}{Q^*}} + (v^\infty)^{\frac{2}{Q^*}} \right]^{\frac{Q^*}{2}}. \quad (3.7)$$

(3.5) and (3.7) imply that

$$\gamma^{-\frac{2}{Q-2}} > \delta (1 - \rho)^{-\frac{Q^*}{2}} \left[ \sum_j (v^j)^{\frac{2}{Q^*}} + (v^\infty)^{\frac{2}{Q^*}} \right]^{\frac{Q^*}{2}}$$

hence  $\gamma^{-\frac{2}{Q-2}} > \delta (1 - \rho)^{-\frac{Q^*}{2}} (\sum_j v^j + v^\infty)$ . Therefore

$$\delta \left( \sum_j v^j + v^\infty \right) < (1 - \rho)^{\frac{Q^*}{2}} \gamma^{-\frac{2}{Q-2}}. \quad (3.8)$$

(3.5), (3.8), and the definition of  $\delta$  imply that

$$\begin{aligned} \gamma^{-\frac{2}{Q-2}} &= \int_{\mathbb{H}^n} K |u|^{Q^*} + \sum_j K(0, t^j) v^j + K(\infty) v^\infty \\ &< \int_{\mathbb{H}^n} K |u|^{Q^*} + (1 - \rho)^{\frac{Q^*}{2}} \gamma^{-\frac{2}{Q-2}}. \end{aligned} \quad (3.9)$$

Hence  $\int_{\mathbb{H}^n} K |u|^{Q^*} > 0$ . Set  $\bar{u} = u \rho^{-1/2} \in M$ ; we have

$$\frac{\int_{\mathbb{H}^n} K |u|^{Q^*}}{\rho^{Q^*/2}} = \int_{\mathbb{H}^n} K |\bar{u}|^{Q^*} < \gamma^{-\frac{2}{Q-2}}. \quad (3.10)$$

From (3.9) and (3.10) we obtain that

$$\gamma^{-\frac{2}{Q-2}} < \gamma^{-\frac{2}{Q-2}} \left[ \rho^{\frac{Q^*}{2}} + (1 - \rho)^{\frac{Q^*}{2}} \right]$$

namely  $1 < \rho^{Q^*/2} + (1 - \rho)^{Q^*/2} \leq \rho + (1 - \rho) = 1$  which is not possible.

Case  $\delta = 0$ . In this case  $K(\infty) \leq 0$  and  $K(0, t) \leq 0$  for any  $t \in \mathbb{R}$  and thus (3.5) and (3.10) imply

$$0 < \gamma^{-\frac{2}{Q-2}} \leq \int_{\mathbb{H}^n} K|u|^{Q^*} \leq \gamma^{-\frac{2}{Q-2}} \left( \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \right)^{\frac{Q^*}{2}}.$$

Hence  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \geq 1$ . Since from (3.6) we have  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \leq 1$ , we can conclude  $\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 = 1$ .

The claim that  $u \in M$  is thereby proved. From (3.6) we obtain also that it must be  $v^j = v^\infty = 0$ . Hence, in view of (3.5), we deduce that the minimum in  $(\mathcal{I}_K)$  is attained by  $u$ . □

*Proof of Theorem 2.1 completed.* Let  $\delta$  be as in Lemma 3.1. If  $K(\infty) \leq 0$  then  $\delta = 0$ . If  $K(\infty) > 0$  then  $\delta = K(\infty) > 0$  and from (2.5) we have

$$\int_{\mathbb{H}^n} KU_{\mu,s}^{Q^*} \geq \delta \int_{\mathbb{H}^n} U_{\mu,s}^{Q^*}$$

and hence

$$\left( \int_{\mathbb{H}^n} KU_{\mu,s}^{Q^*} \right)^{\frac{2-Q}{2}} \leq \left( \int_{\mathbb{H}^n} \delta U_{\mu,s}^{Q^*} \right)^{\frac{2-Q}{2}}.$$

Lemma 3.1 (with  $V = U_{\mu,s}$ ) allows us to conclude. □

### 4. The perturbation problem

In this section, we focus our attention on the case in which  $K$  is close to a constant, namely  $K(z, t) = 1 + \varepsilon k(z, t)$ . We deal with the perturbation problem

$$-\Delta_{\mathbb{H}^n} u = (1 + \varepsilon k) u^{Q^*-1}, \quad u > 0 \quad \text{in } \mathbb{H}^n, \quad (\mathcal{P}_\varepsilon)$$

where  $\varepsilon$  is a small real perturbation parameter and  $k$  is a bounded cylindrically symmetric function on  $\mathbb{H}^n$ . Our approach is based on the finite dimensional reduction method developed by Ambrosetti and Badiale [1,2] and recently applied by Malchiodi and one of the authors [20] to the problem of prescribing the Webster scalar curvature on the unit sphere of  $\mathbb{C}^{n+1}$ . Cylindrically symmetric solutions of  $(\mathcal{P}_\varepsilon)$  can be obtained as critical points on the space  $S_{\text{cyl}}^1(\mathbb{H}^n)$  of the functional

$$f_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_{\mathbb{H}^n} (1 + \varepsilon k) u_+^{Q^*}.$$

Indeed, if  $u$  is a nontrivial critical point of  $f_\varepsilon$ , testing  $f'_\varepsilon(u)$  with  $u_- = \max\{-u, 0\}$  we obtain that  $0 = (f'_\varepsilon(u), u_-) = -\|u_-\|_{S_0^1(\mathbb{H}^n)}^2$  and hence  $u_- = 0$ . From Lemma A.2, [20, Proposition 2] and the Harnack inequality proved in [14, Proposition 5.12] it follows that  $u > 0$ .

For  $\varepsilon = 0$ , the unperturbed functional  $f_0$  has a manifold of critical points  $Z$  given by

$$Z = \{z_{\mu,s} : \mu > 0, s \in \mathbb{R}\} \tag{4.1}$$

where  $z_{\mu,s}$  is given, up to a constant, by the function  $U_{\mu,s}$  defined in (2.3), namely

$$z_{\mu,s}(z, t) = \mu^{-\frac{Q-2}{2}} \omega\left(\frac{z}{\mu}, \frac{t-s}{\mu^2}\right)$$

where

$$\omega(z, t) = (Q - 2)^{\frac{Q-2}{2}} (t^2 + (1 + |z|^2)^2)^{-\frac{Q-2}{4}}. \tag{4.2}$$

#### 4.1. The abstract perturbation method

For the reader’s convenience, here we recall the abstract result that we will use in the sequel, for the proof of which we refer to [2, 3]. Let  $E$  be a Hilbert space and  $f_0, G \in C^2(E, \mathbb{R})$ . Let us denote by  $D^2 f_0(u) \in \mathcal{L}(E, E')$  the second Fréchet derivative of  $f_0$  at  $u$ . Through the Riesz Representation Theorem, we can identify  $D^2 f_0(u)$  with  $f''_0(u) \in \mathcal{L}(E, E)$  given by  $f''_0(u)v = \mathcal{K}(D^2 f_0(u)v)$  where  $\mathcal{K} : E' \rightarrow E, (\mathcal{K}(\varphi), \psi)_E = {}_{E'}\langle \varphi, \psi \rangle_E$ , for any  $\varphi \in E', \psi \in E$ . Suppose that  $f_0$  satisfies

- (a)  $f_0$  has a finite dimensional manifold of critical points  $Z$ ;
- (b) for all  $z \in Z, f''_0(z)$  is a Fredholm operator of index 0;
- (c) for all  $z \in Z$ , there results  $T_z Z = \ker f''_0(z)$ .

Condition (c) is in fact a nondegeneracy condition that is needed to apply the Implicit Function Theorem. The inclusion  $T_z Z \subseteq \ker f''_0(z)$  always holds owing to the criticality of  $Z$ , so that to prove (c) is enough to prove that  $\ker f''_0(z) \subseteq T_z Z$ .

Consider the perturbed functional  $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ , and denote by  $\Gamma$  the functional  $G|_Z$ . Because of assumptions (a), (b), and (c), it is possible to prove (see Lemma 4.4) that there exists, for  $|\varepsilon|$  small, a smooth function  $w_\varepsilon(z) : Z \rightarrow (T_z Z)^\perp$  such that any critical point  $\bar{z} \in Z$  of the functional

$$\Phi_\varepsilon : Z \longrightarrow \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon(z + w_\varepsilon(z))$$

gives rise to a critical point  $u_\varepsilon = \bar{z} + w_\varepsilon(\bar{z})$  of  $f_\varepsilon$ ; in other words, the perturbed manifold  $Z_\varepsilon = \{z + w_\varepsilon(z) : z \in Z\}$  is a *natural constraint* for  $f_\varepsilon$ . Moreover  $\Phi_\varepsilon$  admits an expansion of the type

$$\Phi_\varepsilon(z) = b - \varepsilon \Gamma(z) + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0 \tag{4.3}$$

where  $b = f_0(z)$  for any  $z \in Z$ .

**Theorem 4.1.** *Let  $f_0$  satisfy (a), (b), and (c) and assume that  $\Gamma$  has a proper local maximum or minimum point  $\bar{z}$ . Then for  $|\varepsilon|$  small enough, the functional  $f_\varepsilon$  has a critical point  $u_\varepsilon$  such that  $u_\varepsilon \rightarrow \bar{z}$  as  $\varepsilon \rightarrow 0$ .*

*Remark 4.2.* If  $Z_0 = \{z \in Z : \Gamma(z) = \min_Z \Gamma\}$  is compact, it is still possible to prove that  $f_\varepsilon$  has a critical point near  $Z_0$ . The set  $Z_0$  can also consist of local minimum points; the same holds for maximum points.

#### 4.2. The unperturbed problem

In order to apply the abstract result stated above, we have to prove that the unperturbed functional

$$f_0(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_{\mathbb{H}^n} u_+^{Q^*}, \quad u \in S_{\text{cyl}}^1(\mathbb{H}^n),$$

satisfies (a), (b), and (c). Condition (a) clearly holds. Indeed, as remarked above,  $f_0$  has a two dimensional manifold of critical points  $Z$ , see (4.1). Moreover, it is quite standard to prove that (b) holds. Indeed  $f_0 \in C^2(S_{\text{cyl}}^1(\mathbb{H}^n), \mathbb{R})$ ,

$$(f_0''(u)v, h) = (v, h) - (Q^* - 1) \int_{\mathbb{H}^n} u_+^{Q^*-2} v h \quad \forall u, v, h \in S_{\text{cyl}}^1(\mathbb{H}^n),$$

and for any  $z \in Z$  the operator  $f_0''(z) : S_{\text{cyl}}^1(\mathbb{H}^n) \rightarrow S_{\text{cyl}}^1(\mathbb{H}^n)$  is of the type  $I - C_z$ , where  $I$  is the identity and  $C_z$  is a compact operator and hence  $f_0''(z)$  is a Fredholm operator of index 0.

Let us now prove (c). The tangent space  $T_{z_{\mu,s}}Z$  is given by

$$T_{z_{\mu,s}}Z = \left\{ \alpha \frac{\partial z_{\lambda,t}}{\partial \lambda} \Big|_{\substack{\lambda=\mu \\ t=s}} + \beta \frac{\partial z_{\lambda,t}}{\partial t} \Big|_{\substack{\lambda=\mu \\ t=s}} : \alpha, \beta \in \mathbb{R} \right\}.$$

**Lemma 4.3.** *For any  $\mu > 0$ ,  $s \in \mathbb{R}$ , there holds*

$$T_{z_{\mu,s}}Z = \ker f_0''(z_{\mu,s}).$$

*Proof.* As remarked in the previous subsection, it is enough to prove the inclusion  $\ker f_0''(z_{\mu,s}) \subseteq T_{z_{\mu,s}}Z$ . We can assume  $\mu = 1$  and  $s = 0$ , since  $\ker f_0''(z_{\mu,s})$  is isomorphic to  $\ker f_0''(z_{1,0})$ , owing to the invariance of the problem under dilations and translations along the  $t$ -axis. If  $u \in \ker f_0''(z_{1,0})$ , we have that  $u$  is a solution in the weak sense of  $S_{\text{cyl}}^1(\mathbb{H}^n)$  of the linearized problem

$$-\Delta_{\mathbb{H}^n} u = (Q^* - 1)z_{1,0}^{Q^*-2} u \quad \text{in } S_{\text{cyl}}^1(\mathbb{H}^n).$$

Because of Lemma A.2,  $u$  solves the above equation also in  $S_0^1(\mathbb{H}^n)$ . It has been proved by Malchiodi and one of the authors [20] that any solution  $u$  in  $S_0^1(\mathbb{H}^n)$  of such equation is of the type

$$u(\xi') = \alpha \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda} \Big|_{\substack{\lambda=1 \\ \xi=0}}(\xi') + \sum_{i=1}^{2n+1} \nu_i \frac{\partial \omega_{\lambda,\xi}}{\partial \xi_i} \Big|_{\substack{\lambda=1 \\ \xi=0}}(\xi'), \quad \xi' \in \mathbb{H}^n,$$

for some coefficients  $\alpha \in \mathbb{R}$  and  $\nu = (\nu_1, \dots, \nu_{2n+1}) \in \mathbb{R}^{2n+1}$ , where, for  $\lambda \in \mathbb{R}$  and  $\xi \in \mathbb{H}^n$ ,  $\omega_{\lambda,\xi} = \lambda^{-\frac{Q-2}{2}} \omega \circ \delta_{\lambda^{-1}} \circ \tau_{\xi^{-1}}$ , namely if  $\xi' = (z', t') = (x', y', t')$ ,  $\xi = (z, t) = (x, y, t)$ , then

$$\begin{aligned}
 u(z', t') = u(x', y', t') &= \alpha \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial \lambda} \right|_{(1,0,0,0)}(x', y', t') + \beta \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial t} \right|_{(1,0,0,0)}(x', y', t') \\
 &+ \sum_{i=1}^n \gamma_i \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial x_i} \right|_{(1,0,0,0)}(x', y', t') + \sum_{i=1}^n \tau_i \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial y_i} \right|_{(1,0,0,0)}(x', y', t')
 \end{aligned} \tag{4.4}$$

for some  $\alpha, \beta \in \mathbb{R}$ ,  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ , and  $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ . We claim that the cylindrical symmetry of  $u$  implies that, for any  $i = 1, 2, \dots, n$ ,  $\gamma_i = \tau_i = 0$ . We have

$$\begin{aligned}
 \left. \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda} \right|_{\substack{\lambda=1 \\ \xi=0}}(z', t') &= -\frac{Q-2}{2} \omega(z', t') \\
 &+ (Q-2)^{Q/2} (t'^2 + (1+|z'|^2)^2)^{-\frac{Q+2}{4}} (1+|z'|^2)|z'|^2 \\
 &+ (Q-2)^{Q/2} (t'^2 + (1+|z'|^2)^2)^{-\frac{Q+2}{4}} t'^2, \\
 \left. \frac{\partial \omega_{\lambda,\xi}}{\partial \xi} \right|_{\substack{\lambda=1 \\ \xi=0}}(z', t') &= \frac{(Q-2)^{Q/2}}{(t'^2 + (1+|z'|^2)^2)^{\frac{Q+2}{4}}} \\
 &\times ((1+|z'|^2)x' - t'y', (1+|z'|^2)y' + t'x', t'/2).
 \end{aligned}$$

Therefore  $\left. \frac{\partial \omega_{\lambda,\xi}}{\partial \lambda} \right|_{(1,0)}$  and

$$\left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial t} \right|_{(1,0,0,0)}(x', y', t') = \frac{1}{2} (Q-2)^{Q/2} (t'^2 + (1+|z'|^2)^2)^{-\frac{Q+2}{4}} t'$$

are cylindrically symmetric functions. If  $u$  is cylindrical, in view of the cylindrical symmetry of  $\left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial \lambda} \right|_{(1,0)}$  and  $\left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial t} \right|_{(1,0,0,0)}$ , from (4.4) we deduce that

$$\sum_{i=1}^n \gamma_i \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial x_i} \right|_{(1,0,0,0)}(z', t') + \sum_{i=1}^n \tau_i \left. \frac{\partial \omega_{\lambda,x,y,t}}{\partial y_i} \right|_{(1,0,0,0)}(z', t')$$

must be cylindrical, hence

$$h(x', y', t') = \sum_{i=1}^n \gamma_i ((1+|z'|^2)x'_i - t'_i y'_i) + \tau_i ((1+|z'|^2)y'_i + t'_i x'_i)$$

must be cylindrical. From

$$h(\underbrace{0, \dots, 1, \dots, 0}_{x'}, \underbrace{0, \dots, 0}_{y'}, 0) = h(\underbrace{0, \dots, -1, \dots, 0}_{x'}, \underbrace{0, \dots, 0}_{y'}, 0)$$

it follows that  $2\gamma_i = -2\gamma_i$  and hence  $\gamma_i = 0$  for any  $i = 1, \dots, n$ . In the same way  $\tau_i = 0$  for any  $i = 1, \dots, n$ . The claim is thereby proved. As a consequence, we have that  $\ker f_0''(z_{1,0})$  is contained in  $T_{z_{1,0}}Z$ . □

4.3. Study of  $\Gamma$  and proof of Theorems 2.3 and 2.5

In our case, the reduced functional  $\Gamma$  is given by

$$\begin{aligned} \Gamma(\mu, s) &= \frac{1}{Q^*} \int_{\mathbb{H}^n} k(z, t) z_{\mu, s}^{\mathcal{Q}^*}(z, t) dz dt = \frac{1}{Q^*} \int_{\mathbb{H}^n} k(\mu z, \mu^2 t + s) \omega^{\mathcal{Q}^*}(z, t) dz dt \\ &= \frac{\gamma_n}{Q^*} \int_{\substack{0 < r < \infty \\ t \in \mathbb{R}}} k(\mu r, \mu^2 t + s) \omega^{\mathcal{Q}^*}(r, t) r^{2n-1} dr dt, \end{aligned}$$

where  $\gamma_n$  is the measure of the unit  $(2n - 1)$ -sphere. The function  $\Gamma$  can be extended with continuity to  $\mu = 0$  by setting

$$\Gamma(0, s) = \frac{\gamma_n}{Q^*} k(0, s) \int_{\substack{0 < r < \infty \\ t \in \mathbb{R}}} \omega^{\mathcal{Q}^*}(r, t) r^{2n-1} dr dt = b_0 k(0, s) \tag{4.5}$$

where  $b_0 = \frac{1}{Q^*} \int_{\mathbb{H}^n} \omega^{\mathcal{Q}^*}$ . By the Dominated Convergence Theorem it follows that

$$\Gamma(\infty) := \lim_{(\mu, s) \rightarrow \infty} \Gamma(\mu, s) = b_0 k(\infty). \tag{4.6}$$

Moreover, if  $k = \bar{k} \circ F^{-1}$  with  $\bar{k}$  a smooth function on the sphere  $\mathbb{S}^{2n+1}$ , we have that

$$\begin{aligned} D_\mu \Gamma(\mu, s) &= \frac{1}{Q^*} \int_{\mathbb{H}^n} [\nabla_z k(\mu z, \mu^2 t + s) \cdot z + 2\partial_t k(\mu z, \mu^2 t + s) \mu t] \omega^{\mathcal{Q}^*}(z, t) dz dt \\ D_\mu \Gamma(0, s) &= \frac{1}{Q^*} \int_{\mathbb{H}^n} [\nabla_z k(0, s) \cdot z] \omega^{\mathcal{Q}^*}(z, t) dz dt = 0 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &D_{\mu, \mu}^2 \Gamma(0, s) \\ &= \frac{1}{Q^*} \int_{\mathbb{H}^n} \left[ \sum_{i, j=1}^n \frac{\partial^2 k}{\partial x_i \partial x_j}(0, s) x_i x_j + \sum_{i, j=1}^n \frac{\partial^2 k}{\partial y_i \partial y_j}(0, s) y_i y_j + 2\partial_t k(0, s) t \right] \\ &\qquad \qquad \qquad \cdot \omega^{\mathcal{Q}^*}(x, y, t) dx dy dt \\ &= \frac{1}{2n Q^*} \Delta_{x, y} k(0, s) \int_{\mathbb{H}^n} |z|^2 \omega^{\mathcal{Q}^*}(z, t) dz dt. \end{aligned} \tag{4.8}$$

*Proof of Theorem 2.3.* Assumption (2.7) and (4.6) imply that  $\Gamma(\infty) < \Gamma(\mu, s)$  whereas assumption (2.7) and (4.5) imply that  $\sup_\sigma \Gamma(0, \sigma) < \Gamma(\mu, s)$ , hence  $\Gamma$  must have a compact set of global maximum points in the interior of the half-plane  $\{(\mu, s) : \mu > 0\}$ . From Remark 4.2 we get the conclusion. In the case of (2.8) we have  $\Gamma(\infty) > \Gamma(\mu, s)$  and  $\inf_\sigma \Gamma(0, \sigma) > \Gamma(\mu, s)$  hence  $\Gamma$  must have a compact set of global minimum points. □

*Proof of Theorem 2.5.* The assumptions of Theorem 2.5 imply, in view of (4.5), (4.6), (4.7) and (4.8), that  $\Gamma(0, \bar{s}) = \max_{\sigma} \Gamma(0, \sigma) \geq \Gamma(\infty)$  and  $D_{\mu} \Gamma(0, \bar{s}) = 0$ ,  $D_{\mu, \mu}^2 \Gamma(0, \bar{s}) > 0$  hence  $\Gamma$  must have a compact set of global maximum points in the interior of the half-plane  $\{(\mu, s) : \mu > 0\}$ . The conclusion follows from Remark 4.2.  $\square$

#### 4.4. Study of $\Phi_{\varepsilon}$

To prove Theorem 2.8, the study of the functional  $\Gamma$  is not sufficient since in this case  $\Gamma$  may be constant even if  $k$  is a non-constant function. This fact leads to a loss of information, because the first order expansion (4.3) is not enough to deduce the existence of critical points of  $\Phi_{\varepsilon}$  from the existence of critical points of  $\Gamma$ . Therefore we need to study the function  $\Phi_{\varepsilon}$  directly, which in our case is a function of the two variables  $(\mu, s) \in \mathbb{R}^+ \times \mathbb{R}$ .

For  $\mu > 0$  and  $s \in \mathbb{R}$ , let us define the map

$$\mathcal{U}_{\mu, s} : S_{\text{cyl}}^1(\mathbb{H}^n) \longrightarrow S_{\text{cyl}}^1(\mathbb{H}^n), \quad \mathcal{U}_{\mu, s}(u)(z, t) = \mu^{-\frac{Q-2}{2}} u\left(\frac{z}{\mu}, \frac{t-s}{\mu^2}\right).$$

It is easy to check that  $\|\mathcal{U}_{\mu, s}(u)\|_{S_{\text{cyl}}^1(\mathbb{H}^n)} = \|u\|_{S_{\text{cyl}}^1(\mathbb{H}^n)}$ , for any  $u \in S_{\text{cyl}}^1(\mathbb{H}^n)$ ,  $\mu > 0$ , and  $s \in \mathbb{R}$ , and that  $f_0 = f_0 \circ \mathcal{U}_{\mu, s}$ . Moreover we have  $(\mathcal{U}_{\mu, s})^{-1} = \mathcal{U}_{\mu^{-1}, -\mu^{-2}s} = (\mathcal{U}_{\mu, s})^t$  where  $(\mathcal{U}_{\mu, s})^t$  denotes the adjoint of  $\mathcal{U}_{\mu, s}$ . Differentiating the identity  $f_0 = f_0 \circ \mathcal{U}_{\mu, s}$  we observe that

$$f'_0 = (\mathcal{U}_{\mu, s})^{-1} \circ f'_0 \circ \mathcal{U}_{\mu, s}$$

and

$$f''_0(u) = (\mathcal{U}_{\mu, s})^{-1} \circ f''_0(\mathcal{U}_{\mu, s}(u)) \circ \mathcal{U}_{\mu, s}, \quad \forall u \in S_{\text{cyl}}^1(\mathbb{H}^n). \tag{4.9}$$

Clearly we have  $\mathcal{U}_{\mu, s} : T_{\omega}Z \longrightarrow T_{z_{\mu, s}}Z$  and  $\mathcal{U}_{\mu, s} : (T_{\omega}Z)^{\perp} \longrightarrow (T_{z_{\mu, s}}Z)^{\perp}$ . Because of nondegeneracy, the self adjoint Fredholm operator  $f''_0(\omega)$  maps  $S_{\text{cyl}}^1(\mathbb{H}^n)$  into  $(T_{\omega}Z)^{\perp}$  and  $f''_0(\omega) \in \mathcal{L}((T_{\omega}Z)^{\perp})$ . Moreover (4.9) implies that

$$\|f''_0(\omega)^{-1}\|_{\mathcal{L}((T_{\omega}Z)^{\perp})} = \|f''_0(z)^{-1}\|_{\mathcal{L}((T_zZ)^{\perp})} \quad \forall z \in Z. \tag{4.10}$$

**Lemma 4.4.** *Assume  $k \in L^{\infty}(\mathbb{H}^n)$ . Then there exist constants  $\varepsilon_0, C > 0$  and a smooth function*

$$w = w(\mu, s, \varepsilon) : (0, +\infty) \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \longrightarrow S_{\text{cyl}}^1(\mathbb{H}^n)$$

such that for any  $\mu > 0, s \in \mathbb{R}$ , and  $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$w(\mu, s, \varepsilon) \text{ is orthogonal to } T_{z_{\mu, s}}Z \tag{4.11}$$

$$f'_\varepsilon(z_{\mu, s} + w(\mu, s, \varepsilon)) \in T_{z_{\mu, s}}Z \tag{4.12}$$

$$\|w(\mu, s, \varepsilon)\| \leq C |\varepsilon|. \tag{4.13}$$

*Proof.* Since it will be useful in the sequel, we write the complete proof of the lemma which follows the proofs of analogous results of [2,4]. Let us define

$$H : (0, +\infty) \times \mathbb{R} \times S^1_{\text{cyl}}(\mathbb{H}^n) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow S^1_{\text{cyl}}(\mathbb{H}^n) \times \mathbb{R} \times \mathbb{R}$$

$$(\mu, s, w, \alpha_1, \alpha_2, \varepsilon) \longmapsto (f'_\varepsilon(z_{\mu,s} + w) - \alpha_1 \dot{\xi}_{\mu,s} - \alpha_2 \dot{\zeta}_{\mu,s}, (w, \dot{\xi}_{\mu,s}), (w, \dot{\zeta}_{\mu,s})),$$

where  $\dot{\xi}_{\mu,s}$  (resp.  $\dot{\zeta}_{\mu,s}$ ) denotes the normalized tangent vector  $\frac{\partial}{\partial \mu} z_{\mu,s}$  (resp.  $\frac{\partial}{\partial s} z_{\mu,s}$ ). If  $H(\mu, s, w, \alpha_1, \alpha_2, \varepsilon) = 0$  then  $w$  satisfies (4.11)–(4.12). On the other hand  $H(\mu, s, w, \alpha_1, \alpha_2, \varepsilon) = 0$  if and only if  $(w, \alpha_1, \alpha_2)$  is a fixed point for the map  $F_{\mu,s,\varepsilon}$  defined as

$$F_{\mu,s,\varepsilon}(w, \alpha_1, \alpha_2) := - \left( \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \right)^{-1} H(\mu, s, w, \alpha_1, \alpha_2, \varepsilon) + (w, \alpha_1, \alpha_2).$$

To prove the existence of  $w$  satisfying (4.11) and (4.12) it is enough to prove that  $F_{\mu,s,\varepsilon}$  is a contraction in some ball  $B_\rho(0)$ , with  $\rho = \rho(\varepsilon) > 0$  independent of  $z \in Z$ , whereas the regularity of  $w(\mu, s, \varepsilon)$  follows from the Implicit Function Theorem. We have

$$\begin{aligned} & \left( \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \right) (w, \beta_1, \beta_2) \\ &= (f''_0(z_{\mu,s})w - \beta_1 \dot{\xi}_{\mu,s} - \beta_2 \dot{\zeta}_{\mu,s}, (w, \dot{\xi}_{\mu,s}), (w, \dot{\zeta}_{\mu,s})). \end{aligned}$$

From (b) we deduce that  $\left( \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \right)$  is an injective Fredholm operator of index zero, hence it is invertible and

$$\begin{aligned} & \left( \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \right)^{-1} (w, \beta_1, \beta_2) \\ &= (\beta_1 \dot{\xi}_{\mu,s} + \beta_2 \dot{\zeta}_{\mu,s} + f''_0(z_{\mu,s})^{-1}(w - (w, \dot{\xi}_{\mu,s})\dot{\xi}_{\mu,s} - (w, \dot{\zeta}_{\mu,s})\dot{\zeta}_{\mu,s}), \\ & \quad - (w, \dot{\xi}_{\mu,s}), - (w, \dot{\zeta}_{\mu,s})). \end{aligned}$$

In view of (4.10), we have that

$$\begin{aligned} & \left\| \left( \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \right)^{-1} \right\| \\ & \leq \max(1, \|(f''_0(z_{\mu,s}))^{-1}\|) = \max(1, \|(f''_0(\omega))^{-1}\|). \end{aligned}$$

Set  $C_* = \max(1, \|(f''_0(\omega))^{-1}\|)$ . For any  $(w, \alpha_1, \alpha_2) \in B_\rho(0)$  we have

$$\begin{aligned} \|F_{\mu,s,\varepsilon}(w, \alpha_1, \alpha_2)\| & \leq C_* \|f'_\varepsilon(z_{\mu,s} + w) - f''_0(z_{\mu,s})w\| \\ & \leq C_* \int_0^1 \|f''_0(z_{\mu,s} + tw) - f''_0(z_{\mu,s})\| dt \cdot \|w\| + C_* |\varepsilon| \|G'(z_{\mu,s} + w)\| \\ & \leq C_* \int_0^1 \|f''_0(\omega + t \mathcal{U}_{\mu,s}^{-1}(w)) - f''_0(\omega)\| dt \cdot \|w\| + C_* |\varepsilon| \|G'(z_{\mu,s} + w)\| \\ & \leq C_* \rho \sup_{\|w\| \leq \rho} \|f''_0(\omega + w) - f''_0(\omega)\| + C_* |\varepsilon| \sup_{\|w\| \leq \rho} \|G'(z_{\mu,s} + w)\|. \end{aligned} \tag{4.14}$$

For  $(w_1, \alpha_1, \beta_1), (w_2, \alpha_2, \beta_2) \in B_\rho(0)$

$$\begin{aligned}
 & \frac{\|F_{\mu,s,\varepsilon}(w_1, \alpha_1, \beta_1) - F_{\mu,s,\varepsilon}(w_2, \alpha_2, \beta_2)\|}{C_* \|w_1 - w_2\|} \\
 & \leq \frac{\|f'_\varepsilon(z_{\mu,s} + w_1) - f'_\varepsilon(z_{\mu,s} + w_2) - f''_0(z_{\mu,s})(w_1 - w_2)\|}{\|w_1 - w_2\|} \\
 & \leq \int_0^1 \|f''_0(z_{\mu,s} + w_2 + t(w_1 - w_2)) - f''_0(z_{\mu,s})\| dt \\
 & \quad + |\varepsilon| \int_0^1 \|G''(z_{\mu,s} + w_2 + t(w_1 - w_2))\| dt \\
 & \leq \sup_{\|w\| \leq 3\rho} \|f''_0(\omega + w) - f''_0(\omega)\| + |\varepsilon| \sup_{\|w\| \leq 3\rho} \|G''(z_{\mu,s} + w)\|. \quad (4.15)
 \end{aligned}$$

Choose  $\rho_0 > 0$  such that  $C_* \sup_{\|w\| \leq 3\rho_0} \|f''_0(\omega + w) - f''_0(\omega)\| < 1/2$  and  $\varepsilon_0 > 0$  such that

$$\begin{aligned}
 2\varepsilon_0 & < \left( \sup_{z \in Z, \|w\| \leq 3\rho_0} \|G''(z + w)\| \right)^{-1} C_*^{-1} \quad \text{and} \\
 3\varepsilon_0 & < \left( \sup_{z \in Z, \|w\| \leq \rho_0} \|G'(z + w)\| \right)^{-1} C_*^{-1} \rho_0.
 \end{aligned}$$

With these choices, for any  $z_{\mu,s} \in Z$  and  $|\varepsilon| < \varepsilon_0$  the map  $F_{\mu,s,\varepsilon}$  maps  $B_{\rho_0}(0)$  into itself and is a contraction there such that  $\|F_{\mu,s,\varepsilon}(w_1, \alpha_1, \beta_1) - F_{\mu,s,\varepsilon}(w_2, \alpha_2, \beta_2)\| \leq \lambda \|w_1 - w_2\|$ , where the constant  $\lambda \in (0, 1)$  does not depend on  $\mu, s, \varepsilon$ . Therefore  $F_{\mu,s,\varepsilon}$  has a unique fixed point  $(w(\mu, s, \varepsilon), \alpha_1(\mu, s, \varepsilon), \alpha_2(\mu, s, \varepsilon))$  in  $B_{\rho_0}(0)$ . From (4.14) we also infer that  $F_{\mu,s,\varepsilon}$  maps  $B_\rho(0)$  into  $B_\rho(0)$ , whenever  $\rho \leq \rho_0$  and

$$\rho > 2|\varepsilon| \left( \sup_{\|w\| \leq \rho} \|G'(z_{\mu,s} + w)\| \right) C_*.$$

Consequently for the uniqueness of the fixed point we have

$$\|(w(\mu, s, \varepsilon), \alpha_1(\mu, s, \varepsilon), \alpha_2(\mu, s, \varepsilon))\| \leq 3|\varepsilon| \left( \sup_{\|w\| \leq \rho_0} \|G'(z_{\mu,s} + w)\| \right) C_*,$$

which gives (4.13). □

We are now interested in the behavior of the function

$$\Phi_\varepsilon(\mu, s) = f_\varepsilon(z_{\mu,s} + w(\mu, s, \varepsilon))$$

the critical points of which on  $\mathbb{R}^+ \times \mathbb{R}$  give rise to critical points of  $f_\varepsilon$  on  $S_{\text{cyl}}^1(\mathbb{H}^n)$ , as remarked in Subsection 4.1. In particular we will prove the following proposition.

**Proposition 4.5.** *Assume that  $k$  is cylindrically symmetric and continuous on  $\overline{\mathbb{H}^n}$ . Then for any  $\bar{s} \in \mathbb{R}$  there holds*

$$\begin{aligned}
 \text{(i)} \quad \lim_{(\mu,s) \rightarrow (0,\bar{s})} \Phi_\varepsilon(\mu, s) &= f_0(\omega) (1 + \varepsilon k(0, \bar{s}))^{-\frac{Q-2}{2}} \\
 \text{(ii)} \quad \lim_{(\mu,s) \rightarrow \infty} \Phi_\varepsilon(\mu, s) &= f_0(\omega) (1 + \varepsilon k(\infty))^{-\frac{Q-2}{2}}.
 \end{aligned}$$

*Remark 4.6.* Thanks to the above proposition we can extend  $\Phi_\varepsilon$  to the axis  $\{\mu = 0\}$  by setting

$$\Phi_\varepsilon(0, s) := f_0(\omega)(1 + \varepsilon k(0, s))^{-\frac{Q-2}{2}}$$

and to infinity by setting

$$\Phi_\varepsilon(\infty) := f_0(\omega)(1 + \varepsilon k(\infty))^{-\frac{Q-2}{2}}$$

thus obtaining a continuous function on  $\{(\mu, s) : \mu \geq 0\} \cup \{\infty\}$ .

For  $\mu > 0, s \in \mathbb{R}$ , let us consider the functional  $f_\varepsilon^{\mu,s} = f_\varepsilon \circ \mathcal{U}_{\mu,s}$  i. e.

$$f_\varepsilon^{\mu,s}(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dz dt - \frac{1}{Q^*} \int_{\mathbb{H}^n} (1 + \varepsilon k(\mu z, \mu^2 t + s)) u_+^{Q^*} dz dt.$$

There results  $(f_\varepsilon^{\mu,s})' = (\mathcal{U}_{\mu,s})^{-1} \circ f'_\varepsilon \circ \mathcal{U}_{\mu,s}$  and

$$(f_\varepsilon^{\mu,s})''(u) = (\mathcal{U}_{\mu,s})^{-1} \circ f''_\varepsilon(u) \circ \mathcal{U}_{\mu,s},$$

for any  $u \in S^1_{\text{cyl}}(\mathbb{H}^n)$ . Let us consider the map

$$\begin{aligned} \tilde{H}^{\mu,s} : S^1_{\text{cyl}}(\mathbb{H}^n) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} &\longrightarrow S^1_{\text{cyl}}(\mathbb{H}^n) \times \mathbb{R} \times \mathbb{R}, \\ (w, \alpha_1, \alpha_2, \varepsilon) &\longmapsto ((f_\varepsilon^{\mu,s})'(\omega + w) - \alpha_1 \dot{\xi}_0 - \alpha_2 \dot{\zeta}_0, (w, \dot{\xi}_0), (w, \dot{\zeta}_0)) \end{aligned}$$

where  $\dot{\xi}_0$  (resp.  $\dot{\zeta}_0$ ) is normalized tangent vector  $\frac{\partial}{\partial \mu} z_{\mu,s} \Big|_{\mu=1,s=0}$  (resp.  $\frac{\partial}{\partial s} z_{\mu,s} \Big|_{\mu=1,s=0}$ ).

We have

$$\frac{\partial \tilde{H}^{\mu,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0) = \frac{\partial H}{\partial (w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0, 0) \Big|_{\substack{\mu=1 \\ s=0}}$$

hence  $\frac{\partial \tilde{H}^{\mu,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0)$  is invertible and  $\|(\frac{\partial \tilde{H}^{\mu,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0))^{-1}\| \leq C_*$ . The map

$$F_\varepsilon^{\mu,s}(w, \alpha_1, \alpha_2) := -\left(\frac{\partial \tilde{H}^{\mu,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0)\right)^{-1} \tilde{H}^{\mu,s}(w, \alpha_1, \alpha_2, \varepsilon) + (w, \alpha_1, \alpha_2)$$

satisfies

$$\|F_\varepsilon^{\mu,s}(w, \alpha_1, \alpha_2)\| \leq C_* \|f'_\varepsilon(z_{\mu,s} + \mathcal{U}_{\mu,s}(w)) - f''_0(z_{\mu,s})\mathcal{U}_{\mu,s}(w)\|$$

and

$$\begin{aligned} &\frac{\|F_\varepsilon^{\mu,s}(w_1, \alpha_1, \beta_1) - F_\varepsilon^{\mu,s}(w_2, \alpha_2, \beta_2)\|}{C_* \|w_1 - w_2\|} \\ &\leq \frac{\|f'_\varepsilon(z_{\mu,s} + \mathcal{U}_{\mu,s}(w_1)) - f'_\varepsilon(z_{\mu,s} + \mathcal{U}_{\mu,s}(w_2))\| - f''_0(z_{\mu,s})(\mathcal{U}_{\mu,s}(w_1 - w_2))\|}{\|\mathcal{U}_{\mu,s}(w_1) - \mathcal{U}_{\mu,s}(w_2)\|} \end{aligned}$$

which imply, in view of (4.14) and (4.15), that  $F_\varepsilon^{\mu,s}$  is a contraction in the same ball where  $F_{\mu,s,\varepsilon}$  is a contraction (see the proof of Lemma 4.4). Hence  $F_\varepsilon^{\mu,s}$  has a fixed point  $(w_\varepsilon^{\mu,s}, \alpha_{1,\varepsilon}^{\mu,s}, \alpha_{2,\varepsilon}^{\mu,s})$  such that  $\tilde{H}^{\mu,s}(w_\varepsilon^{\mu,s}, \alpha_{1,\varepsilon}^{\mu,s}, \alpha_{2,\varepsilon}^{\mu,s}, \varepsilon) = 0$ .

From the uniqueness of the fixed point of  $F_\varepsilon^{\mu,s}$  and from the fact that

$$(f_\varepsilon^{\mu,s})'(\omega + (\mathcal{U}_{\mu,s})^{-1}w(\mu, s, \varepsilon)) \in T_\omega Z$$

and  $(\mathcal{U}_{\mu,s})^{-1}w(\mu, s, \varepsilon) \in (T_\omega Z)^\perp$ , it follows that  $w_\varepsilon^{\mu,s} = (\mathcal{U}_{\mu,s})^{-1}(w(\mu, s, \varepsilon))$ , where  $w(\mu, s, \varepsilon)$  is given in Lemma 4.4. Assume now that  $k$  is continuous on  $\overline{\mathbb{H}^n}$  and fix  $s \in \mathbb{R}$ . Let us consider the functional

$$f_\varepsilon^{0,s} = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} (1 + \varepsilon k(0, s)) \int_{\mathbb{H}^n} u \frac{Q^*}{+}.$$

For  $w_\varepsilon^{0,s} = (t_\varepsilon(s) - 1)\omega$  where  $t_\varepsilon(s) = (1 + \varepsilon k(0, s))^{-\frac{Q-2}{4}}$  we have

$$(f_\varepsilon^{0,s})'(\omega + w_\varepsilon^{0,s}) = 0$$

and

$$(f_\varepsilon^{0,s})(\omega + w_\varepsilon^{0,s}) = (1 + \varepsilon k(0, s))^{-\frac{Q-2}{2}} \left( \frac{1}{2} - \frac{1}{Q^*} \right) \int_{\mathbb{H}^n} \omega \frac{Q^*}{+}$$

and hence  $\tilde{H}^{0,s}(w_\varepsilon^{0,s}, 0, 0, \varepsilon) = 0$  where

$$\tilde{H}^{0,s}(w, \alpha_1, \alpha_2, \varepsilon) = ((f_\varepsilon^{0,s})'(\omega + w) - \alpha_1 \dot{\xi}_0 - \alpha_2 \dot{\zeta}_0, (w, \dot{\xi}_0), (w, \dot{\zeta}_0)).$$

We have  $\frac{\partial \tilde{H}^{0,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0) = \frac{\partial H}{\partial (w, \alpha_1, \alpha_2)}(\mu, s, 0, 0, 0) \Big|_{\substack{\mu=1 \\ s=0}}$  and hence  $(w_\varepsilon^{0,s}, 0, 0)$  is a fixed point of the map

$$F_\varepsilon^{0,s}(w, \alpha_1, \alpha_2) = \left( \frac{\partial \tilde{H}^{0,s}}{\partial (w, \alpha_1, \alpha_2)}(0, 0, 0, 0) \right)^{-1} \tilde{H}^{0,s}(w, \alpha_1, \alpha_2, \varepsilon) + (w, \alpha_1, \alpha_2).$$

It is easy to check that  $F_\varepsilon^{0,s}$  is a contraction in some ball of radius  $O(|\varepsilon|)$ . Hence  $w_\varepsilon^{0,s}$  is the unique fixed point of  $F_\varepsilon^{0,s}$  in such a ball.

Let us also set  $w_\varepsilon^\infty = (t_\varepsilon^\infty - 1)\omega$ ,  $t_\varepsilon^\infty = (1 + \varepsilon k(\infty))^{-\frac{Q-2}{4}}$ .

**Lemma 4.7.** *For any  $\bar{s} \in \mathbb{R}$  there holds*

$$w_\varepsilon^{\mu,s} \rightarrow w_\varepsilon^{0,\bar{s}}, \quad as \ (\mu, s) \rightarrow (0, \bar{s}), \quad (4.16)$$

$$w_\varepsilon^{\mu,s} \rightarrow w_\varepsilon^\infty, \quad as \ (\mu, s) \rightarrow (0, \infty). \quad (4.17)$$

*Proof.* We have

$$\begin{aligned} & \|F_\varepsilon^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0) - F_\varepsilon^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0)\| \\ & \leq C_* \|\tilde{H}^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0, \varepsilon) - \tilde{H}^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0, \varepsilon)\| \\ & \leq C_* \|(f_\varepsilon^{\mu,s})'(\omega + w_\varepsilon^{0,\bar{s}}) - (f_\varepsilon^{0,\bar{s}})'(\omega + w_\varepsilon^{0,\bar{s}})\| \\ & = C_* \|(f_\varepsilon^{\mu,s})'(t_\varepsilon(\bar{s})\omega) - (f_\varepsilon^{0,\bar{s}})'(t_\varepsilon(\bar{s})\omega)\|. \end{aligned}$$

Because of (2.2) and the Hölder inequality we have

$$\begin{aligned} & \left| \left( (f_\varepsilon^{\mu,s})'(t_\varepsilon(\bar{s})\omega) - (f_\varepsilon^{0,\bar{s}})'(t_\varepsilon(\bar{s})\omega), v \right) \right| \\ &= \left| \int_{\mathbb{H}^n} \varepsilon [k(\mu z, \mu^2 t + s) - k(0, \bar{s})](t_\varepsilon(\bar{s})\omega)^{Q^*-1} v \right| \\ &\leq S^{-1/2} \|v\| \left[ \int_{\mathbb{H}^n} \varepsilon \frac{Q^*}{Q^*-1} |k(\mu z, \mu^2 t + s) - k(0, \bar{s})| \frac{Q^*}{Q^*-1} t_\varepsilon(\bar{s})^{Q^*} \omega^{Q^*} \right]^{\frac{Q^*-1}{Q^*}} \end{aligned}$$

and hence by the Dominated Convergence Theorem,

$$\begin{aligned} & \left\| (f_\varepsilon^{\mu,s})'(t_\varepsilon(\bar{s})\omega) - (f_\varepsilon^{0,\bar{s}})'(t_\varepsilon(\bar{s})\omega) \right\| \\ &\leq c \left[ \int_{\mathbb{H}^n} |k(\mu z, \mu^2 t + s) - k(0, \bar{s})| \frac{Q^*}{Q^*-1} \omega^{Q^*} \right]^{\frac{Q^*-1}{Q^*}} \xrightarrow{(\mu,s) \rightarrow (0,\bar{s})} 0. \end{aligned}$$

Therefore

$$F_\varepsilon^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0) \rightarrow F_\varepsilon^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0), \quad \text{as } (\mu, s) \rightarrow (0, \bar{s}). \tag{4.18}$$

Since  $F_\varepsilon^{\mu,s}$  is a contraction with a contraction factor  $0 < \lambda < 1$  independent of  $\mu, s$ , and  $\varepsilon$  we have

$$\begin{aligned} \left\| w_\varepsilon^{\mu,s} - w_\varepsilon^{0,\bar{s}} \right\| &\leq \left\| (w_\varepsilon^{\mu,s}, \alpha_{1,\varepsilon}^{\mu,s}, \alpha_{2,\varepsilon}^{\mu,s}) - (w_\varepsilon^{0,\bar{s}}, 0, 0) \right\| \\ &= \left\| F_\varepsilon^{\mu,s}(w_\varepsilon^{\mu,s}, \alpha_{1,\varepsilon}^{\mu,s}, \alpha_{2,\varepsilon}^{\mu,s}) - F_\varepsilon^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0) \right\| \\ &\leq \left\| F_\varepsilon^{\mu,s}(w_\varepsilon^{\mu,s}, \alpha_{1,\varepsilon}^{\mu,s}, \alpha_{2,\varepsilon}^{\mu,s}) - F_\varepsilon^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0) \right\| \\ &\quad + \left\| F_\varepsilon^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0) - F_\varepsilon^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0) \right\| \\ &\leq \lambda \left\| w_\varepsilon^{\mu,s} - w_\varepsilon^{0,\bar{s}} \right\| + \left\| F_\varepsilon^{\mu,s}(w_\varepsilon^{0,\bar{s}}, 0, 0) - F_\varepsilon^{0,\bar{s}}(w_\varepsilon^{0,\bar{s}}, 0, 0) \right\| \end{aligned}$$

and hence from (4.18) we obtain (4.16). The proof of (4.17) is analogous. □

*Proof of Proposition 4.5.* By definition of  $\Phi_\varepsilon$  and  $f_\varepsilon^{\mu,s}$ , we have

$$\Phi_\varepsilon(\mu, s) = f_\varepsilon(z_{\mu,s} + w(\mu, s, \varepsilon)) = f_\varepsilon^{\mu,s}(\omega + w_\varepsilon^{\mu,s}). \tag{4.19}$$

Moreover

$$\begin{aligned} & f_\varepsilon^{\mu,s}(\omega + w_\varepsilon^{\mu,s}) \\ &= f_\varepsilon^{\mu,s}(\omega + w_\varepsilon^{\mu,s}) - f_\varepsilon^{0,\bar{s}}(\omega + w_\varepsilon^{\mu,s}) + f_\varepsilon^{0,\bar{s}}(\omega + w_\varepsilon^{\mu,s}) \\ &= \frac{\varepsilon}{Q^*} \int_{\mathbb{H}^n} (k(0, \bar{s}) - k(\mu z, \mu^2 t + s)) (\omega + w_\varepsilon^{\mu,s})_+^{Q^*} dz dt + f_\varepsilon^{0,\bar{s}}(\omega + w_\varepsilon^{\mu,s}) \end{aligned}$$

and hence by (4.16) and the Dominated Convergence Theorem

$$f_\varepsilon^{\mu,s}(\omega + w_\varepsilon^{\mu,s}) \xrightarrow{(\mu,s) \rightarrow (0,\bar{s})} f_\varepsilon^{0,\bar{s}}(\omega + w_\varepsilon^{0,\bar{s}}). \tag{4.20}$$

On the other hand we have that

$$f_\varepsilon^{0,\bar{s}}(\omega + w_\varepsilon^{0,\bar{s}}) = f_\varepsilon^{0,\bar{s}}(t_\varepsilon(\bar{s})\omega) = f_0(\omega)(1 + \varepsilon k(0, \bar{s}))^{-\frac{Q-2}{2}}. \tag{4.21}$$

From (4.19), (4.20), and (4.21), (i) follows. In an analogous way, using (4.17), it is easy to prove (ii).  $\square$

Thanks to Proposition 4.5 it is now quite easy to give some conditions on  $k$  in order to have critical points of  $\Phi_\varepsilon$ . In particular the knowledge of  $k$  on the axis  $\{(0, s) : s \in \mathbb{R}\}$  and at  $\infty$  gives exact information about the behavior of  $\Phi_\varepsilon$  on the axis  $\{(0, s) : s \in \mathbb{R}\}$  and at  $\infty$ .

*Proof of Theorem 2.8.* As remarked in Subsection 4.1, it is enough to prove that  $\Phi_\varepsilon(\mu, s) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  has a critical point. In view of Proposition 4.5,  $k(0, t) = k(\infty) \forall t \in \mathbb{R}$  implies that  $\Phi_\varepsilon(0, t) = \Phi_\varepsilon(\infty) \forall t \in \mathbb{R}$ . Hence, either  $\Phi_\varepsilon$  is constant (and we have infinitely many critical points) or it has a global maximum or minimum point  $(\bar{\mu}, \bar{s}), \bar{\mu} > 0$ . In any case,  $\Phi_\varepsilon$  has a critical point which provides a solution to  $(\mathcal{P}_\varepsilon)$ .  $\square$

### Appendix

In the first part of this appendix, we prove some technical lemmas about the properties of cylindrically symmetric functions of the Folland–Stein Sobolev space  $S_0^1(\mathbb{H}^n)$ .

*Remark A.1.* If  $u(z, t) = \tilde{u}(|z|, t), v(z, t) = \tilde{v}(|z|, t)$  are in  $S_{\text{cyl}}^1(\mathbb{H}^n)$ , then  $\tilde{u}, \tilde{v} \in H_{\text{loc}}^1(\{(r, t) \in \mathbb{R}^2 | r > 0\})$ . Moreover the following formula holds a.e.

$$\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v \rangle(z, t) = (\partial_r \tilde{u} \partial_r \tilde{v} + 4r^2 \partial_t \tilde{u} \partial_t \tilde{v})(|z|, t). \tag{A.1}$$

*Proof.* It is straightforward to verify that formula (A.1) holds for smooth functions. In order to extend it to general  $u, v \in S_{\text{cyl}}^1(\mathbb{H}^n)$ , we choose two sequences of cylindrically symmetric functions  $\phi_k, \psi_k \in C_0^\infty(\mathbb{H}^n)$ , converging in  $S_0^1$  to  $u, v$ , respectively. By a cylindrical change of coordinates, it is then easy to see that  $\tilde{\phi}_k, \tilde{\psi}_k$  are Cauchy sequences in  $H^1(\Omega)$  for every  $\Omega \subset\subset ]0, \infty[ \times \mathbb{R}$ . Moreover, since  $\phi_k \rightarrow u, \psi_k \rightarrow v$ , pointwise a.e. (up to subsequences), the limits of  $\tilde{\phi}_k, \tilde{\psi}_k$  in  $H^1(\Omega)$  are necessarily  $\tilde{u}, \tilde{v}$ . As a consequence, formula (A.1) (which we know to hold for  $\phi_k, \psi_k$ ) extends to  $u, v$  by means of the a.e. pointwise convergences  $\nabla_{\mathbb{H}^n} \phi_k \rightarrow \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \psi_k \rightarrow \nabla_{\mathbb{H}^n} v, (\partial_r \tilde{\phi}_k, \partial_t \tilde{\phi}_k) \rightarrow (\partial_r \tilde{u}, \partial_t \tilde{u}), (\partial_r \tilde{\psi}_k, \partial_t \tilde{\psi}_k) \rightarrow (\partial_r \tilde{v}, \partial_t \tilde{v})$ .  $\square$

**Lemma A.2.** *Let  $K(z, t) = \tilde{K}(|z|, t)$ , with  $\tilde{K}$  bounded and locally Hölder continuous in  $]0, \infty[ \times \mathbb{R}$ , and let  $u \in S_{\text{cyl}}^1(\mathbb{H}^n)$  be a (nonnegative) weak solution of  $-\Delta_{\mathbb{H}^n} u = Ku^{Q^*-1}$  in the  $S_{\text{cyl}}^1(\mathbb{H}^n)$  sense (i.e., with respect to  $S_{\text{cyl}}^1(\mathbb{H}^n)$  test functions). Then  $u$  is a weak solution of the same equation in the  $S_0^1(\mathbb{H}^n)$  sense. An analogous result also holds for weak solutions of the equation  $-\Delta_{\mathbb{H}^n} u = (Q^* - 1)U_{\mu, s}^{Q^*-2}u$ .*

*Proof.* Let us prove the statement related to the equation  $-\Delta_{\mathbb{H}^n} u = Ku^{Q^*-1}$  (the same proof works also for the other equation). Recalling Remark A.1, for every

test function  $\tilde{\phi} \in C_0^\infty(]0, \infty[ \times \mathbb{R})$  we have

$$\begin{aligned} c_n \int_{]0, \infty[ \times \mathbb{R}} \tilde{K}\tilde{u}^{Q^*-1} \tilde{\phi} r^{2n-1} dr dt &= \int_{\mathbb{H}^n} Ku^{Q^*-1} \phi = \int_{\mathbb{H}^n} \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \phi \rangle \\ &= c_n \int_{]0, \infty[ \times \mathbb{R}} (\partial_r \tilde{u} \partial_r \tilde{\phi} + 4r^2 \partial_t \tilde{u} \partial_t \tilde{\phi}) r^{2n-1} dr dt \end{aligned}$$

where  $u(z, t) = \tilde{u}(|z|, t)$ . Hence  $\tilde{u}$  is a weak solution of the elliptic equation

$$-\partial_r(r^{2n-1} \partial_r \tilde{u}) - \partial_t(4r^{2n+1} \partial_t \tilde{u}) = r^{2n-1} \tilde{K}\tilde{u}^{Q^*-1}$$

in  $]0, \infty[ \times \mathbb{R}$ . Now, using a classical bootstrap elliptic argument, one can easily see that  $\tilde{u} \in C^2(]0, \infty[ \times \mathbb{R})$ . As a consequence  $u \in C^2(\{(z, t) \in \mathbb{H}^n \mid z \neq 0\})$  is a classical solution of the equation  $-\Delta_{\mathbb{H}^n} u = Ku^{Q^*-1}$  in  $\{z \neq 0\}$  and we can argue as follows. Let us fix a test function  $\phi \in C_0^\infty(\mathbb{H}^n)$  with support contained in  $\mathbb{R}^{2n} \times [-T, T]$  and let us set  $\Omega_\varepsilon = \{(z, t) \in \mathbb{H}^n : |z| < \varepsilon, |t| < T\}$ ,  $\delta_\varepsilon = \{(z, t) \in \mathbb{H}^n : |z| = \varepsilon, |t| \leq T\}$ . Let us also choose a vanishing sequence of positive numbers  $\varepsilon_k$  such that

$$\int_{\delta_{\varepsilon_k}} |\nabla_{\mathbb{H}^n} u|^2 dH_{2n} = o\left(\frac{1}{\varepsilon_k}\right), \quad \text{as } k \rightarrow \infty$$

(such a sequence does exist since  $|\nabla_{\mathbb{H}^n} u| \in L^2(\mathbb{H}^n)$ ). Then, setting

$$A = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix},$$

we have by the Divergence Theorem

$$\begin{aligned} \left| \int_{\mathbb{H}^n \setminus \Omega_{\varepsilon_k}} (\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \phi \rangle - Ku^{Q^*-1} \phi) \right| &= \left| \int_{\mathbb{H}^n \setminus \Omega_{\varepsilon_k}} \operatorname{div}(\phi A \nabla u) \right| \\ &= \left| \int_{\delta_{\varepsilon_k}} \langle \phi A \nabla u, \nabla(-z) \rangle dH_{2n} \right| = \left| \int_{\delta_{\varepsilon_k}} \phi \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n}(z) \rangle dH_{2n} \right| \\ &\leq c \int_{\delta_{\varepsilon_k}} |\nabla_{\mathbb{H}^n} u| dH_{2n} \leq c \left( \int_{\delta_{\varepsilon_k}} |\nabla_{\mathbb{H}^n} u|^2 dH_{2n} \right)^{1/2} \varepsilon_k^{(2n-1)/2} \\ &= o(\varepsilon_k^{n-1}), \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Since  $\left| \int_{\mathbb{H}^n \setminus \Omega_{\varepsilon_k}} (\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \phi \rangle - Ku^{Q^*-1} \phi) \right| \rightarrow \left| \int_{\mathbb{H}^n} (\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \phi \rangle - Ku^{Q^*-1} \phi) \right|$ , this proves that  $\int_{\mathbb{H}^n} \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \phi \rangle = \int_{\mathbb{H}^n} Ku^{Q^*-1} \phi$  holds for every  $\phi \in C_0^\infty(\mathbb{H}^n)$  and thus for every  $\phi \in S_0^1(\mathbb{H}^n)$ .  $\square$

**Lemma A.3.** *Let  $u_m$  be a sequence weakly converging in  $S_{\text{cyl}}^1(\mathbb{H}^n)$  to some function  $u \in S_{\text{cyl}}^1(\mathbb{H}^n)$ . Then (up to subsequences)  $u_m \rightarrow u$  in  $L^q(C)$  for any set  $C$  of the type  $\{z : 0 < c_1 \leq |z| \leq c_2\} \times [-c_3, c_3]$  (and hence on any compact set away from the axis  $\{z = 0\}$ ) and for any  $1 \leq q < +\infty$ .*

*Proof.* Let  $C = \{z : c_1 \leq |z| \leq c_2\} \times [-c_3, c_3]$ . From Remark A.1 we have that for any function  $w \in S_{\text{cyl}}^1(\mathbb{H}^n)$

$$|\nabla_{\mathbb{H}^n} w|^2 = |\partial_r w|^2 + 4r^2 |\partial_t w|^2. \tag{A.2}$$

Let now  $u_m$  be a sequence weakly converging to  $u$  in  $S_{\text{cyl}}^1(\mathbb{H}^n)$  (and so in  $S_0^1(\mathbb{H}^n)$  and in  $L^{Q^*}(\mathbb{H}^n)$ ). Thanks to (A.2) we can write

$$\begin{aligned} \text{const} &\geq \int_C |\nabla_{\mathbb{H}^n} u_m|^2 = \gamma_n \int_{\substack{c_1 \leq r \leq c_2 \\ |t| \leq c_3}} (|\partial_r u_m|^2 + 4r^2 |\partial_t u_m|^2) r^{2n-1} dr dt \\ &\geq \gamma_n \min\{1, 4c_1^2\} c_1^{2n-1} \int_{[c_1, c_2] \times [-c_3, c_3]} (|\partial_r u_m|^2 + |\partial_t u_m|^2) dr dt \end{aligned}$$

and analogously for the  $L^2$  norm. Hence the sequence  $u_m(r, t)$  is bounded in the space  $H^1([c_1, c_2] \times [-c_3, c_3])$  which is compactly embedded in  $L^q([c_1, c_2] \times [-c_3, c_3])$  for any  $1 \leq q < +\infty$ . Therefore, up to a subsequence,  $u_m(r, t) \rightarrow u(r, t)$  in  $L^q([c_1, c_2] \times [-c_3, c_3])$ . Consequently, we get

$$\begin{aligned} \int_C |u_m - u|^q &= \gamma_n \int_{\substack{c_1 \leq r \leq c_2 \\ |t| \leq c_3}} |u_m(r, t) - u(r, t)|^q r^{2n-1} dr dt \\ &\leq \gamma_n c_2^{2n-1} \int_{[c_1, c_2] \times [-c_3, c_3]} |u_m(r, t) - u(r, t)|^q dr dt \longrightarrow 0. \end{aligned}$$

Lemma A.3 is thereby established. □

Let us now state the P.L. Lions concentration–compactness principle in  $\mathbb{H}^n$ . Since the proof does not present further difficulties with respect to the euclidean case (see [17] and [18]), we omit it. Let  $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \{\infty\}$  be the compactification of  $\mathbb{H}^n$ . Let us denote by  $\mathcal{M}(\overline{\mathbb{H}^n})$  the Banach space of finite signed regular Borel measures on  $\overline{\mathbb{H}^n}$ , endowed with the total variation norm. In view of the Riesz representation theorem, the space  $\mathcal{M}(\overline{\mathbb{H}^n})$  can be identified with the dual of the Banach space  $C(\overline{\mathbb{H}^n})$ . We say that a sequence of measures  $\mu_m$  weakly converges to  $\mu$  in  $\mathcal{M}(\overline{\mathbb{H}^n})$  if for any  $f \in C(\overline{\mathbb{H}^n})$  (i.e., continuous on  $\mathbb{H}^n$  with finite limit at  $\infty$ )

$$\int_{\mathbb{H}^n} f d\mu_m \longrightarrow \int_{\mathbb{H}^n} f d\mu.$$

In this case we will use the notation  $\mu_m \xrightarrow{\mathcal{M}} \mu$ .

**Theorem A.4 (Concentration-compactness).** *Let  $\{u_m\}$  be a sequence weakly converging to  $u$  in  $S_0^1(\mathbb{H}^n)$ . Then, up to subsequences,*

- (i)  $|\nabla_{\mathbb{H}^n} u_m|^2$  weakly converges in  $\mathcal{M}(\overline{\mathbb{H}^n})$  to a nonnegative measure  $\mu$ ,
- (ii)  $|u_m|^{Q^*}$  weakly converges in  $\mathcal{M}(\overline{\mathbb{H}^n})$  to a nonnegative measure  $\nu$ .

Moreover there exist an at most countable index set  $J$ , a sequence  $(z_j, t_j) \in \mathbb{H}^n$ ,  $v^j, v^\infty \in (0, \infty)$  such that

$$v = |u|^{Q^*} + \sum_{j \in J} v^j \delta_{(z_j, t_j)} + v^\infty \delta_\infty$$

$$\mu \geq |\nabla_{\mathbb{H}^n} u|^2 + \sum_{j \in J} S(v^j)^{\frac{2}{Q^*}} \delta_{(z_j, t_j)} + S(v^\infty)^{\frac{2}{Q^*}} \delta_\infty.$$

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## References

1. Ambrosetti, A., Badiale, M.: Variational perturbative methods and bifurcation of bound states from the essential spectrum. Proc. R. Soc. Edinb., Sect. A, Math. **128**, 1131–1161 (1998)
2. Ambrosetti, A., Badiale, M.: Homoclinics: Poincaré-Melnikov type results via a variational approach. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **15**, 233–252 (1998)
3. Ambrosetti, A., Garcia Azorero, J., Peral, I.: Perturbation of  $\Delta u + u^{(N+2)/(N-2)} = 0$ , the scalar curvature problem in  $\mathbb{R}^N$ , and related topics. J. Funct. Anal. **165**, 117–149 (1999)
4. Ambrosetti, A., Malchiodi, A.: On the symmetric scalar curvature problem on  $S^n$ . J. Differ. Equations **170**, 228–245 (2001)
5. Bianchi, G., Egnell, H.: Local existence and uniqueness of positive solutions of the equation  $\Delta u + (1 + \epsilon \varphi(r))u^{(n+2)/(n-2)} = 0$ , in  $\mathbb{R}^n$  and a related equation. In: Nonlinear diffusion equations and their equilibrium states. Vol. 3, pp. 111–128 (Gregynog, 1989), Progress in Nonlinear Differ. Equations Applications, Vol. 7. Boston, MA: Birkhäuser 1992
6. Bianchi, G., Egnell, H.: A variational approach to the equation  $\Delta u + Ku^{(n+2)/(n-2)} = 0$  in  $\mathbb{R}^n$ . Arch. Ration. Mech. Anal. **122**, 159–182 (1993)
7. Brandolini, L., Rigoli, M., Setti, A.G.: Positive solutions of Yamabe-type equations on the Heisenberg group. Duke Math. J. **91**, 241–296 (1998)
8. Chang, S.A., Yang, P.: A perturbation result in prescribing scalar curvature on  $S^n$ . Duke Math. J. **64**, 27–69 (1991)
9. Folland, G.B., Stein, E.M.: Estimates for the  $\bar{\partial}_b$  complex and analysis on the Heisenberg group. Commun. Pure Appl. Math. **27**, 429–522 (1974)
10. Gamara, N.: The CR Yamabe conjecture the case  $n = 1$ . J. Eur. Math. Soc. (JEMS) **3**, 105–137 (2001)
11. Gamara N., Yacoub, R.: CR Yamabe conjecture – the conformally flat case. Pac. J. Math. **201**, 121–175 (2001)
12. Garofalo, N., Lanconelli, E.: Existence and nonexistence results for semilinear equations on the Heisenberg group. Indiana Univ. Math. J. **41**, 71–98 (1992)
13. Hebey, E.: Changements de métriques conformes sur la sphère – Le problème de Nirenberg. Bull. Sci. Math. **114**, 215–242 (1990)
14. Jerison D., Lee, J.M.: Yamabe problem on CR manifolds. J. Differ. Geom. **25**, 167–197 (1987)
15. Jerison, D., Lee, J.M.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. Am. Math. Soc. **1**, 1–13 (1988)
16. Jerison, D., Lee, J.M.: Intrinsic CR normal coordinates and the CR Yamabe problem. J. Differ. Geom. **29**, 303–343 (1989)

17. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoam.* **1**, 145–201 (1985)
18. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoam.* **1**, 45–121 (1985)
19. Lu, G., Wei, J.: On positive entire solutions to the Yamabe-type problem on the Heisenberg and stratified groups. *Electron. Res. Announc. Am. Math. Soc.* **3**, 83–89 (1997)
20. Malchiodi, A., Uguzzoni, F.: A perturbation result for the Webster scalar curvature problem on the CR sphere. *J. Math. Pure Appl.* **81**, 983–997 (2002)
21. Uguzzoni, F.: A note on Yamabe-type equations on the Heisenberg group. *Hiroshima Math. J.* **30**, 179–189 (2000)
22. Webster, S.M.: Pseudo-Hermitian structures on a real hypersurface. *J. Differ. Geom.* **13**, 25–41 (1978)