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Families of diffeomorphic sub-Laplacians and free Carnot groups*

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Abstract. Motivated by some problems coming from non-linear subelliptic PDE's, we investigate the equivalence of sub-Laplacians on Carnot groups. We show that all the sub-Laplacians are equivalent if the group is free, while this is not true in the general case. We directly construct diffeomorphisms turning general sub-Laplacians into the canonical one, deriving uniform estimates when the operators vary in suitable classes.

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1 Introduction and main results

The study of second order linear PDE's sum of squares of vector fields, started with Hörmander's paper [Ho], has significatively developed after the works by Folland [F1] and by Rothschild-Stein [RS]. In [RS], it has been shown that any Hörmander operator can be locally approximated by a sub-Laplacian on a free stratified Lie group. Since the appearance of this result, the study of stratified groups, also known as Carnot groups, has received great impulse. Many authors have investigated on this topic from different points of view and with different scopes (see the references below). The present paper is the first part of a project aimed to apply analysis on Carnot groups to the study of *non-linear* subelliptic PDE's arising in geometric theory of several complex variables such as the Levi-curvature equation, which has achieved rising concern in the last few years (see [ST], [HuK], [CM], [CLM], [M]). The starting point of this programme is to study the fundamental solution for parabolic-type operators in *non-divergence* form

(1.1) $$\sum_{i,j} a_{i,j}(x) X_i X_j - \partial_t,$$

being $\{X_i\}_i$ a stratified system of Hörmander vector fields and $a_{i,j}$ Hölder continuous functions. This study is also meant to obtain Harnack inequalities for such operators.

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The fundamental solution for (1.1) could be constructed via the Levi-parametrix method, provided suitable uniform estimates of the fundamental solutions for the frozen operators may be established. Despite the wide scientific production on related topics ([RS], [JS], [KS1], [KS2], [VSC], [BB]), at the authors' knowledge these uniform estimates are not available in literature. The aim of this paper is to settle the basis in order to establish such uniform estimates (see (1.3)). The main steps towards this goal are Theorem 1.1 and Theorem 1.2 below.

We next give a plan of the paper along with a descriptive summary of our main results. We start by giving the definition of a Carnot group. Our definition may seem slightly different from the ones given in literature, but it is indeed equivalent, as we observe below. Let $\circ$ be an assigned Lie group law on $\mathbb{R}^N$. We suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$(1.2) \qquad \delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}).$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \dots + N_r = N$. We denote by $\mathfrak{g}$ the Lie algebra of $(\mathbb{R}^N, \circ)$ i.e. the Lie algebra of left-invariant vector fields on $\mathbb{R}^N$. For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in $\mathfrak{g}$ that agrees at the origin with $\partial/\partial x_i$. We make the following assumption: the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$. With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *homogeneous Carnot group*. We also say that $\mathbb{G}$ is of *step* r and has $m := N_1$ *generators*. We denote by $Q = \sum_{j=1}^r jN_j$ the *homogeneous dimension* of $\mathbb{G}$. The *canonical sub-Laplacian* on $\mathbb{G}$ is the second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2.$$

If $Y_1, \dots, Y_m$ is any basis for $\operatorname{span}\{X_1, \dots, X_m\}$, the second order differential operator $\mathscr{L} = \sum_{i=1}^m Y_i^2$ will be called a *sub-Laplacian* on $\mathbb{G}$. We explicitly remark that $\mathscr{L}$ is hypoelliptic since $Y_1, \dots, Y_m$ Lie-generate $\mathfrak{g}$ and hence they satisfy Hörmander's condition

$$\operatorname{rank}(\operatorname{Lie}\{Y_1, \dots, Y_m\}(x)) = N, \quad \forall x \in \mathbb{R}^N.$$

The simplest example of homogeneous Carnot group is $(\mathbb{R}^N, +)$ (in this case the canonical sub-Laplacian is the classical Laplace operator). The most simple non-abelian example is the Heisenberg group $\mathbb{H}^n$ (with the Kohn-Laplace operator).

In literature a *Carnot group* (or *stratified group*) $\mathbb{H}$ is defined as a connected and simply connected Lie group whose Lie algebra $\mathfrak{h}$ admits a *stratification* $\mathfrak{h} = V_1 \oplus \dots \oplus V_r$ with $[V_1, V_i] = V_{i+1}$, $[V_1, V_r] = \{0\}$. It is not difficult to recognize that any homogeneous Carnot group is a Carnot group according to the classical definition. On the other hand, up to isomorphism, the opposite implication is also true (see [BU] for a detailed proof).

In order to give the main motivation beyond this paper, let us introduce the parabolic-type constant coefficients operators

$$\mathscr{H}_A = \mathscr{L}_A - \partial_t = \sum_{i,j=1}^{m} a_{i,j} X_i X_j - \partial_t, \quad A = (a_{i,j})_{i,j} \in \mathscr{M}_\Lambda,$$

where $\mathscr{M}_\Lambda$ is the set of $m \times m$ symmetric matrices A such that $\Lambda^{-1}|\xi|^2 \le \langle A\xi, \xi \rangle \le \Lambda|\xi|^2$ ($\Lambda \ge 1$ being a fixed constant). The Levi-parametrix method for the operators (1.1) requires the following estimates:

$$(1.3) \quad |X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_A(x,t) - X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma_B(x,t)|$$
$$\le \mathbf{c}_{\Lambda,p,q} \|A - B\|^{1/r} t^{-(Q+p+2q)/2} \exp\left(-\frac{d_{\mathbb{G}}^2(x)}{\mathbf{c}_\Lambda t}\right), \quad x \in \mathbb{G}, t > 0, A, B \in \mathscr{M}_\Lambda,$$

where we have denoted by Γ_A the fundamental solution for $\mathscr{H}_A$. Here and throughout the paper $d_{\mathbb{G}}$ is a *fixed* homogeneous norm on $\mathbb{G}$, i.e. a continuous function $d_{\mathbb{G}} : \mathbb{R}^N \to [0, \infty[$, smooth away from the origin, such that $d_{\mathbb{G}}(\delta_\lambda(x)) = \lambda\, d_{\mathbb{G}}(x)$, $d_{\mathbb{G}}(x^{-1}) = d_{\mathbb{G}}(x)$ and $d_{\mathbb{G}}(x) = 0$ iff $x = 0$. We remark that for every compact set $K \subset \mathbb{R}^N$ there exists a constant $\mathbf{c}_K > 0$ such that

$$(1.4) \quad d_{\mathbb{G}}(y^{-1} \circ x) \le \mathbf{c}_K |x - y|^{1/r}, \quad \forall x, y \in K.$$

A natural question in approaching (1.3) (whose proof will be given in [BLU]) is to ask whether the sub-Laplacians $\mathscr{L}_A$'s are all diffeomorphic to the canonical operator $\Delta_{\mathbb{G}}$, via a change of variables. This naïve idea is straightforwardly fitting in the case of constant coefficients elliptic operators. Otherwise the problem seems to be nontrivial. Here we point out that the core of the question lies inside the structure of the Lie algebra generated by the vector fields $\{X_i\}_i$. Indeed in Section 2, we prove that such change of variables always exists if $\mathbb{G}$ is a *free* Carnot group. We say that $\mathbb{G}$ is free if its Lie algebra is isomorphic to $\mathfrak{f}_{m,r}$ (the free nilpotent Lie algebra of step r and with m generators), for some m and r.

Theorem 1.1. *Let* $\mathbb{G}$ *be a free homogeneous Carnot group and let* $A \in \mathscr{M}_\Lambda$. *Then, there exists a Lie group automorphism* T_A *of* $\mathbb{G}$ *such that*

$$\left(\sum_{j=1}^{m} (A^{1/2})_{i,j} X_j\right)(u \circ T_A) = (X_i u) \circ T_A, \quad i = 1, \ldots, m,$$

for every smooth function u. As a consequence, we also have

$$\mathscr{L}_A(u \circ T_A) = (\Delta_{\mathbb{G}} u) \circ T_A.$$

Moreover, T_A *has polynomial component functions and it commutes with the dilations of* $\mathbb{G}$.

We stress that the hypothesis $\mathbb{G}$ free in the above theorem *cannot be removed* (see Remark 2.4 and Remark 2.5). Theorem 1.1 allows us to obtain the fundamental solution Γ_A for $\mathscr{H}_A$ as the composition of T_A with the fundamental solution $\Gamma_{\mathbb{G}}$ for $\mathscr{H}_{\mathbb{G}}$ $(=\Delta_{\mathbb{G}} - \partial_t)$. Indeed, (if $\mathbb{G}$ is free) it turns out that

$$(1.5) \qquad \Gamma_A(x,t;\xi,\tau) = |\det \mathscr{J}_{T_A}(x)| \Gamma_{\mathbb{G}}(T_A(x),t;T_A(\xi),\tau), \quad x,\xi \in \mathbb{R}^N, t,\tau \in \mathbb{R},$$

($\mathscr{J}_{T_A}$ denotes the Jacobian matrix of T_A). A crucial step in order to obtain the uniform estimates in (1.3) is then to establish *ad hoc* uniform estimates for T_A. This is done in Theorem 2.7. We remark that T_A plays the role of a classical change of basis, but in our context it is not necessarily a linear application (see Remark 2.3).

In order to handle the case of an arbitrary Carnot group $\mathbb{G}$, our main tool is to *lift* $\mathbb{G}$ to a free Carnot group $\tilde{\mathbb{G}}$ in such a way that $\Delta_{\mathbb{G}}$ is lifted to $\Delta_{\tilde{\mathbb{G}}}$. The lifting technique has been introduced by Rothschild and Stein [RS] in a very general setting and different proofs of their result have been given by several authors (see Hormander-Melin [HM], Folland [F2], Goodman [Go2]). However, we warn the reader that the result we need is not explicitly given in none of the above papers. Actually it can be deduced using the arguments in [RS]. Nevertheless we believe it could be of interest to furnish a direct and simple proof of it. This is done in [BU].

Theorem 1.2 [BU]. *Let $\mathbb{G}$ be a homogeneous Carnot group on $\mathbb{R}^N$. Then, there exists a free homogeneous Carnot group $\tilde{\mathbb{G}}$ on $\mathbb{R}^H$ (with $H \geq N$) such that, denoting by $\pi : \mathbb{R}^H \to \mathbb{R}^N$ the projection on the first N coordinates (up to a permutation of the coordinates of $\mathbb{R}^H$), we have*

$$\tilde{X}_i(u \circ \pi) = (X_i u) \circ \pi, \quad \forall u \in C^\infty(\mathbb{R}^N),$$

where $\sum_{i=1}^m X_i^2$ and $\sum_{i=1}^m \tilde{X}_i^2$ are the canonical sub-Laplacians $\Delta_{\mathbb{G}}$ and $\Delta_{\tilde{\mathbb{G}}}$, respectively. Moreover $\pi : \tilde{\mathbb{G}} \to \mathbb{G}$ is a Lie group morphism.

This theorem, together with Theorem 1.1, will allow to prove (1.3) in the general case. We briefly describe the way this is accomplished in [BLU]. A family of sub-Laplacians $\{\mathscr{L}_A\}$ on an arbitrary Carnot group $\mathbb{G}$ is lifted to a family of sub-Laplacians $\{\tilde{\mathscr{L}}_A\}$ on a free $\tilde{\mathbb{G}}$. The related family $\{\tilde{\Gamma}_A\}$ of fundamental solutions fulfills identity (1.5). By means of the uniform estimates of T_A, this allows to derive (1.3) for $\{\tilde{\Gamma}_A\}$. Finally, the fundamental solutions Γ_A are explicitly represented by integrating $\tilde{\Gamma}_A$ over the added variables. As a consequence estimate (1.3) can be obtained on $\mathbb{G}$.

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2 Construction and estimates of the diffeomorphism

Let $\mathbb{G}$ be a homogeneous Carnot group and let $\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2$ be the canonical sub-Laplacian on $\mathbb{G}$ $(m = N_1)$. Suppose it is given a positive-definite symmetric matrix

$A = (a_{i,j})_{1 \le i,j \le m}$. It is natural to ask the question whether there exists a diffeomorphism $T = T_A : \mathbb{G} \to \mathbb{G}$ such that, in the new coordinate system defined by T, the operator

$$(2.1) \qquad \mathscr{L}_A = \sum_{i,j=1}^{m} a_{i,j} X_i X_j$$

is turned into $\Delta_{\mathbb{G}}$, i.e. such that $\mathscr{L}_A(u \circ T) = (\Delta_{\mathbb{G}} u) \circ T$ for every $u \in C^\infty(\mathbb{G})$. We explicitly remark that any sub-Laplacian on $\mathbb{G}$ is of the form (2.1) and vice-versa any operator $\mathscr{L}_A$ in that form is a sub-Laplacian

$$(2.2) \qquad \mathscr{L}_A = \sum_{i=1}^{m} Y_i^2, \quad \text{with } Y_i = \sum_{j=1}^{m} (A^{1/2})_{i,j} X_j.$$

Hence it is natural to look for a diffeomorphism T turning Y_i into X_i, for $i = 1, \dots, m$. In the simple case when $X_i = \partial/\partial x_i$, this problem always has a solution, but when the X_i's have no constant coefficients, classical changes of basis may fail to apply. Moreover, simple examples show that T may not exist in the general case (see Remark 2.4 and Remark 2.5), but rather than restricting the form of the matrix A, it seems natural to make an additional assumption on the group $\mathbb{G}$. Indeed, we shall prove the existence of such T whenever $\mathbb{G}$ is a *free Carnot group* (see the definition below). The free group setting turns out to be the most natural one in order to generalize the classical Euclidean case. We stress that T is not necessarily a linear application, in the general case (see Remark 2.3 below).

We now recall the definition of $\mathfrak{f}_{m,r}$, the free nilpotent Lie algebra of step r with m (≥ 2) generators $x_1, \dots, x_m$. By definition, $\mathfrak{f}_{m,r}$ is the unique (up to isomorphism) nilpotent Lie algebra of step r generated by m of its elements $x_1, \dots, x_m$, such that for every nilpotent Lie algebra $\mathfrak{n}$ of step r and for every map φ from $\{x_1, \dots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) Lie algebra morphism $\tilde{\varphi}$ from $\mathfrak{f}_{m,r}$ to $\mathfrak{n}$ extending φ. The construction of such a Lie algebra $\mathfrak{f}_{m,r}$ is classical (see e.g. [V] and [VSC]). We say that the Carnot group $\mathbb{G}$ is a *free Carnot group* if its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{f}_{m,r}$, for some m and r. $\mathbb{R}^N$ equipped with the ordinary abelian structure is an example of free Carnot group. The Heisenberg group $\mathbb{H}^1$ is also a free Carnot group, while $\mathbb{H}^n$ is not free, for any $n \ge 2$, as can be seen by a dimensional argument. We refer the reader to [H] for the construction of a basis for $\mathfrak{f}_{m,r}$ (see also [GG]). The following simple lemma will be relevant for our purposes.

Lemma 2.1. *Let $\mathfrak{f}_{m,r}$ be the free nilpotent Lie algebra of step r and m generators $x_1, \dots, x_m$ and let $\varphi : \operatorname{span}\{x_1, \dots, x_m\} \to \operatorname{span}\{x_1, \dots, x_m\}$ be a bijective linear map. Then, there exists a (unique) Lie algebra automorphism $\tilde{\varphi}$ of $\mathfrak{f}_{m,r}$ extending φ.*

We are now in the position to prove our main result.

Theorem 2.2. *Let $\mathbb{G}$ be a free homogeneous Carnot group and let A be a given positive-definite symmetric matrix. Then, there exists a Lie group automorphism T_A of $\mathbb{G}$ such that (with the notations introduced in* (2.1) *and* (2.2)*)*

$$(2.3) \qquad Y_i(u \circ T_A) = (X_i u) \circ T_A, \quad i = 1, \dots, m,$$

$$(2.4) \qquad \mathscr{L}_A(u \circ T_A) = (\Delta_{\mathbb{G}} u) \circ T_A,$$

for every smooth function $u : \mathbb{G} \to \mathbb{R}$. Moreover, T_A has polynomial component functions and it commutes with the dilations of $\mathbb{G}$.

Proof. First of all, we remark that (2.4) is an immediate consequence of (2.3) which, in turn, follows by proving

$$(2.5) \qquad X_i(u \circ T_A) = \sum_{j=1}^{m} (A^{-1/2})_{i,j} (X_j u) \circ T_A, \quad i = 1, \dots, m.$$

Moreover, if $T = T_A$ is a Lie group morphism, then it is sufficient to prove that (2.5) holds at the origin. Indeed, suppose that

$$(2.6) \qquad X_i(u \circ T)(0) = \sum_{j=1}^{m} b_{i,j} (X_j u)(0), \quad \forall u \in C^\infty(\mathbb{R}^N), \quad i = 1, \dots, m,$$

where we have set for brevity $B = (b_{i,j})_{i,j} = A^{-1/2}$. We fix $y \in \mathbb{G}$, $v \in C^\infty(\mathbb{R}^N)$ and we apply (2.6) to $u(x) = v(T(y) \circ x)$. Since T is a Lie group morphism and X_i is left-invariant, we have

$$\begin{aligned} X_i(v \circ T)(y) &= X_i(v(T(y \circ \cdot)))(0) = X_i(v(T(y) \circ T(\cdot)))(0) \\ &= \sum_{j=1}^{m} b_{i,j} X_j(v(T(y) \circ \cdot))(0) = \sum_{j=1}^{m} b_{i,j} (X_j v)(T(y)), \end{aligned}$$

i.e. (2.5) holds.

We now observe that, by the chain rule and since B is symmetric, (2.6) is equivalent to

$$(2.7) \qquad \mathscr{J}_T(0)(XI)(0) = (XI)(0)B.$$

Here we have denoted by I the identity map on $\mathbb{G}$, by $XI = (X_1 I \cdots X_m I)$ the $(N \times m)$-matrix whose i-th column is $X_i I$ and by $\mathscr{J}_T$ the Jacobian matrix of T. We now construct a Lie group automorphism T of $\mathbb{G}$ satisfying (2.7). This will complete the proof of the first statement of the theorem. Consider the linear map defined as follows

$$\varphi : \text{span}\{X_1, \dots, X_m\} \to \text{span}\{X_1, \dots, X_m\}, \quad X_i \mapsto \sum_{j=1}^{m} b_{i,j} X_j.$$

Since $X_1, \dots, X_m$ are linearly independent, φ is well posed. Moreover, B being an invertible matrix, φ is a bijective linear map. Since $\mathbb{G}$ is a free Carnot group and its Lie algebra $\mathfrak{g}$ is nilpotent of step r and generated by $\{X_1, \dots, X_m\}$, clearly $\mathfrak{g}$ is isomorphic to $\mathfrak{f}_{m,r}$. Then, by Lemma 2.1, there exists a unique Lie algebra automorphism $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}$ extending φ. For the simplicity of notations, we also denote $\tilde{\varphi}$ by φ. We are now in position to define T. We set

$$(2.8) \qquad T : \mathbb{G} \overset{\text{Log}}{\longrightarrow} \mathfrak{g} \overset{\varphi}{\longrightarrow} \mathfrak{g} \overset{\text{Exp}}{\longrightarrow} \mathbb{G}, \quad T := \text{Exp} \circ \varphi \circ \text{Log},$$

where Exp denotes the exponential map and Log its inverse function. T is a Lie group automorphism of $\mathbb{G}$: indeed, when $\mathfrak{g}$ is equipped with the Campbell-Hausdorff group law, Exp, Log and any Lie algebra morphism of $\mathfrak{g}$ are Lie group morphisms.

Let us prove the matrix identity (2.7). We recall that, by definition,

$$(XI)(0) = \begin{pmatrix} \mathbb{I}_m \\ 0 \end{pmatrix}$$

where $\mathbb{I}_m$ is the identity matrix of order m and 0 is the null matrix of order $(N - m) \times m$. The following notation will be used henceforth: we call the *Jacobian basis* of $\mathfrak{g}$ the basis of vector fields in $\mathfrak{g}$ agreeing at the origin with the coordinate partial derivatives. With the choice of this basis, we have $\mathscr{J}_{\text{Log}}(0) = (\mathscr{J}_{\text{Exp}}(0))^{-1} = \mathbb{I}_N$, whence $\mathscr{J}_T(0) = \mathscr{J}_{\text{Exp}}(0)\mathscr{J}_\varphi(0)\mathscr{J}_{\text{Log}}(0) = \mathscr{J}_\varphi(0)$. Thus, in order to prove (2.7), it is enough to prove that the $(N \times m)$-matrix of the first m-columns of $\mathscr{J}_\varphi(0)$ is equal to

$$\begin{pmatrix} \mathbb{I}_m \\ 0 \end{pmatrix} B = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

which straightforwardly follows from the definition of φ and from the fact that $X_1, \dots, X_m$ are the first m vectors of the Jacobian basis.

We now turn to the proof of the last statement of the theorem. We recall that Exp and Log have polynomial components. Hence T has polynomial component functions since φ is linear. Finally, we prove that T commutes with δ_λ, the dilations on $\mathbb{G}$. Let us denote by $Z_1^{(1)}, \dots, Z_{N_1}^{(1)}, \dots, Z_1^{(r)}, \dots, Z_{N_r}^{(r)}$ the Jacobian basis of $\mathfrak{g}$ and let δ_λ also denote the map

$$\mathfrak{g} \ni \sum_{i=1}^{r} \sum_{j=1}^{N_i} \xi_j^{(i)} Z_j^{(i)} \mapsto \sum_{i=1}^{r} \sum_{j=1}^{N_i} \lambda^i \xi_j^{(i)} Z_j^{(i)} \in \mathfrak{g}.$$

With this notation, both Exp and Log commute with δ_λ. Thus, we only have to show $\varphi \circ \delta_\lambda = \delta_\lambda \circ \varphi$. Now, $Z_j^{(k)}$ is a δ_λ-homogeneous vector field of degree k, hence a homogeneous Lie polynomial of degree k in the generators $X_1, \dots, X_m$. By the definition of the Lie algebra morphism φ, $\varphi(Z_j^{(k)})$ is a homogeneous Lie polynomial of degree k in $X_1, \dots, X_m$ as well. Hence, there exist scalars $c_{i,j}^{(k)}$ such that

$$\varphi(Z_j^{(k)}) = \sum_{i=1}^{N_k} c_{i,j}^{(k)} Z_i^{(k)}. \tag{2.9}$$

Hence, for every $k = 1, \dots, r$ and $j = 1, \dots, N_k$ we have

$$(\varphi \circ \delta_\lambda)(Z_j^{(k)}) = \varphi(\lambda^k Z_j^{(k)}) = \lambda^k \sum_{i=1}^{N_k} c_{i,j}^{(k)} Z_i^{(k)} = \sum_{i=1}^{N_k} c_{i,j}^{(k)} \delta_\lambda (Z_i^{(k)}) = (\delta_\lambda \circ \varphi)(Z_j^{(k)}).$$

This completes the proof. □

Remark 2.3. The diffeomorphism T_A constructed in Theorem 2.2 may fail to be linear.

Proof. We consider the free homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^3, \circ, \delta_\lambda)$, where $x \circ y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2)$ and $\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. The Jacobian basis for $\mathfrak{g}$ is given by $X_1 = \partial_1$, $X_2 = \partial_2 + x_1 \partial_3$, $X_3 = [X_1, X_2] = \partial_3$. Moreover, we have

$$\mathrm{Exp}(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = \left(\xi_1, \xi_2, \xi_3 + \frac{1}{2}\xi_1 \xi_2\right),$$

$$\mathrm{Log}(x) = x_1 X_1 + x_2 X_2 + \left(x_3 - \frac{1}{2} x_1 x_2\right) X_3.$$

Let now $B = (b_{i,j})_{i,j \le 2}$ be an assigned symmetric matrix. Since $\mathbb{G}$ is free, there exists a unique Lie algebra morphism φ from $\mathfrak{g}$ into itself which maps the generators X_1, X_2 respectively in $b_{1,1} X_1 + b_{1,2} X_2$, $b_{1,2} X_1 + b_{2,2} X_2$. In Jacobian coordinates, φ is represented by the block-diagonal matrix $\mathrm{diag}\{B, \det B\}$. We now set $T = \mathrm{Exp} \circ \varphi \circ \mathrm{Log}$. A direct computation gives

$$T(x) = \Big(b_{1,1} x_1 + b_{1,2} x_2, b_{1,2} x_1 + b_{2,2} x_2,$$
$$\det(B) x_3 + \frac{1}{2} b_{1,1} b_{1,2} x_1^2 + \frac{1}{2} b_{1,2} b_{2,2} x_2^2 + b_{1,2}^2 x_1 x_2\Big).$$

In particular, if we choose $b_{1,1} = 2$, $b_{1,2} = b_{2,1} = 1$, $b_{2,2} = 4$, we obtain $T(x) = (2x_1 + x_2, x_1 + 4x_2, 7x_3 + x_1^2 + 2x_2^2 + x_1 x_2)$. With the notation in the proof of Theorem 2.2, we see that, choosing $A = B^{-2}$, then the diffeomorphism $T = T_A$ turns $Y_1 = \frac{4}{7} X_1 - \frac{1}{7} X_2$ and $Y_2 = -\frac{1}{7} X_1 + \frac{2}{7} X_2$ respectively into X_1 and X_2. As a consequence, T_A turns the sub-Laplacian $\mathscr{L}_A = \frac{17}{49} X_1^2 - \frac{6}{49} X_1 X_2 - \frac{6}{49} X_2 X_1 + \frac{5}{49} X_2^2$ into the canonical sub-Laplacian $\Delta_{\mathbb{G}} = X_1^2 + X_2^2$. We remark that T_A is not linear. □

We observe that the group $\mathbb{G}$ in the above remark is isomorphic to $\mathbb{H}^1$. However, on $\mathbb{H}^1$, T_A is always linear. This follows from the linearity of Exp in this latter case.

Remark 2.4. If $\mathbb{G}$ is not free, a diffeomorphism T_A satisfying (2.3) of Theorem 2.2 may not exist.

Proof. We consider the group $\mathbb{G} = \mathbb{H}^2$, the Heisenberg group on $\mathbb{R}^5$. If the point of $\mathbb{H}^2$ is denoted by (a, b, c), with $a, b \in \mathbb{R}^2$, $c \in \mathbb{R}$, we have

$$(a, b, c) \circ (\alpha, \beta, \gamma) = (a + \alpha, b + \beta, c + \gamma + 2\langle b, \alpha\rangle - 2\langle a, \beta\rangle),$$

$$\delta_\lambda(a, b, c) = (\lambda a, \lambda b, \lambda^2 c),$$

and the canonical sub-Laplacian on $\mathbb{H}^2$ is given by $\sum_{j=1}^2 (A_j^2 + B_j^2)$, where

$$A_j = \partial_{a_j} + 2b_j\partial_c, \quad B_j = \partial_{b_j} - 2a_j\partial_c, \quad j = 1, 2.$$

$(\mathbb{H}^2, \circ, \delta_\lambda)$ is then a homogeneous Carnot group of step 2 with 4 generators. The dimension of the Lie algebra of $\mathbb{H}^2$ is 5 whereas $\dim(\mathfrak{f}_{4,2}) = 10$, hence $\mathbb{H}^2$ is not free. We observe that there does not exist any diffeomorphism on $\mathbb{H}^2$ turning the set of vector fields $\mathscr{Y} = \{A_1 + B_2, A_2, B_1, B_2\}$ into the set $\mathscr{X} = \{A_1, A_2, B_1, B_2\}$. Indeed, each vector field in $\mathscr{X}$ has exactly one non-vanishing commutator with any other vector field in $\mathscr{X}$; on the contrary $A_1 + B_2 \in \mathscr{Y}$ has two non-vanishing commutators with the other vector fields in $\mathscr{Y}$. □

Remark 2.5. If $\mathbb{G}$ is not free, a diffeomorphism T_A satisfying (2.4) of Theorem 2.2 may not exist.

Proof. We only give a sketch of the proof which makes use of some Liouville-type theorems for sub-Laplacians (see [BL]). Our counterexample is rather elaborated. First, if $A = (a_{i,j})_{i,j \le m}$ is a fixed symmetric matrix and $X_i = \sum_{k=1}^N \alpha_k^{(i)}(x)\partial_k$ $(i = 1, \dots, m)$ are smooth vector fields on $\mathbb{R}^N$, then a map $T \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ satisfies

$$\text{(2.10)} \qquad \sum_{i,j=1}^m a_{i,j} X_i X_j (u \circ T) = \left(\sum_{i=1}^m X_i^2 u\right) \circ T, \quad \forall u \in C^\infty(\mathbb{R}^N)$$

if and only if the following system of quasi-linear PDE's is satisfied

$$\text{(2.11)} \qquad \begin{cases} \displaystyle\sum_{i,j=1}^m a_{i,j} X_i X_j T_k = \sum_{i=1}^m (X_i \alpha_k^{(i)}) \circ T, & \forall k = 1, \dots, N, \\ \displaystyle\sum_{i,j=1}^m a_{i,j} X_i T_l X_j T_k = \sum_{i=1}^m (\alpha_k^{(i)} \alpha_l^{(i)}) \circ T, & \forall k, l = 1, \dots, N. \end{cases}$$

We now consider the special case of the Heisenberg group $\mathbb{H}^2$ (whose points are denoted by $(x_1, \dots, x_5)$). With the natural choice of $X_1, \dots, X_4$, from the above system it follows that $\mathscr{L}_A T_k = 0$ $(k = 1, \dots, 5)$ and $\mathscr{L}_A((T_k)^2) = 2$ $(k = 1, \dots, 4)$. If A is positive definite, from [BL, Theorem 1.3] and [BL, Theorem 1.4], it follows that T_k $(k = 1, \dots, 4)$ is linear and only depends on the first four variables (we are supposing $T(0) = 0$, which is non-restrictive). We set $M = \partial(T_1, \dots, T_4)/\partial(x_1, \dots, x_4)$. By analogous arguments it then follows that T_5 is a polynomial of the form

$$T_5(x) = c_5 x_5 + \langle (x_1, x_2, x_3, x_4), C \cdot {}^t(x_1, x_2, x_3, x_4) \rangle, \quad \text{where } {}^tC = C.$$

After several direct computations, one obtains from (2.11) the following system

$$(2.12) \quad \begin{cases} M \cdot A \cdot {}^tM = \mathbb{I}_4 \\ \hat{C} = {}^tM \cdot \hat{I} \cdot M, \end{cases} \quad \text{where } \hat{C} = C + c_5 \hat{I} \text{ and } \hat{I} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}.$$

Now, since C is symmetric and ${}^tM \cdot \hat{I} \cdot M$ is skew-symmetric, the second equation in (2.12) holds only if C is the null matrix. Hence (2.12) gives in particular

$$(2.13) \quad \begin{cases} M \cdot A \cdot {}^tM = \mathbb{I}_4 \\ c_5 \hat{I} = {}^tM \cdot \hat{I} \cdot M. \end{cases}$$

We now choose A in the form $A = S \cdot {}^tS$, where S is a 4×4 invertible matrix. From the first equation in (2.13), we then have $M = O \cdot S^{-1}$, where O is an orthogonal matrix. As a consequence, the second equation in (2.13) yields

$$(2.14) \quad {}^tS \cdot \hat{I} \cdot S = \frac{1}{c_5} {}^tO \cdot \hat{I} \cdot O.$$

We finally choose $S = (s_{i,j})_{i,j \le 4}$ with $s_{1,1} = s_{2,2} = s_{3,3} = s_{4,4} = s_{1,4} = 1$, $s_{i,j} = 0$ otherwise. With this choice, (2.14) becomes

$$(2.15) \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = \frac{1}{c_5} {}^tO \cdot \hat{I} \cdot O, \quad \text{where } O \text{ is an orthogonal matrix.}$$

This gives a contradiction, as follows by comparing the eigenvalues of the matrix on the left hand-side of (2.15) to the eigenvalues of $\frac{1}{c_5}\hat{I}$. As a consequence, with the above choice of S and consequently of A, *it does not exist any map* $T \in C^2(\mathbb{H}^2, \mathbb{H}^2)$ *satisfying* (2.10). □

Throughout the end of this section, $\Lambda \ge 1$ will be a fixed constant and $\mathscr{M}_\Lambda$ will denote the set of symmetric $m \times m$ constant matrices A such that

(2.16) $\Lambda^{-1}|\xi|^2 \le \langle A\xi, \xi \rangle \le \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^m.$

Moreover, we shall denote by $\mathbf{c}_\Lambda$ any positive constant depending only on Λ and the structure of $\mathbb{G}$. Finally, $\|A\|$ will stand for the matrix norm $\max_{|\xi|=1}|A\xi|$. Simple arguments of Linear Algebra prove the following lemma.

Lemma 2.6. *With the above notations we have:*

(i) *If $A \in \mathscr{M}_\Lambda$, then $A^{-1} \in \mathscr{M}_\Lambda$.*

(ii) *If $A \in \mathscr{M}_\Lambda$ then*

(2.17) $\Lambda^{-1} \le \|A\|, \|A^{-1}\| \le \Lambda \quad$ *and* $\quad \Lambda^{-1/2} \le \|A^{1/2}\|, \|A^{-1/2}\| \le \Lambda^{1/2}.$

(iii) *For every $A, B \in \mathscr{M}_\Lambda$*

(2.18) $\|A^{1/2} - B^{1/2}\|, \|A^{-1} - B^{-1}\|, \|A^{-1/2} - B^{-1/2}\| \le \mathbf{c}_\Lambda \|A - B\|.$

We now establish some more properties of the diffeomorphism T_A constructed above, which are uniform in $A \in \mathscr{M}_\Lambda$. These properties will be used throughout [BLU] in order to establish uniform estimates for the fundamental solutions Γ_A of the operators $\mathscr{H}_A = \mathscr{L}_A - \partial_t$.

Theorem 2.7. *We set, for $x \in \mathbb{G}$, $J_A(x) = |\det \mathscr{J}_{T_A}(x)|$ (we recall that $\mathscr{J}_{T_A}$ denotes the Jacobian matrix of T_A). Then, J_A turns out to be constant in x. Moreover,*

(2.19) $(\mathbf{c}_\Lambda)^{-1} \le J_A \le \mathbf{c}_\Lambda,$

(2.20) $|J_{A_1} - J_{A_2}| \le \mathbf{c}_\Lambda \|A_1 - A_2\|,$

(2.21) $(\mathbf{c}_\Lambda)^{-1}\, d_{\mathbb{G}}(x) \le d_{\mathbb{G}}(T_A(x)) \le \mathbf{c}_\Lambda\, d_{\mathbb{G}}(x),$

(2.22) $d_{\mathbb{G}}((T_{A_2}(x))^{-1} \circ T_{A_1}(x)) \le \mathbf{c}_\Lambda \|A_1 - A_2\|^{1/r}\, d_{\mathbb{G}}(x),$

for every $A, A_1, A_2 \in \mathscr{M}_\Lambda$ and $x \in \mathbb{G}$. We recall that $d_{\mathbb{G}}$ is a fixed homogeneous norm on $\mathbb{G}$ and r is defined by (1.2).

Proof. Let $A \in \mathscr{M}_\Lambda$ and set $A^{-1/2} =: (b_{i,j})_{i,j}$. We recall that T_A is defined in (2.8), where $\varphi = \varphi_A$ is the Lie algebra automorphism of $\mathfrak{g}$ such that $\varphi_A(X_i) = \sum_{j=1}^m b_{i,j} X_j$, for every $i = 1, \dots, m$. As we showed in (2.9), φ_A is a linear map represented in Jacobian coordinates by a block-diagonal matrix whose entries are polynomials in the $b_{i,j}$'s. Moreover, since $(\varphi_A \circ \varphi_{A^{-1}})(X_i) = X_i$ for all $i = 1, \dots, m$, it easily follows that $(\varphi_A)^{-1} = \varphi_{A^{-1}}$ whence $(T_A)^{-1} = T_{A^{-1}}$.

By the definition of T_A, we have

$$J_A(x) = |\det \mathscr{J}_{\mathrm{Exp}}(\varphi_A(\mathrm{Log}\, x)) \det \mathscr{J}_{\varphi_A} \det \mathscr{J}_{\mathrm{Log}}(x)| = |\det \mathscr{J}_{\varphi_A}|,$$

since $\det \mathscr{J}_{\mathrm{Exp}} = \det \mathscr{J}_{\mathrm{Log}} \equiv 1$. Hence $J_A(x)$ does not depend on x.

Moreover $\det \mathscr{J}_{\varphi_A}$ is a polynomial of degree at most Q in the entries of $A^{-1/2}$. Then, by (2.17)

$$(2.23) \quad J_A \le \mathbf{c}_\Lambda, \quad \forall A \in \mathscr{M}_\Lambda.$$

From $(J_A)^{-1} = J_{A^{-1}}$ and (2.23), it immediately follows (2.19).

Now, let $A_1, A_2 \in \mathscr{M}_\Lambda$. We have already observed that J_A is in the form $J_A = |\Psi(A^{-1/2})|$, being Ψ a polynomial function in the entries of $A^{-1/2}$. Hence, by the mean value Theorem there exists $t \in [0,1]$ such that

$$|J_{A_1} - J_{A_2}| \le \|\mathscr{J}_\Psi(tA_1^{-1/2} + (1-t)A_2^{-1/2})\| \, |A_1^{-1/2} - A_2^{-1/2}|.$$

Therefore, recalling (2.17) and (2.18), we obtain (2.20).

Since T_A commutes with the dilations of $\mathbb{G}$ (see Theorem 2.2) and $d_{\mathbb{G}}$ is δ_λ-homogeneous of degree 1, it is sufficient to prove (2.21) and (2.22) only for $x \in S_{\mathbb{G}} := \{\xi \in \mathbb{G} \mid d_{\mathbb{G}}(\xi) = 1\}$. We remark that $S_{\mathbb{G}}$ is a compact subset of $\mathbb{G}$ not containing the origin. If we consider $\mathscr{M}_\Lambda$ as a subset of $\mathbb{R}^{m^2}$, from (2.17) it follows that also $\mathscr{M}_\Lambda$ is a compact set. We now define the following map:

$$(2.24) \quad \mathscr{M}_\Lambda \times S_{\mathbb{G}} \ni (A, \xi) \mapsto T_A(\xi).$$

It is not difficult to recognize that $T_A(\xi)$ is a polynomial function both in ξ and in the entries of $A^{-1/2}$. Hence, by (2.18) the function in (2.24) is continuous. In particular, $K := \{T_A(\xi) \mid A \in \mathscr{M}_\Lambda, \xi \in S_{\mathbb{G}}\}$ is a compact set. Moreover, $d_{\mathbb{G}}(T_A(\xi)) > 0$ for all $\xi \in S_{\mathbb{G}}$, being T_A a group automorphism. Since also $d_{\mathbb{G}}$ is continuous, we obtain

$$(\mathbf{c}_\Lambda)^{-1} \le d_{\mathbb{G}}(T_A(\xi)) \le \mathbf{c}_\Lambda, \quad \forall \xi \in S_{\mathbb{G}}, \forall A \in \mathscr{M}_\Lambda.$$

Finally, applying (1.4) to $d_{\mathbb{G}}$ and K, we obtain that

$$d_{\mathbb{G}}((T_{A_2}(x))^{-1} \circ T_{A_1}(x)) \le \mathbf{c}_\Lambda |T_{A_1}(x) - T_{A_2}(x)|^{1/r}, \quad \forall A_1, A_2 \in \mathscr{M}_\Lambda, \forall x \in S_{\mathbb{G}}.$$

On the other hand, arguing as in the proof of (2.20), it is easy to see that $|T_{A_1}(x) - T_{A_2}(x)| \le \mathbf{c}_\Lambda \|A_1 - A_2\|$. This completes the proof. □

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