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Nonlinear Liouville theorems for some critical problems on H-type groups[☆]

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Abstract

In this paper, we provide a non-existence result for a semilinear sub-elliptic Dirichlet problem with critical growth on the half-spaces of any group of Heisenberg-type. Our result improves a recent theorem in (Math. Ann. 315 (3) (2000) 453).

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1. Introduction and main results

In the last years, a great interest has been paid to non-linear Liouville theorems within degenerate-elliptic contexts. These results play a crucial rôle in applying blow-up techniques in order to obtain existence results for nonlinear subelliptic equations. Particularly relevant is the study of critical semilinear equations on stratified groups in connection with the Yamabe problem and the Webster scalar curvature problem on CR manifolds [11,24,25,33–35,42,43,47].

At the present time, the greatest part of the related literature is devoted to the Heisenberg group \mathbb{H}^k , the simplest case of a non-abelian stratified group. For instance, in a series of papers by Birindelli et al. [4,5] (see also [3,6,15]), sub-critical

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semilinear equations on \mathbb{H}^k are investigated: a priori estimates and existence theorems are established via non-existence results of “Gidas and Spruck type”. We also refer to the recent papers [7,18] for other non-linear Liouville theorems. A different approach is followed in some papers by Lanconelli and one of the authors [38–40,45], where non-linear Liouville theorems in \mathbb{H}^k are established for variational solutions. The aim of these papers was to provide some basic ingredients for studying semilinear problems with critical growth by means of variational techniques. Following this project, existence of solutions has been obtained in [17] making use of the uniqueness results of Jerison and Lee [34] and of the non-existence results in [39,45], for critical problems at infinity on unbounded domains of \mathbb{H}^k .

It would be interesting to deal with the case of general stratified groups \mathbb{G} , whose relevance is highlighted by the celebrated paper of Rothschild and Stein [44]. The variational setting seems to be appropriate in order to obtain existence results also in this wider case, provided suitable non-linear Liouville theorems are established. More precisely, in using variational techniques, one is led to characterize the energy levels of the variational solutions of the following semilinear Dirichlet problem with critical growth

$$\begin{cases} -\Delta_{\mathbb{G}} u = u^{\frac{Q+2}{Q-2}}, \\ u \in S_0^1(\Omega), \quad u > 0, \end{cases} \quad (1.1)$$

when Ω is a half-space of \mathbb{G} or the whole space. Here, $\Delta_{\mathbb{G}}$ is a sub-Laplacian on \mathbb{G} , Q denotes the homogeneous dimension of \mathbb{G} and $S_0^1(\Omega)$ is the appropriate subelliptic Sobolev space (we refer to Section 2 for all the notation and definitions).

The classical analogue of problem (1.1), when \mathbb{G} is the Euclidean group $(\mathbb{R}^N, +)$ and $\Delta_{\mathbb{G}}$ is the ordinary Laplace operator, has been intensively studied starting from the early 80s (see e.g., [1,2,13,19,41]). With respect to the classical setting, uniqueness and non-existence results for (1.1) present new and significant difficulties, even in the “simplest” case of the Heisenberg group \mathbb{H}^k [39,45]. These difficulties are mainly due to the lack of good a priori estimates for the Lie derivatives of the solutions along the directions of higher commutators. The case of general stratified groups presents further complications and, at the authors’ knowledge, only very partial results have been obtained. This is true even in the case of the so-called H-type groups, a remarkable class of stratified groups of step two introduced by Kaplan [36] and widely studied in the latest literature. In the same paper [36, Eq. (17)], Kaplan exhibited an explicit (cylindrically symmetric) solution to (1.1) when \mathbb{G} is a H-type group and $\Omega = \mathbb{G}$. Moreover, in the recent paper [28], Garofalo and Vassilev have established a uniqueness result for cylindrically symmetric solutions to (1.1) when \mathbb{G} is a H-type group of Iwasawa-type and Ω is the whole \mathbb{G} . Furthermore, in [27] Garofalo and Vassilev deal with the non-existence problem on half-spaces. The techniques in [27] are based on the use of a Kelvin-type transform: this forces the authors to work only in the case of the Iwasawa-type groups, a particular sub-class of H-type groups, where the Kelvin transform possesses several useful properties. Besides, in [27] only a certain class of half-spaces are covered.

The aim of this paper is to exhaust the problem for *all* half-spaces and *all* H-type groups. Indeed, we prove the following result.

Theorem 1.1. *Let \mathbb{G} be a H-type group and let Ω be any half-space of \mathbb{G} . Then the Dirichlet problem (1.1) has no solution.*

Some of the arguments in the proof of Theorem 1.1 (see Remark 4.5) also allow to prove a non-existence result on general step-two stratified groups, for a certain class of non-characteristic half-spaces.

Theorem 1.2. *Let \mathbb{G} be a step-two stratified group. Let Ω be any half-space of \mathbb{G} whose boundary is parallel to the center of \mathbb{G} . Then the Dirichlet problem (1.1) has no solution.*

We explicitly remark that not every non-characteristic half-space of the step-two group \mathbb{G} has the form in the assertion of Theorem 1.2, if \mathbb{G} is not a H-type group.

Though our techniques in approaching Theorem 1.1 are inspired by the ideas contained in the papers [39,45], we stress that the case of general H-type groups present several new difficulties. Broadly speaking, these complications are mainly due to the structure of the second layer in the stratification of the Lie algebra of \mathbb{G} , which (when the group is not \mathbb{H}^k) always has dimension strictly larger than one. In particular, the different geometry of \mathbb{G} makes it harder to construct explicit barrier functions. Moreover, one has to face with the more general form of the group composition law at different levels.

We next give a descriptive plan of the paper. The core of our proof consists in finding suitable asymptotic estimates for the second-layer derivatives of the solutions u of (1.1), which will allow us to apply some general Rellich and Pohozaev-type identities. After recalling a few basic notation and definitions (Section 2), in Section 3 we obtain asymptotic estimates at infinity of u in terms of the fundamental solution of $\Delta_{\mathbb{G}}$. The main tool here is a result in [46], that some summability properties of u allow to apply. The proof of the needed L^p estimates is post-posed in Appendix B: we use boot-strap and iteration techniques inspired to those of Brezis and Kato [12]; however, since we aim to obtain global L^p summability of u with p strictly lower than the critical exponent, a significant modification of the standard machinery is needed (see Theorem 8.4).

We henceforth have to distinguish between two classes of half-spaces, whose different geometric structures require ad hoc approaches. In Section 4, the needed estimates for the second-layer derivatives of u are obtained in the case of non-characteristic half-spaces Ω . The main argument (adapted from an idea introduced in [39]) is based on the representation of the $\Delta_{\mathbb{G}}$ -harmonic part of such derivatives as the limit of a sequence of iterated mean-value operators modelled on the geometry of Ω . In Section 5, the characteristic case is investigated. A rather elaborated argument is exploited in obtaining the needed estimates. This argument is based on the delicate construction of explicit barrier functions. We highlight that the lack of compactness

of the characteristic set adds remarkable complications in this construction, in comparison to the case of \mathbb{H}^k . Section 6 is devoted to the derivation of the needed Pohozaev-type identities and to the conclusion of the proof of Theorem 1.1. Finally, in Appendix A we collect a few results on H-type groups. In particular we give an “explicit” characterization of such groups (see Theorem 7.2, see also Remark 7.4) which turns out to be very convenient in an analytical framework. As a consequence, we are also able to write explicit formulas for the sub-Laplacian on a H-type group (see (7.11), (7.12), (7.13)).

2. Notation and definitions

The aim of this section is to provide the basic notation and definitions about stratified groups (and in particular about H-type groups) that we shall use throughout the paper. We start by giving an operative definition of *Carnot group*. Our definition may seem slightly different from the ones given in literature, but it is indeed equivalent, as we observe below. Let \circ be an assigned Lie group law on \mathbb{R}^N . We suppose \mathbb{R}^N is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}). \tag{2.1}$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \dots + N_r = N$. We denote by \mathfrak{g} the Lie algebra of (\mathbb{R}^N, \circ) i.e., the Lie algebra of \circ -left-invariant vector fields on \mathbb{R}^N . For $i = 1, \dots, N_1$, let X_i be the (unique) vector field in \mathfrak{g} that agrees at the origin with $\partial/\partial x_i^{(1)}$, i.e., $X_i(x) = (d/dh)_{h=0}(x \circ (he_i))$ (where e_i is the i th versor of the canonical basis of \mathbb{R}^N). We make the following assumption:

the Lie algebra generated by X_1, \dots, X_{N_1} is the whole \mathfrak{g} .

With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a (*homogeneous*) *Carnot group*. We also say that \mathbb{G} is of *step* r and has $m := N_1$ *generators*. We denote by $Q = \sum_{j=1}^r jN_j$ the *homogeneous dimension* of \mathbb{G} . The (*canonical*) *sub-Laplacian* on \mathbb{G} is the second-order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2.$$

We explicitly remark that in literature a Carnot group (or *stratified group*) \mathbb{G} is defined as a connected and simply connected Lie group whose Lie algebra \mathfrak{g} admits a *stratification* $\mathfrak{g} = \mathfrak{G}_1 \oplus \dots \oplus \mathfrak{G}_r$ with $[\mathfrak{G}_1, \mathfrak{G}_i] = \mathfrak{G}_{i+1}$, $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$ (we say that \mathfrak{G}_i is the i th layer of the stratification). It is not difficult to recognize that any homogeneous Carnot group is a Carnot group according to the classical definition. On the other hand, up to isomorphism, the opposite implication is also true (see [10])

for a detailed proof). Our operative definition of homogeneous Carnot group is not only more convenient to deal with in an analytic context, but also allows us to obtain a somewhat explicit knowledge of the group law \circ . For example, N -dimensional Carnot groups of step *two* and m generators are characterized by being (canonically isomorphic to) (\mathbb{R}^N, \circ) with the following Lie group law ($N = m + n$, $x^{(1)} \in \mathbb{R}^m$, $x^{(2)} \in \mathbb{R}^n$)

$$(x^{(1)}, x^{(2)}) \circ (y^{(1)}, y^{(2)}) = \left(\begin{array}{l} x_j^{(1)} + y_j^{(1)}, \quad j = 1, \dots, m \\ x_j^{(2)} + y_j^{(2)} + \frac{1}{2} \langle x^{(1)}, U^{(j)} y^{(1)} \rangle, \quad j = 1, \dots, n \end{array} \right), \quad (2.2)$$

where the $U^{(j)}$'s are $m \times m$ linearly independent skew-symmetric matrices (see Remark 7.4).

We next give a list of known results about homogeneous Carnot groups. Since X_1, \dots, X_m generate the whole \mathfrak{g} (which has rank N at any point), $\Delta_{\mathbb{G}}$ satisfies the Hörmander's hypoellipticity condition

$$\text{rank}(\text{Lie}\{X_1, \dots, X_m\}(x)) = N \quad \forall x \in \mathbb{R}^N.$$

Moreover, the vector fields X_1, \dots, X_m are homogeneous of degree one w.r.t. δ_λ and X_j^* (the adjoint operator of X_j) is $-X_j$. In particular, $\Delta_{\mathbb{G}}$ is a self-adjoint operator in divergence form. Moreover, the Lebesgue measure is invariant w.r.t. the left and right translations on \mathbb{G} and $\text{meas}(\delta_\lambda(E)) = \lambda^Q \text{meas}(E)$ for any measurable set E . In the sequel, we denote by $\nabla_{\mathbb{G}} = (X_1, \dots, X_m)$ the subelliptic gradient operator related to the sub-Laplacian $\Delta_{\mathbb{G}}$. If $\Omega \subseteq \mathbb{G}$ is a smooth open set, we recall that the *characteristic set* of Ω is the set

$$\{x \in \partial\Omega \mid X_i(x) \in T_x(\partial\Omega), \quad i = 1, \dots, m\},$$

$T_x(\partial\Omega)$ being the tangent space to $\partial\Omega$ at the point x . We explicitly remark that in our setting the problem of regularity of solutions up to the boundary is very delicate, especially near characteristic points. In the classical paper [37], Kohn and Nirenberg proved a general result of regularity away from the characteristic set. Moreover, in the papers [31,32], Jerison investigated in a more explicit way the case of Heisenberg groups \mathbb{H}^k . In particular, the following assertion is proved in [31] when $\mathbb{G} = \mathbb{H}^k$ (here $\Gamma^{j,\alpha}$ denotes the appropriate Folland and Stein Hölder space).

Let $\Omega \subset \mathbb{G}$ be a smooth domain and consider $f \in \Gamma_{\text{loc}}^{j,\alpha}(\overline{\Omega})$, $j \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$. For every $x_0 \in \partial\Omega$ not belonging to the characteristic set of Ω , there exists a neighborhood U of x_0 such that a solution u to $\Delta_{\mathbb{G}} u = f$ in Ω , $u = 0$ on $\partial\Omega$, belongs to $\Gamma^{j+2,\alpha}(\overline{\Omega} \cap U)$.

In [27] it is conjectured that the same result holds true for bounded domains of arbitrary Carnot groups \mathbb{G} . Indeed, this is a very natural conjecture, in light of the results of Kohn and Nirenberg, but no precise reference is provided in literature. Hence, following the analogous assumption made in [27, (2.8)], we also shall assume such result (when \mathbb{G} is a H-type group, see the definition below). We do not insist on this topic here, since the aim of this paper is to focus on other kind of problems.

When $Q \geq 3$ (which will always be assumed throughout the paper), Carnot groups possess the following remarkable property: there exists a homogeneous norm d on \mathbb{G} such that

$$\Gamma(x, y) = d^{2-Q}(y^{-1} \circ x), \quad x, y \in \mathbb{G}, \tag{2.3}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$ (see [21,23]; see also [9] for a detailed proof). We recall that a *homogeneous norm* on \mathbb{G} is a continuous function $d : \mathbb{G} \rightarrow [0, \infty)$, smooth away from the origin, such that $d(\delta_\lambda(x)) = \lambda d(x)$, $d(x^{-1}) = d(x)$, and $d(x) = 0$ iff $x = 0$. Hereafter, we also denote $d(y^{-1} \circ x)$ by $d(x, y)$. The following *quasi-triangle inequality* holds for every homogeneous norm d on \mathbb{G} ,

$$d(x, y) \leq \beta(d(x, z) + d(z, y)), \quad x, y, z \in \mathbb{G}, \tag{2.4}$$

for a suitable constant β . In the sequel, we shall denote by $B_d(x, r)$ the d -ball with radius $r > 0$ and center $x \in \mathbb{G}$. We trivially have $B_d(x, r) = x \circ \delta_r(B_d(0, 1))$ and $|B_d(x, r)| = r^Q |B_d(0, 1)|$.

We now recall the definition of group of *Heisenberg-type* (H-type, henceforth). A *H-type* group is a Carnot group \mathbb{G} of step two with the following property: the Lie algebra \mathfrak{g} of \mathbb{G} is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that, if \mathfrak{z} is the center of \mathfrak{g} , then $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and moreover, for every fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \rightarrow \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle \quad \forall w \in \mathfrak{z}^\perp,$$

is an orthogonal map whenever $\langle z, z \rangle = 1$. Similar to general Carnot groups, any H-type group can be identified with a homogeneous Carnot group with explicit properties, as we state below (all the details can be found in Appendix A). Indeed, if $m = \dim(\mathfrak{z}^\perp)$ and $n = \dim(\mathfrak{z})$, then the H-type group \mathbb{G} is characterized by being (canonically isomorphic to) \mathbb{R}^{m+n} with the following group law (throughout the paper, (x, t) denotes the point of the H-type group \mathbb{G} , being $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$)

$$(x, t) \circ (\xi, \tau) = \left(\begin{array}{l} x_j + \xi_j, \quad j = 1, \dots, m \\ t_j + \tau_j + \frac{1}{2} \langle x, U^{(j)} \xi \rangle, \quad j = 1, \dots, n \end{array} \right), \tag{2.5}$$

where the $U^{(j)}$'s are *orthogonal skew-symmetric* $m \times m$ matrices satisfying the following property:

$$U^{(r)} U^{(s)} + U^{(s)} U^{(r)} = 0, \quad \text{for every } r, s \in \{1, \dots, n\} \text{ with } r \neq s.$$

Moreover, the canonical sub-Laplacian on \mathbb{G} takes the following form:

$$\Delta_{\mathbb{G}} = \sum_{j=1}^m \left(\frac{\partial}{\partial x_j} \right)^2 + \frac{1}{4} |x|^2 \sum_{s=1}^n \left(\frac{\partial}{\partial t_s} \right)^2 + \sum_{s=1}^n \sum_{i,j=1}^m x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s}. \tag{2.6}$$

Furthermore, $(\mathbb{R}^{m+n}, \circ)$ with the family of dilations $\delta_\lambda(x, t) = (\lambda x, \lambda^2 t)$ is a homogeneous Carnot group. The homogeneous norm d realizing the fundamental

solution for $\Delta_{\mathbb{G}}$ as in (2.3) has the following remarkable explicit coordinate expression (see [36, Theorem 2]):

$$d(x, t) = c(|x|^4 + 16|t|^2)^{1/4}, \tag{2.7}$$

for a suitable constant $c > 0$. We explicitly remark that, whenever the center of a H-type group \mathbb{G} is one dimensional, then \mathbb{G} is canonically isomorphic to the Heisenberg group on \mathbb{R}^{m+1} (see Remark 7.7 in Appendix A). For this reason, since Theorem 1.1 for the Heisenberg group was proved in [39,45], throughout the paper we shall suppose $n \geq 2$. Furthermore, since m is always even (see the proof of Theorem 7.2 in Appendix A) and $m = 2$ gives back the Heisenberg group on \mathbb{R}^3 , it is non-restrictive to suppose that $Q = m + 2n \geq 8$. Motivated by our interest in half-spaces, we explicitly remark that the general half-space of the H-type group \mathbb{G} (here $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$)

$$\Pi = \{(x, t) \in \mathbb{G} \mid \langle a, x \rangle + \langle b, t \rangle > c\}$$

possesses characteristic points if and only if $b \neq 0$ (see Appendix A for the details).

We finally fix the notation for the Dirichlet problem (1.1). Let \mathbb{G} be an arbitrary Carnot group and let Q denote its homogeneous dimension. We set

$$2^\star = \frac{2Q}{Q-2}.$$

The exponent $2^\star - 1 = (Q + 2)/(Q - 2)$ is a critical exponent for the semilinear Dirichlet problem (1.1) as well as the exponent $(N + 2)/(N - 2)$ is critical for the classical semilinear Poisson equation in \mathbb{R}^N , $N \geq 3$. A basic rôle in the functional analysis on \mathbb{G} is played by the following Sobolev-type inequality (see for example [21,22]):

$$\|\varphi\|_{2^\star}^2 \leq S_Q \|\nabla_{\mathbb{G}} \varphi\|_2^2 \quad \forall \varphi \in C_0^\infty(\mathbb{G}), \tag{2.8}$$

for a suitable constant $S_Q > 0$ (whose best value, for the case of \mathbb{H}^k , has been determined in [34]). Hereafter $\|\cdot\|_p$ will denote the usual L^p -norm. If Ω is an open subset of \mathbb{G} , we shall denote by $S^1(\Omega)$ the Sobolev space of the functions $u \in L^{2^\star}(\Omega)$ such that $\nabla_{\mathbb{G}} u \in L^2(\Omega)$. The norm in $S^1(\Omega)$ is given by

$$\|u\|_{S^1(\Omega)} = \|u\|_{2^\star} + \|\nabla_{\mathbb{G}} u\|_2. \tag{2.9}$$

We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to (2.9). By means of (2.8), this norm is equivalent in $S_0^1(\Omega)$ to the norm generated by the inner product $\langle u, v \rangle_{S_0^1} = \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} v \rangle$. Thus $S_0^1(\Omega)$ is a Hilbert space. We emphasize that, for general unbounded domains, the space $S_0^1(\Omega)$ is not embedded in $L^2(\Omega)$. A solution

to the Dirichlet problem (1.1) is, by definition, a function $u \in S_0^1(\Omega)$, $u > 0$, such that

$$\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle = \int_{\Omega} u^{2^{\star}-1} \varphi \quad \forall \varphi \in S_0^1(\Omega). \tag{2.10}$$

We explicitly remark that every classical solution of the equation in (1.1) satisfies the integral identity (2.10) since $(\nabla_{\mathbb{G}})^{\star} = -\nabla_{\mathbb{G}}$. The exponent 2^{\star} is the critical Sobolev exponent for $\Delta_{\mathbb{G}}$ since, even if Ω is bounded, the continuous embedding $S_0^1(\Omega) \hookrightarrow L^{2^{\star}}(\Omega)$ is not compact.

3. Some asymptotic estimates

Throughout the sequel, Ω will be an open subset of an arbitrary Carnot group \mathbb{G} and u will denote any fixed solution of the boundary value problem (1.1). Moreover, d will denote the homogeneous norm on \mathbb{G} introduced in (2.3). The main result of this section is estimate (3.8) of u at infinity. Moreover, we give some preliminary results in order to provide asymptotic estimates for the derivatives of u in the first and second layer of stratification. We remark that the estimate of the second-layer derivatives will be derived under the assumption that \mathbb{G} has step two.

The starting point of our approach is given by the following a priori L^p estimates, which will be proved in Section 8.

Theorem 3.1. *We have $u \in L^p(\Omega)$ for every $p \in (2^{\star}/2, \infty]$ and u vanishes at infinity. Moreover, if Ω satisfies the following boundary regularity condition:*

$$\exists \delta, r_0 > 0 : |B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \quad \forall \xi \in \partial\Omega, \quad \forall r \in (0, r_0), \tag{3.1}$$

then u is a classical solution of

$$\begin{cases} -\Delta_{\mathbb{G}} u = u^{\frac{Q+2}{Q-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u(\xi) \rightarrow 0 & \text{as } d(\xi) \rightarrow \infty. \end{cases}$$

Furthermore, if we continue u on \mathbb{G} by setting $u = 0$ outside Ω , then there exists $\alpha \in (0, 1)$ such that $u \in \Gamma^{\alpha}(\mathbb{G})$.

We now fix some notation which will be used throughout the section. If u is set to be zero outside Ω and $f = u^{2^{\star}-1}$, we introduce the function

$$w = \Gamma * f : \mathbb{G} \rightarrow \mathbb{R}, \quad w(x) = \int_{\mathbb{G}} \Gamma(x, y) f(y) \, dy, \tag{3.2}$$

where Γ denotes the fundamental solution for $\Delta_{\mathbb{G}}$ as in (2.3). From the L^p estimates of u in Theorem 3.1, it follows that $f \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. By means of the results of Rothschild and Stein [44], we get

$$w \in L^p(\mathbb{G}) \quad \forall p \in (2^\star/2, \infty), \quad -\Delta_{\mathbb{G}} w = f \text{ weakly in } \mathbb{G}. \tag{3.3}$$

Moreover, $w \geq 0$, $w \in C^\infty(\Omega)$, $\Delta_{\mathbb{G}}(u - w) = 0$ in Ω and $(u - w)^+ \in S_0^1(\Omega)$. Then $(u - w)^+ = 0$ in Ω , i.e.

$$0 \leq u \leq w \quad \text{in } \Omega. \tag{3.4}$$

We first need the following result.

Lemma 3.2. *For every $s \in (0, Q - 2)$, we have $w(x) = \mathcal{O}(d(x)^{-s})$ as $d(x) \rightarrow \infty$.*

Proof. This is a consequence of Theorem 1.1 in [46]: according to this result, if v is a non-negative weak solution in O of $-\Delta_{\mathbb{G}} v \leq Vv$ (where O is an open neighborhood of infinity and the potential V has L^q -summability for every q in a neighborhood of $Q/2$) such that $v \in L^p(O)$ (being $p \geq 2^\star/2$) then we have $v(x) = \mathcal{O}(d(x)^{-Q/p})$ at infinity. Our lemma is then proved by choosing $O = \mathbb{G}$, $V = u^{\frac{Q+2}{Q-2}-1}$, $v = w$ and by taking into account Theorem 3.1 (in particular we exploit the non-trivial fact that $u \in L^p(\Omega)$ for $p \in (2^\star/2, 2^\star]$) and (3.3). \square

We explicitly remark that the proof of [46, Theorem 1.1] cited above is a consequence of some mean representation formulas related to $\Delta_{\mathbb{G}}$: we hereafter recall these formulas since we shall employ them in the proof of Proposition 3.7 below and in Section 4. If $\varphi \in C^2(\Omega)$ and $B_d(z, r) \Subset \Omega$, we have the following representation formula (for details, see [8,16]):

$$\varphi(z) = M_r(\varphi)(z) - N_r(\Delta_{\mathbb{G}}\varphi)(z), \tag{3.5}$$

where (here m_Q and n_Q are suitable positive constants only depending on Q)

$$\begin{aligned} M_r(\varphi)(z) &= \frac{m_Q}{r^Q} \int_{B_d(z,r)} K(z^{-1} \circ \zeta) \varphi(\zeta) \, d\zeta, \\ N_r(\psi)(z) &= \frac{n_Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(z,\varrho)} (d^{2-Q}(z^{-1} \circ \zeta) - \varrho^{2-Q}) \psi(\zeta) \, d\zeta \right) d\varrho. \end{aligned} \tag{3.6}$$

In (3.6), d is the homogeneous norm on \mathbb{G} introduced in (2.3) and $K := |\nabla_{\mathbb{G}} d|^2$. We explicitly remark that the kernel K is a non-negative smooth function on $\mathbb{G} \setminus \{0\}$, homogeneous of degree zero with respect to the dilations of \mathbb{G} .

We are now able to establish the following “optimal” asymptotic behavior for w and then to derive the analogous estimate for u .

Proposition 3.3. *We have $w(x) = \mathcal{O}(d(x)^{2-\mathcal{Q}})$, as $d(x) \rightarrow \infty$.*

Proof. From (3.4) and Lemma 3.2, choosing $s = \mathcal{Q}/(2^\star - 1)$ it follows that there exist $M, R > 0$ such that $f(y) \leq Md(y)^{s(2^\star - 1)} = Md(y)^\mathcal{Q}$ if $d(y) > R$. Hence, for $d(x) > 2\beta R$ (where β is the constant appearing in (2.4)) we obtain

$$\sup_{B_d\left(x, \frac{d(x)}{2\beta}\right)} f \leq M(2\beta)^\mathcal{Q} d(x)^{-\mathcal{Q}}, \tag{3.7}$$

which yields, since $f \in L^1(\mathbb{G})$ and by (2.3)

$$\begin{aligned} w(x) &= \left(\int_{d(x,y) \geq \frac{d(x)}{2\beta}} + \int_{d(x,y) < \frac{d(x)}{2\beta}} \right) \Gamma(x, y) f(y) \, dy \\ &\leq \frac{\mathbf{c}}{d(x)^{\mathcal{Q}-2}} \|f\|_1 + \frac{\mathbf{c}}{d(x)^\mathcal{Q}} \int_{B_d\left(x, \frac{d(x)}{2\beta}\right)} \frac{dy}{d(x, y)^{\mathcal{Q}-2}} = \frac{c_f}{d(x)^{\mathcal{Q}-2}}. \end{aligned}$$

This ends the proof. \square

Theorem 3.4. *Let u be a solution of (1.1). Then there exists $M > 0$ such that*

$$u(x) \leq M \min\{1, d(x)^{2-\mathcal{Q}}\}. \tag{3.8}$$

Proof. It directly follows from (3.4), $u \in L^\infty(\Omega)$ (see Theorem 3.1) and Proposition 3.3. \square

We now turn to prove some preliminary results in order to obtain the asymptotic estimates of the derivatives of u . First, in Proposition 3.6 we prove the estimate of the derivatives of w . Then, in Proposition 3.7 we give a general result for $\Delta_\mathbb{G}$ -harmonic functions which will be applied in estimating $v := w - u$ (the $\Delta_\mathbb{G}$ -harmonic part of u). We shall denote by \mathfrak{g} the Lie algebra of \mathbb{G} and by $\nabla_\mathbb{G} = (X_1, \dots, X_m)$ the subelliptic gradient operator related to the fixed sub-Laplacian $\Delta_\mathbb{G} = \sum_{i=1}^m X_i^2$ on \mathbb{G} . We recall that \mathfrak{g} admits the stratification $\mathfrak{g} = \mathfrak{G}_1 \oplus \dots \oplus \mathfrak{G}_r$, where $\mathfrak{G}_1 = \text{span}\{X_1, \dots, X_m\}$, $\mathfrak{G}_{i+1} = [\mathfrak{G}_1, \mathfrak{G}_i]$.

If Ω satisfies the boundary regularity condition (3.1), then by Theorem 3.1 and by the results in [44] it follows that there exists $\alpha \in (0, 1)$ such that (being $f = u^{2^\star - 1}$ and w as in (3.2))

$$f \in \Gamma^\alpha(\mathbb{G}), \quad w \in \Gamma_{\text{loc}}^{2+\alpha}(\mathbb{G}), \tag{3.9}$$

i.e., $XYw \in \Gamma_{\text{loc}}^\alpha(\mathbb{G})$ for every $X, Y \in \mathfrak{G}_1$. In the following Lemma, $\Gamma(x, y) = d^{2-\mathcal{Q}}(y^{-1} \circ x)$ denotes as usual the fundamental solution for $\Delta_\mathbb{G}$.

Lemma 3.5. *Let \mathbb{G} be a Carnot group of step two. Suppose $\Omega \subset \mathbb{G}$ is an open set satisfying (3.1). Then, for every $Y \in \mathfrak{G}_1$ and for every $Z \in \mathfrak{G}_2$, we have*

$$\begin{aligned}
 Yw(x_0) &= \int_{\mathbb{G}} Y\Gamma(x_0, y)f(y) \, dy, \\
 Zw(x_0) &= \int_{B_d(x_0, r)} Z\Gamma(x_0, y)(f(y) - f(x_0)) \, dy + \int_{\mathbb{G} \setminus B_d(x_0, r)} Z\Gamma(x_0, y)f(y) \, dy,
 \end{aligned}$$

for every $x_0 \in \mathbb{G}$ and $r > 0$.

Proof. Since the proofs of the representation of Yw and Zw follow the same lines, we only consider the latter case. We remark that the hypothesis on the step of \mathbb{G} only intervenes in this last case. Indeed, we shall use the following fact: recalling the notations introduced in Section 2 for groups of step two, we have $Z = Z_x = \sum_{j=1}^n z_j(\partial/\partial x_j^{(2)})$, for suitable constants z_1, \dots, z_n . As a consequence, considering the form of operation (2.2) in a group of step two, for any regular function g we have

$$Z_x(g(y^{-1} \circ x)) = -Z_y(g(y^{-1} \circ x)). \tag{3.10}$$

Let us now fix $x_0 \in \mathbb{G}$ and $r > 0$. If β denotes the constant appearing in the quasi-triangle inequality (2.4), we set $B = B_d(x_0, r/(2\beta))$, $B_0 = B_d(x_0, r)$, $f_0 = f\chi_{B_0}$, $f_1 = f - f_0$. Let $\eta \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\eta \equiv 0$ in $[0, 1]$, $\eta \equiv 1$ in $[2, \infty)$ and for every $\varepsilon > 0$ let us set $\eta_\varepsilon = \eta(d/\varepsilon)$; we shall also write $\eta_\varepsilon(x, y) = \eta_\varepsilon(y^{-1} \circ x)$. We finally set $w_0 = \Gamma * f_0$, $w_1 = \Gamma * f_1$, $w_{0,\varepsilon} = (\Gamma\eta_\varepsilon) * f_0$. Since f is bounded, it is immediate to verify that

$$\lim_{\varepsilon \rightarrow 0} w_{0,\varepsilon} = w_0 \quad \text{uniformly on the compact sets of } B. \tag{3.11}$$

Moreover, $w_{0,\varepsilon} \in C^\infty(B)$ and, for every $x \in B$ and $\varepsilon < r/(4\beta)$, we have (by means of (3.10))

$$\begin{aligned}
 Zw_{0,\varepsilon}(x) &= \int_{B_0} Z_x(\Gamma\eta_\varepsilon(x, y))f_0(y) \, dy \\
 &= \int_{B_0} Z_x(\Gamma\eta_\varepsilon(x, y))(f_0(y) - f_0(x)) \, dy - f_0(x) \int_{B_0} Z_y(\Gamma\eta_\varepsilon(x, y)) \, dy \\
 &= \int_{B_0} Z(\Gamma\eta_\varepsilon)(x, y)(f_0(y) - f_0(x)) \, dy \\
 &\quad - f_0(x) \int_{\partial B_0} \Gamma(x, y) \langle Z(y), \nu(y) \rangle \, d\sigma(y).
 \end{aligned}$$

The last equality follows from the left-invariance of Z and from the divergence theorem (we have denoted by $d\sigma$ the surface measure and by ν the outer unit normal

on ∂B_0). We set $\bar{w}_0 : B \rightarrow \mathbb{R}$,

$$\bar{w}_0(x) = \int_{B_0} Z\Gamma(x, y)(f_0(y) - f_0(x)) \, dy - f_0(x) \int_{\partial B_0} \Gamma(x, y) \langle Z(y), \nu(y) \rangle \, d\sigma(y).$$

From the Hölder continuity of f given by (3.9) and the estimate $|Z\Gamma| \leq c d^{-Q}$ (since Γ is δ_λ -homogeneous of degree $2 - Q$ and $Z \in \mathfrak{G}_2$), we obtain

$$\int_{B_0} |Z\Gamma(x, y)(f_0(y) - f_0(x))| \, dy \leq c \int_{B_0} d(x, y)^{\alpha-Q} < \infty,$$

which proves that \bar{w}_0 is well-posed on B . We now claim that

$$\lim_{\varepsilon \rightarrow 0} Z w_{0,\varepsilon} = \bar{w}_0 \quad \text{uniformly on the compact sets of } B. \tag{3.12}$$

Indeed, from the above representation of $Z w_{0,\varepsilon}$ and from the definition of \bar{w}_0 , we obtain

$$\begin{aligned} |\bar{w}_0(x) - Z w_{0,\varepsilon}(x)| &= \left| \int_{B_d(x, 2\varepsilon)} Z(\Gamma(1 - \eta_\varepsilon))(x, y)(f_0(y) - f_0(x)) \, dy \right| \\ &\leq c \int_{B_d(x, 2\varepsilon)} (\Gamma(x, y) \|\dot{\eta}\|_\infty |Zd(x, y)|\varepsilon^{-1} + |Z\Gamma(x, y)|) d^\alpha(x, y) \, dy \\ &\leq \tilde{c} \varepsilon^\alpha \int_{B_d(0, 2)} (d(x, y)^{\alpha-Q} + d(x, y)^{\alpha+1-Q}) \, dy, \end{aligned}$$

for every $x \in B$ and $\varepsilon < r/(4\beta)$ (here we used the fact that Zd and $Z\Gamma$ are δ_λ -homogeneous of degree -1 and $-Q$, respectively). From (3.11) and (3.12), we deduce that $Z w_0 = \bar{w}_0$ in B . On the other hand, since $f \in L^1(\mathbb{G})$ we have (differentiating under the integral sign)

$$Z w_1(x) = \int_{\mathbb{G} \setminus B_0} Z\Gamma(x, y) f_1(y) \, dy, \quad x \in B.$$

Finally, taking $x = x_0$ we obtain

$$\begin{aligned} Z w(x_0) &= Z(w_0 + w_1)(x_0) = \bar{w}_0(x_0) + Z(w_1)(x_0) \\ &= \int_{B_0} Z\Gamma(x_0, y)(f(y) - f(x_0)) \, dy + \int_{\mathbb{G} \setminus B_0} Z\Gamma(x_0, y) f(y) \, dy, \end{aligned}$$

since the divergence theorem ensures that

$$\begin{aligned} \int_{\partial B_0} \Gamma(x_0, y) \langle Z(y), v(y) \rangle \, d\sigma(y) &= cr^{2-Q} \int_{\partial B_0} \langle Z(y), v(y) \rangle \, d\sigma(y) \\ &= cr^{2-Q} \int_{B_0} Z(1) \, dy = 0. \end{aligned}$$

This completes the proof. \square

Proposition 3.6. *Let \mathbb{G} be a Carnot group of step two and suppose $\Omega \subset \mathbb{G}$ is an open set satisfying (3.1). Let w be as in (3.2). Then, for every $Y \in \mathfrak{G}_1$ and for every $Z \in \mathfrak{G}_2$, we have*

$$Yw(x) = \mathcal{O}(d(x)^{1-Q}), \quad Zw(x) = \mathcal{O}(d(x)^{2-Q}) \quad \text{as } d(x) \rightarrow \infty. \quad (3.13)$$

Proof. If $Y \in \mathfrak{G}_1$, from the first identity in Lemma 3.5, we get (since $f \in L^1(\mathbb{G})$ and $Y\Gamma$ is δ_λ -homogeneous of degree $1 - Q$)

$$\begin{aligned} |Yw(x)| &= \left| \int_{B_d(x,r)} Y\Gamma(x, y)f(y) \, dy + \int_{\mathbb{G} \setminus B_d(x,r)} Y\Gamma(x, y)f(y) \, dy \right| \\ &\leq c \sup_{B_d(x,r)} f \int_{B_d(x,r)} d(x, y)^{1-Q} \, dy + c \int_{\mathbb{G} \setminus B_d(x,r)} d(x, y)^{1-Q} f(y) \, dy \\ &\leq c'r \sup_{B_d(x,r)} f + c'r^{1-Q} \|f\|_1. \end{aligned} \quad (3.14)$$

Let α be as in (3.9) and set $\gamma = 4/(Q + 2 + \alpha)$. If $Z \in \mathfrak{G}_2$, from the second identity in Lemma 3.5, we get (since $\gamma < 1$, $f \in L^1 \cap L^\infty \cap \Gamma^\alpha(\mathbb{G})$ and $Z\Gamma$ is δ_λ -homogeneous of degree $-Q$)

$$\begin{aligned} |Zw(x)| &\leq c \int_{B_d(x,r)} d(x, y)^{-Q} |f(y) - f(x)|^\gamma \left(\sup_{B_d(x,r)} f \right)^{1-\gamma} \, dy \\ &\quad + c \int_{\mathbb{G} \setminus B_d(x,r)} d(x, y)^{-Q} f(y) \, dy \\ &\leq c' \left(\sup_{B_d(x,r)} f \right)^{1-\gamma} \int_{B_d(x,r)} d(x, y)^{\alpha\gamma-Q} \, dy + c'r^{-Q} \|f\|_1 \\ &= c'' r^{\alpha\gamma} \left(\sup_{B_d(x,r)} f \right)^{1-\gamma} + c'' r^{-Q}. \end{aligned} \quad (3.15)$$

From Theorem 3.4, it follows that $f = \mathcal{O}(d^{-Q-2})$ at infinity. Choosing $r = d(x)/(2\beta)$ in (3.14) and (3.15) (β as in the quasi-triangle inequality (2.4)), for large $d(x)$

we get respectively

$$\begin{aligned}
 |Yw(x)| &\leq c_f(d(x)^{-1-Q} + d(x)^{1-Q}) \leq c'_f d(x)^{1-Q}, \\
 |Zw(x)| &\leq c_f(d(x)^{\alpha_f} d(x)^{(\gamma-1)(Q+2)} + d(x)^{-Q}) \\
 &= c'_f(d(x)^{2-Q} + d(x)^{-Q}) \leq c''_f d(x)^{2-Q}.
 \end{aligned}$$

This ends the proof of the theorem. \square

In what follows, we call $\Delta_{\mathbb{G}}$ -harmonic on Ω any smooth function v such that $\Delta_{\mathbb{G}}v = 0$ in the open set Ω . The following result provides an estimate for the Lie derivatives of a general $\Delta_{\mathbb{G}}$ -harmonic function.

Proposition 3.7. *Let $X \in \mathfrak{g}$ be a Lie derivative of order k (i.e., $X \in \mathfrak{G}_k$). Let v be a $\Delta_{\mathbb{G}}$ -harmonic function on Ω . Then there exists a positive constant c such that*

$$|Xv(z)| \leq cr^{-k} \sup_{B_d(z,r)} |v|, \tag{3.16}$$

for every $B_d(z,r) \Subset \Omega$. The constant c only depends on X (and the structure of \mathbb{G}) and not on v , $r > 0$ or $z \in \Omega$.

Proof. We begin by recalling a variant of the representation formula (3.5). If $\varphi \in C^2(\Omega)$ and $B_d(z,r) \Subset \Omega$, we have the following representation formula on the annulus $B_d(z,r) \setminus \overline{B_d(z,r/2)}$ (for details, see [8,16])

$$\varphi(z) = \tilde{M}_r(\varphi)(z) - \tilde{N}_r(\Delta_{\mathbb{G}}\varphi)(z), \tag{3.17}$$

where (here \tilde{m}_Q and \tilde{n}_Q are suitable positive constants only depending on Q)

$$\begin{aligned}
 \tilde{M}_r(\varphi)(z) &= \frac{\tilde{m}_Q}{r^Q} \int_{r/2 < |z^{-1} \circ \zeta| < r} K(z^{-1} \circ \zeta) \varphi(\zeta) \, d\zeta \\
 \tilde{N}_r(\psi)(z) &= \frac{\tilde{n}_Q}{r^Q} \int_{r/2}^r \varrho^{Q-1} \left(\int_{B_d(z,\varrho)} (d^{2-Q}(z^{-1} \circ \zeta) - \varrho^{2-Q}) \psi(\zeta) \, d\zeta \right) d\varrho.
 \end{aligned} \tag{3.18}$$

In (3.18), K still denotes the kernel $K = |\nabla_{\mathbb{G}}d|^2$. If v is $\Delta_{\mathbb{G}}$ -harmonic in Ω , from (3.17) we obtain $v(z) = \tilde{M}_r(v)(z)$. We then fix $\vartheta \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\text{supp}(\vartheta) \subseteq (1/2, 1)$ and $\int_{\mathbb{R}} \vartheta = 1$. A simple superposition argument gives

$$v(z) = \int_{\mathbb{R}^N} v(\zeta) \Theta(\delta_{1/r}(\zeta^{-1} \circ z)) r^{-Q} \, d\zeta, \quad \text{where } \Theta(\eta) = \tilde{m}_Q K(\eta^{-1}) \int_{d(\eta)}^{2d(\eta)} \frac{\vartheta(t)}{t^Q} \, dt.$$

We explicitly remark that the kernel Θ is smooth on \mathbb{G} and $\text{supp}(\Theta) \subseteq B_d(0, 1) \setminus B_d(0, 1/2)$. Finally, let $X \in \mathfrak{G}_k$. Since X is left-invariant on \mathbb{G}

and homogeneous of degree k w.r.t. the dilations on \mathbb{G} , after differentiating the above representation of v we obtain

$$\begin{aligned}
 |Xv(z)| &= \left| \int_{\mathbb{R}^N} v(\zeta) (1/r)^k (X\Theta)(\delta_{1/r}(\zeta^{-1} \circ z)) r^{-Q} d\zeta \right| \\
 &\leq r^{-k} \sup_{B_d(z,r)} |v| \int_{\mathbb{R}^N} |X\Theta|(\eta) d\eta.
 \end{aligned}$$

This completes the proof. \square

4. Estimates of second-layer derivatives: the non-characteristic case

Throughout this section, \mathbb{G} will denote a fixed H-type group and we shall use the notation introduced in Section 2 for such groups (see (2.5)). We shall denote by $\Pi^{(1)}$ any half-space of \mathbb{G} without characteristic points for the sub-Laplacian $\Delta_{\mathbb{G}}$: such a half-space is by necessity of the following form (see Appendix A for the details):

$$\Pi^{(1)} = \{(x, t) \in \mathbb{G} \mid \langle a, x \rangle > c\}, \tag{4.1}$$

where $c \in \mathbb{R}$ and $a \in \mathbb{R}^m \setminus \{0\}$ are fixed. Moreover, u will denote any fixed solution of the boundary value problem (1.1) in $\Pi^{(1)}$. We are interested in obtaining a priori estimates at infinity for the derivative Zu , where Z belongs to $[\mathfrak{g}, \mathfrak{g}]$, being \mathfrak{g} the Lie algebra of \mathbb{G} . We shall follow an idea in [39, Section 4], where analogous estimates are proved for the Heisenberg group. As in the previous section, we shall denote by w the function

$$w(z) = (\Gamma * f)(z) = \int_{\mathbb{G}} \Gamma(z, \zeta) f(\zeta) d\zeta, \quad f = u^{2^*-1}.$$

Moreover, v will denote the $\Delta_{\mathbb{G}}$ -harmonic part of u , i.e.

$$v = w - u. \tag{4.2}$$

Since any half-space in \mathbb{G} satisfies (3.1), from (3.3), (3.9), Theorem 3.1 and the hypoellipticity of $\Delta_{\mathbb{G}}$, it follows that v is a classical solution of

$$\Delta_{\mathbb{G}} v = 0 \text{ in } \Pi^{(1)}, \quad v = w \text{ in } \partial\Pi^{(1)}. \tag{4.3}$$

If $Z \in [\mathfrak{g}, \mathfrak{g}]$ is fixed, we then claim that Zv is a classical solution of the Dirichlet problem

$$\Delta_{\mathbb{G}}(Zv) = 0 \text{ in } \Pi^{(1)}, \quad Zv = Zw \text{ in } \partial\Pi^{(1)}.$$

This is an immediate consequence of the following facts: since Z is a second-layer derivative and \mathbb{G} has step two, the operators $\Delta_{\mathbb{G}}$ and Z commute; moreover, $Z \in [\mathfrak{g}, \mathfrak{g}]$

also yields $Z = \sum_{j=1}^n z_j \partial_{t_j}$, for suitable scalars z_1, \dots, z_n ; $\partial\Pi^{(1)}$ is invariant with respect to the Euclidean translations along the t -coordinates (see (4.1)); finally $u = 0$ on $\partial\Pi^{(1)}$. Our main tool is then to use these properties for representing Zv as a fixed point for a suitable mean-value operator modelled on the geometry of $\Pi^{(1)}$, which we hereafter introduce.

Let $(x, t) \in \Pi^{(1)}$ be fixed. Since $(x, 0)$ still belongs to $\Pi^{(1)}$ (see (4.1)), there exists a continuous function $r(x) > 0$ only depending on x such that $B_d((x, 0), r(x)) \Subset \Pi^{(1)}$ (we can choose, e.g., $r(x) = \frac{1}{2} \inf\{d((x, 0)^{-1} \circ (\xi, \tau)) : (\xi, \tau) \notin \Pi^{(1)}\}$). Recalling from (2.5) of the composition law on a H-type group, we explicitly remark that the left translations by $(0, t)$ coincide with the Euclidean translations by $(0, t)$. As a consequence, if $r(x)$ is as above, we have

$$\begin{aligned} B_d((x, t), r(x)) &= (0, t) \circ B_d((x, 0), r(x)) \\ &= (0, t) + B_d((x, 0), r(x)) \Subset (0, t) + \Pi^{(1)} = \Pi^{(1)}. \end{aligned}$$

In what follows, we shall then assume that $r : \mathbb{R}^m \rightarrow (0, \infty)$ is a fixed continuous function such that

$$B_d((x, t), r(x)) \Subset \Pi^{(1)}, \quad \forall (x, t) \in \Omega. \tag{4.4}$$

For every $\omega \in L^1_{\text{loc}}(\Pi^{(1)})$ we finally define

$$T\omega : \Pi^{(1)} \rightarrow \mathbb{R}, \quad (T\omega)(x, t) = (M_{r(x)}\omega)(x, t), \tag{4.5}$$

where M_r is the mean-value operator introduced in (3.6) of Section 3. We explicitly remark that (4.4) ensures that T is well defined. The following result holds.

Proposition 4.1. *T is a linear operator with the following properties:*

(i) *T maps $L^1_{\text{loc}}(\Pi^{(1)})$ into $C(\Pi^{(1)})$. Hence we can define, by induction, $T^{k+1}\omega = T(T^k(\omega))$ for every $\omega \in L^1_{\text{loc}}(\Pi^{(1)})$ and every $k \in \mathbb{N}$.*

(ii) *T is an increasing operator, i.e.*

$$(\omega_1, \omega_2 \in L^1_{\text{loc}}(\Pi^{(1)}), \omega_1 \leq \omega_2) \Rightarrow (T\omega_1 \leq T\omega_2).$$

(iii) *If $\omega \in C^2(\Pi^{(1)})$ and $\Delta_{\mathbb{G}}\omega = 0$ then $T\omega = \omega$.*

(iv) *If $\omega \in C^2(\Pi^{(1)})$ and $\Delta_{\mathbb{G}}\omega \leq 0$ then $T\omega \leq \omega$ and $(T^k\omega)_{k \in \mathbb{N}}$ is a non-increasing sequence.*

Proof. We first prove (i). We show that, for every fixed $\omega \in L^1_{\text{loc}}(\Pi^{(1)})$ and $z_0 = (x_0, t_0) \in \Pi^{(1)}$,

$$|T\omega(z) - T\omega(z_0)| \leq |M_{r(x)}\omega(z) - M_{r(x)}\omega(z_0)| + |M_{r(x)}\omega(z_0) - M_{r(x_0)}\omega(z_0)| \rightarrow 0,$$

as $z = (x, t) \rightarrow z_0$. On the one hand, a change of variable gives

$$\begin{aligned} |M_{r(x)}\omega(z) - M_{r(x)}\omega(z_0)| &= \frac{m_Q}{rQ(x)} \left| \int_{B_d(z_0, r(x))} K(z_0^{-1} \circ \eta) (\omega(z \circ z_0^{-1} \circ \eta) - \omega(\eta)) \, d\eta \right| \\ &\leq \frac{m_Q \|K\|_\infty}{rQ(x)} \int_{B_d(z_0, r(x))} |\omega(z \circ z_0^{-1} \circ \eta) - \omega(\eta)| \, d\eta. \end{aligned}$$

The continuity of r and the L^1 -continuity theorem ensure that this last term goes to zero as $z \rightarrow z_0$. On the other hand, the continuity of r also yields $\lim_{z \rightarrow z_0} M_{r(x)}\omega(z) = M_{r(x_0)}\omega(z_0)$.

(ii) Follows from the non-negativity of the kernel K . From (3.5) we immediately get (iii). Finally, (iv) is a consequence of (3.5) and (ii). \square

Proposition 4.2. *If $Z \in [\mathfrak{g}, \mathfrak{g}]$ is any second-layer derivative, the operators T and Z commute. More precisely, if there exists $Z\omega \in C(\Pi^{(1)})$ then there also exists $Z(T\omega) \in C(\Pi^{(1)})$ and it is $Z(T\omega) = T(Z\omega)$.*

Proof. By means of a change of variable, we obtain

$$(T\omega)(x, t) = m_Q \int_{B_d(0,1)} K(\xi, \tau) \omega((x, t) \circ \delta_{r(x)}(\xi, \tau)) \, d\xi \, d\tau.$$

Here we denoted by $\{\delta_\lambda\}_{\lambda>0}$ the group of natural dilations on \mathbb{G} and we used the fact that K is homogeneous of degree zero w.r.t. δ_λ . We now explicitly remark that, by (2.5), we have

$$(x, t) \circ \delta_{r(x)}(\xi, \tau) = \left(\begin{array}{l} x_j + r(x)\xi_j, \quad j = 1, \dots, m \\ t_j + r^2(x)\tau_j + \frac{1}{2}r(x)\langle x, U^{(j)}\xi \rangle, \quad j = 1, \dots, n \end{array} \right).$$

Hence, recalling that $Z \in [\mathfrak{g}, \mathfrak{g}]$ is a linear combination of partial derivatives in the t -coordinates, we have

$$Z(\omega((x, t) \circ \delta_{r(x)}(\xi, \tau))) = (Z\omega)((x, t) \circ \delta_{r(x)}(\xi, \tau)).$$

The proposition now follows by differentiating under the integral sign, taking into account the continuity of $Z\omega$ and Proposition 4.1(i). \square

Lemma 4.3. *$T^k u$ is a non-increasing sequence pointwise convergent to zero in $\Pi^{(1)}$ as $k \rightarrow \infty$.*

Proof. Since $\Delta_{\mathbb{G}}u \leq 0$, from Proposition 4.1(iv) we deduce the existence of a function $h : \Pi^{(1)} \rightarrow \mathbb{R}$ such that $T^k u \searrow h$. Moreover, being $u \geq 0$, from Proposition 4.1(ii) we obtain

$$0 \leq h \leq u. \tag{4.6}$$

Hence $h \in L^1_{\text{loc}}(\Pi^{(1)})$. By the definition of T , for every $z = (x, t) \in \Pi^{(1)}$ we have

$$\begin{aligned} h(z) &= \lim_{k \rightarrow \infty} T^{k+1}u(z) = \lim_{k \rightarrow \infty} \frac{m_Q}{r^Q} \int_{B_d(z,r)} K(z^{-1} \circ \zeta) T^k u(\zeta) \, d\zeta \\ &= \frac{m_Q}{r^Q} \int_{B_d(z,r)} K(z^{-1} \circ \zeta) h(\zeta) \, d\zeta = Th(\zeta). \end{aligned}$$

Therefore, Proposition 4.1(i) provides

$$h = Th \in C(\Pi^{(1)}). \tag{4.7}$$

Moreover, since u is continuous on the closure of $\Pi^{(1)}$, $u \equiv 0$ in $\partial\Pi^{(1)}$ and $u(z) \rightarrow 0$ as $d(z) \rightarrow \infty$ (see Theorem 3.1), (4.6) yields

$$h \in C(\overline{\Pi^{(1)}}), \quad h \equiv 0 \text{ in } \partial\Pi^{(1)}, \quad h(z) \rightarrow 0 \text{ as } d(z) \rightarrow \infty. \tag{4.8}$$

Let us now assume by contradiction that h is not identically 0. From (4.6) and (4.8) there exists $z_0 \in \Pi^{(1)}$ such that $h(z_0) = \max\{h(z) \mid z \in \overline{\Pi^{(1)}}\} > 0$. Hence the set $A = h^{-1}(\{h(z_0)\})$ is closed and non-empty. Moreover, for every $z \in A$, (4.7) yields

$$0 = h(z) - Th(z) = \frac{m_Q}{r^Q} \int_{B_d(z,r(x))} K(z^{-1} \circ \zeta) (h(z_0) - h(\zeta)) \, d\zeta. \tag{4.9}$$

Here we used the fact that $M_r(1) = 1$ for every $r > 0$ (see (3.5)). We now remark that the set $\{\eta \in \mathbb{G} \mid K(\eta) = 0\}$ has empty interior. Indeed, suppose by contradiction that $K(\eta) = 0$ for every η in a neighborhood of a suitable $\eta_0 \neq 0$. Then, since $K = |(X_1 d, \dots, X_m d)|^2$ and the Lie algebra generated by X_1, \dots, X_m has dimension $\dim(\mathbb{G})$ at any point, d would be constant in an open set containing η_0 . As a consequence, the function $\lambda \mapsto d(\delta_\lambda(\eta_0)) = \lambda d(\eta_0)$ would be constant near $\lambda = 1$, contradicting the assumption $d(\eta_0) \neq 0$. From (4.9), the above remark and the choice of z_0 , we immediately derive that $h(\zeta) = h(z_0)$ for every $\zeta \in B_d(z, r(x))$, i.e., $B_d(z, r(x)) \subseteq A$. Therefore A is also open. This yields $A = \Pi^{(1)}$ since $\Pi^{(1)}$ is connected. Equivalently, $h \equiv h(z_0) > 0$ in $\Pi^{(1)}$, contradicting (4.8). Hence it has to be $h \equiv 0$ in $\Pi^{(1)}$, and this completes the proof. \square

Theorem 4.4. *Let $\Pi^{(1)}$ be a non-characteristic half-space of the H-type group \mathbb{G} . Let u be a solution of (1.1) in $\Pi^{(1)}$. Then, for every second-layer derivative $Z \in [\mathfrak{g}, \mathfrak{g}]$ it is*

$$|Zu(x)| = \mathcal{O}(d(x)^{2-Q}), \quad \text{as } d(x) \rightarrow \infty \text{ in } \Pi^{(1)}.$$

Proof. Let $Z \in [\mathfrak{g}, \mathfrak{g}]$ be fixed. Let v be as in (4.2). From (4.3) and Proposition 4.1(iii), it is $Tv = v$. Lemma 4.3 and the linearity of T give

$$T^k w = T^k v + T^k u = v + T^k u \searrow v, \quad \text{as } k \rightarrow \infty. \tag{4.10}$$

Moreover, the continuity of w (see (3.9)) and a repeated application of Proposition 4.2 immediately give $T^k(Zw) = Z(T^k w)$, for every $k \in \mathbb{N}$. We fix $\Phi \in C_0^\infty(\Pi^{(1)})$. From (4.10) and the fact that the adjoint operator of Z is $-Z$, it follows

$$\begin{aligned} \int_{\Pi^{(1)}} \Phi T^k(Zw) &= \int_{\Pi^{(1)}} \Phi Z(T^k w) = - \int_{\Pi^{(1)}} (Z\Phi) T^k w \xrightarrow{k \rightarrow \infty} - \int_{\Pi^{(1)}} (Z\Phi)v \\ &= \int_{\Pi^{(1)}} \Phi Zv. \end{aligned} \tag{4.11}$$

Let $z_0 \notin \Pi^{(1)}$ be fixed. From Proposition 3.6 and the continuity of Zw , there exists $M > 0$ such that

$$-M\Gamma(z_0, z) \leq Zw \leq M\Gamma(z_0, z), \quad \forall z \in \mathbb{G}, \tag{4.12}$$

where Γ is the fundamental solution of $\Delta_{\mathbb{G}}$ (see (2.3)). Now, since the map $\Gamma(z_0, \cdot)$ is $\Delta_{\mathbb{G}}$ -harmonic on $\Pi^{(1)}$, from (4.12) we obtain

$$\begin{aligned} -M\Gamma(z_0, \cdot) &= T^k(-M\Gamma(z_0, \cdot)) \leq T^k(Zw) \leq T^k(M\Gamma(z_0, \cdot)) \\ &= M\Gamma(z_0, \cdot) \quad \text{in } \Pi^{(1)}. \end{aligned} \tag{4.13}$$

Here we exploited Proposition 4.1(ii, iii). From (4.11) and (4.13) we finally obtain

$$\left| \int_{\Pi^{(1)}} \Phi Zv \right| \leq M \int_{\Pi^{(1)}} \Phi \Gamma(z_0, \cdot) \quad \forall \Phi \in C_0^\infty(\Pi^{(1)}), \Phi \geq 0,$$

which implies $|Zv(z)| \leq M\Gamma(z_0, z)$ for every $z \in \Pi^{(1)}$. By the definition of v and by (4.12), we then infer that $|Zu| \leq 2M\Gamma(z_0, \cdot)$ in $\Pi^{(1)}$. This completes the proof. \square

Remark 4.5. The results in this section (as well as the proof of the non-existence Theorem 6.3) hold true (with the same proofs) when \mathbb{G} is a general step-two Carnot group and $\Pi^{(1)} = \{(x^{(1)}, x^{(2)}) \in \mathbb{G} \mid \langle a, x^{(1)} \rangle > c\}$ (we follow the notation in (2.2)). We warn the reader that not every non-characteristic half-space has this form, if \mathbb{G} is not a H -type group.

5. Estimates of second-layer derivatives: the characteristic case

Throughout this section, \mathbb{G} will denote a fixed H -type group and we shall use the notation introduced in Section 2 for such groups (see (2.5)). In particular, Γ will always denote the fundamental solution for the sub-Laplacian $\Delta_{\mathbb{G}}$ as in (2.3), (2.7). Furthermore, we shall denote by $\Pi^{(2)}$ any half-space of \mathbb{G} with characteristic points for $\Delta_{\mathbb{G}}$. It can easily be proved (see Appendix A for the details) that such a half-space

can always be left-translated into a half-space of the following form:

$$\Pi^{(2)} = \{(x, t) \in \mathbb{G} \mid \langle b, t \rangle > 0\}, \tag{5.1}$$

where $b \in \mathbb{R}^n$ is a fixed unit vector. Moreover, u will denote any fixed non-negative weak solution of the boundary value problem (1.1) in $\Pi^{(2)}$. We explicitly remark that, since any half-space in \mathbb{G} satisfies (3.1), then Theorem 3.1 holds for u . In particular $u \in \Gamma^{2\beta_0}(\mathbb{G})$ for a suitable $\beta_0 \in (0, \frac{1}{2})$ and $u = 0$ in $\partial\Pi^{(2)}$. Moreover, by a suitable version of the Kohn and Nirenberg Theorem for H-type groups, u is smooth up to $\partial\Pi$ away from the characteristic set $\text{char}(\Pi) = \{(x, t) \in \mathbb{G} \mid x = 0, \langle b, t \rangle = 0\}$. In this section, we are interested in obtaining global a priori estimates for the second-layer partial derivative $\langle b, \nabla_t u \rangle$. This will be done by a very delicate construction of barrier functions modelled on the geometry of $\Pi^{(2)}$. For this purpose, we need to introduce some notation which will be used throughout the section.

Let $\{b, b'_1, \dots, b'_{n-1}\}$ be a chosen orthonormal system of vectors in \mathbb{R}^n , where b is the unit vector as in (5.1). The system of Cartesian coordinates w.r.t. this new basis of \mathbb{R}^n will be denoted by $s = \langle b, t \rangle, t'_1, \dots, t'_{n-1}$. There obviously exists an orthogonal matrix O of order n such that $t = O \cdot (s, t'_1, \dots, t'_{n-1})^T$. The following notation will be used throughout the present section and the next one:

$$\begin{aligned} r &= |x|, \quad s = \langle b, t \rangle, \quad \varrho = |t'|, \quad d = (r^4 + 16s^2 + 16\varrho^2)^{1/4}, \\ \tilde{d} &= (r^4 + 16s^2)^{1/4}. \end{aligned} \tag{5.2}$$

We are interested in finding an estimate for the partial derivative $\partial_s u$ that we shall need in Section 6 in order to apply Pohozaev-type identities. Namely we want to prove the following result.

Theorem 5.1. *Let $\Pi^{(2)}$ be a characteristic half-space of the H-type group \mathbb{G} (see (5.1)). Let u be a solution of (1.1) in $\Pi^{(2)}$. Then, following the notation in (5.2), there exists a constant $M > 0$ such that*

$$|\partial_s u| \leq M \cdot \min\{1, \Gamma\} (1 + \tilde{d}^{-1}) \quad \text{in } \Pi^{(2)}.$$

Proof. The proof will directly follow from Proposition 3.6 and from Lemmas 5.7–5.8. \square

If we consider functions which have radial symmetry both in x and in $t' = (t'_1, \dots, t'_{n-1})$, i.e. (following (5.2)) $u = u(r, s, \varrho)$, then the canonical sub-Laplacian of \mathbb{G} has the following useful and remarkable form (see Appendix A for the details):

$$\Delta_{\mathbb{G}} = \partial_r^2 + \frac{m-1}{r} \partial_r + \frac{1}{4} r^2 \partial_s^2 + \frac{1}{4} r^2 \left(\partial_{\varrho}^2 + \frac{n-2}{\varrho} \partial_{\varrho} \right). \tag{5.3}$$

For the sake of brevity, we shall also denote the characteristic half-space $\Pi^{(2)}$ simply by Π . With the notation introduced in (5.2), we have the decomposition: $\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$, where

$$\begin{aligned} \Pi_1 &= \{(x, t) \in \mathbb{G} \setminus \{(0, 0)\} \mid r^2 \leq s \leq 1\}, & \Pi_2 &= \{(x, t) \in \mathbb{G} \mid 0 < s \leq r^2 \leq 1\}, \\ \Pi_3 &= \{(x, t) \in \mathbb{G} \mid 0 < s \leq r, r \geq 1\}, & \Pi_4 &= \{(x, t) \in \mathbb{G} \mid r \leq s, s \geq 1\}. \end{aligned} \tag{5.4}$$

The first a priori estimate of u that we need is the following one.

Theorem 5.2. *With the notation in (5.2), there exists a constant $M > 0$ such that*

$$u \leq M \min\{1, \Gamma\} \quad \text{in } \{0 < s < 1\}. \tag{5.5}$$

Proof. We first observe that, for some positive constant c , we have

$$u \leq cs^{\beta_0} \quad \text{in } \Pi. \tag{5.6}$$

Indeed, recalling the explicit expression (2.5) of the group law, from $u \in \Gamma^{2\beta_0}(\mathbb{G})$ it follows that

$$u(x, s, t') = u(x, s, t') - u(x, 0, t') \leq cd((x, 0, t')^{-1} \circ (x, s, t'))^{2\beta_0} = cd(0, s, 0)^{2\beta_0} = cs^{\beta_0}.$$

We now consider the family of barrier functions

$$F_{\alpha,\beta} = (\sin s)^\beta (1 + r^2)^{-\alpha}, \quad \alpha > 0, \quad 0 < \beta \leq 1,$$

in the strip $\{0 < s < 1\}$. A direct computation (that makes use of the explicit expression (5.3)) gives

$$-\Delta_{\mathbb{G}} F_{\alpha,\beta} = \left(\frac{\beta(1-\beta)r^2}{4 \sin^2 s} + \frac{2\alpha(m-2-2\alpha)r^2 + 2\alpha m}{(1+r^2)^2} + \frac{\beta^2 r^2}{4} \right) F_{\alpha,\beta}.$$

Setting $\alpha_0 = \frac{Q-2}{2}$, recalling estimate (3.8) of u , and taking r_0 large enough, we obtain

$$-\Delta_{\mathbb{G}} u = u^{\frac{Q+2}{Q-2}} \leq cr^{2-Q} \leq -c\Delta_{\mathbb{G}} F_{\alpha_0,\beta_0} \quad \text{in } \{r > r_0, 0 < s < 1\}.$$

Moreover, from (5.6) and (3.8), it follows that $u \leq cF_{\alpha_0,\beta_0}$ on the boundary of $\{r > r_0, 0 < s < 1\}$ and at infinity. Therefore, the weak maximum principle for $\Delta_{\mathbb{G}}$ gives $u \leq cF_{\alpha_0,\beta_0}$ in $\{r > r_0, 0 < s < 1\}$. Recalling again that also (5.6) holds, we immediately obtain that

$$u \leq cF_{\alpha_0,\beta_0} \quad \text{in } \{0 < s < 1\}.$$

We now define $(\mathcal{P})_{\alpha,\beta} = \{\exists c > 0 : u \leq cF_{\alpha,\beta} \text{ in } \{0 < s < 1\}\}$. We set $\alpha_1 = \frac{m-2}{2}$ (we recall that $m \geq 4$ since we are assuming that \mathbb{G} is not a Heisenberg group \mathbb{H}^k) and we want to prove that $(\mathcal{P})_{\alpha_1,1}$ holds. Since we have already proved $(\mathcal{P})_{\alpha_0,\beta_0}$ and obviously $(\mathcal{P})_{\alpha,\beta} \Rightarrow (\mathcal{P})_{\alpha',\beta'}$ for every $0 < \alpha' \leq \alpha, 0 < \beta' \leq \beta$, we only need to prove that

$$(\mathcal{P})_{\alpha_1, \left(\frac{Q-2}{Q+2}\right)\beta} \Rightarrow (\mathcal{P})_{\alpha_1,\beta} \quad \forall \beta \in (0, 1]. \tag{5.7}$$

From $(\mathcal{P})_{\alpha_1, \left(\frac{Q-2}{Q+2}\right)\beta}$, it follows that

$$\begin{aligned} -\Delta_{\mathbb{G}}u &= u^{\frac{Q+2}{Q-2}} \leq c(\sin s)^\beta (1+r^2)^{-\alpha_1} \left(\frac{Q+2}{Q-2}\right) \\ &\leq c(\sin s)^\beta (r^2(1+r^2)^{-\alpha_1} + (1+r^2)^{-\alpha_1-2}) \leq -c\Delta_{\mathbb{G}}F_{\alpha_1,\beta} \quad \text{in } \{0 < s < 1\}. \end{aligned}$$

On the other hand, $u \leq cF_{\alpha_1,\beta}$ on the boundary of $\{0 < s < 1\}$ and at infinity, by means of (3.8) (recall that $Q = m + 2n > m$). Therefore, from the weak maximum principle for $\Delta_{\mathbb{G}}$, we obtain $(\mathcal{P})_{\alpha_1,\beta}$. This proves (5.7) and completes the proof of $(\mathcal{P})_{\alpha_1,1}$. More explicitly, we have proved the following estimate:

$$u \leq c(1+r^2)^{\frac{2-m}{2}} \sin s \quad \text{in } \{0 < s < 1\}. \tag{5.8}$$

In order to improve the above estimate, we now consider the family of barrier functions

$$G_{\beta,\gamma} = (\sin s)^\beta (r^4 + q^2)^{\frac{2-Q}{4}} (1+r^{-\gamma}), \quad 0 < \beta \leq 1, \quad 0 \leq \gamma \leq 2,$$

in the sets $A_R = \{0 < s < \min\{1, r^2\}, r^4 + q^2 > R^4\}$. A direct computation (recall (5.3)) gives

$$\begin{aligned} -\Delta_{\mathbb{G}}G_{\beta,\gamma} &= \sum_{i=0}^1 (\sin s)^\beta (r^4 + q^2)^{\frac{2-Q}{4}} r^{2-i\gamma} \left(\frac{\beta(1-\beta)}{4 \sin^2 s} + \frac{\beta^2}{4} + \frac{i\gamma(m-2-i\gamma)}{r^4} \right) \\ &\quad + \frac{(Q-2)(m-2i\gamma + (n+15)/8)}{r^4 + q^2} - \frac{(Q-2)(Q+2)(r^4 + q^2/16)}{(r^4 + q^2)^2}. \end{aligned}$$

We first want to prove that for every fixed $\beta \in (0, 1)$, there exist positive constants c, R such that

$$u \leq cG_{\beta,2\beta} \quad \text{in } A_R. \tag{5.9}$$

Using (3.8) and recalling that $m \geq 4$, we see that (for R large enough) we have

$$-\Delta_{\mathbb{G}}u = u^{\frac{Q+2}{Q-2}} \leq c\Gamma^{\frac{Q+2}{Q-2}} \leq c\Gamma \leq c(\sin s)^{\beta-2} r^{2-2\beta} \Gamma \leq -c\Delta_{\mathbb{G}}G_{\beta,2\beta} \quad \text{in } A_R.$$

Moreover, from (5.8) and (3.8) it follows that $u \leq cG_{\beta,2\beta}$ in $\partial A_R \cup \{\infty\}$. By the weak maximum principle, we then deduce (5.9). We now define the barrier functions

$$\Phi_\beta = s^\beta \varrho^{\frac{2-Q}{2}} \exp(-r^2), \quad 0 < \beta \leq 1,$$

in the sets $D_R = \{0 < s < 1, r < 1, \varrho > R\}$. A direct computation based on (5.3) gives

$$-\Delta_G \Phi_\beta = \left(2m - 4r^2 + \frac{\beta(1-\beta)r^2}{4s^2} - \frac{(m+4)(Q-2)r^2}{16\varrho^2} \right) \Phi_\beta.$$

We define $(\mathcal{P})_{R,\beta} = (\exists c > 0: u \leq c\Phi_\beta \text{ in } D_R)$. We want to show that

$$\text{for every } \beta \in (0, 1) \text{ there exists } R > 0 \text{ such that } (\mathcal{P})_{R,\beta} \text{ holds.} \quad (5.10)$$

We first observe that $(\mathcal{P})_{R,\beta} \Rightarrow (\tilde{\mathcal{P}})_{R,\beta}$, where

$$(\tilde{\mathcal{P}})_{R,\beta} = (\exists c > 0: u \leq cs^\beta (\varrho^{\frac{2-Q}{2}})^{\frac{Q-2}{Q+2}} \exp(-r^2) \text{ in } D_R).$$

Let us now prove that, for every sufficiently large R and for every $\beta \in (0, 1]$, the following implication holds:

$$\begin{aligned} & ((\tilde{\mathcal{P}})_{R, \frac{Q-2}{Q+2}\beta}, \exists c > 0: u \leq c\Phi_\beta \text{ in } \{0 < s < 1, r = 1, \varrho > R\}) \\ & \Rightarrow ((\mathcal{P})_{R,\beta'} \quad \forall \beta' \in (0, \beta]). \end{aligned} \quad (5.11)$$

From the hypothesis $(\tilde{\mathcal{P}})_{R, \frac{Q-2}{Q+2}\beta}$, it follows that (recall that $m \geq 4$)

$$-\Delta_G u = u^{\frac{Q+2}{Q-2}} \leq cs^\beta \varrho^{\frac{2-Q}{2}} \exp(-r^2) \leq -c\Delta_G \Phi_\beta \quad \text{in } D_R$$

(if R is large enough). Moreover, from (3.8), (5.8) and the hypothesis, we infer that $u \leq c\Phi_\beta$ in $\partial D_R \cup \{\infty\}$. Therefore, by the weak maximum principle, we obtain $(\mathcal{P})_{R,\beta}$ which immediately gives $(\mathcal{P})_{R,\beta'}$ for every $\beta' \in (0, \beta]$. This proves (5.11). We now observe that $(\tilde{\mathcal{P}})_{R,\beta_1}$ holds when $\beta_1 = \frac{4}{Q+2}$ and R is large enough. Indeed (5.8) and (3.8) give

$$u = u^{\frac{Q-2}{Q+2}} u^{\frac{4}{Q+2}} \leq c\Gamma^{\frac{Q-2}{Q+2}} s^{\frac{4}{Q+2}} \leq cs^{\frac{4}{Q+2}} (\varrho^{\frac{2-Q}{2}})^{\frac{Q-2}{Q+2}} \exp(-r^2) \quad \text{in } D_R.$$

Moreover, from (5.9) it follows that for every $\beta \in (0, 1)$ there exists $R > 0$ such that

$$u \leq cG_{\beta,2\beta} = c(\sin s)^\beta (1 + \varrho^2)^{\frac{2-Q}{4}} \leq c\Phi_\beta, \quad \text{in } \{0 < s < 1, r = 1, \varrho > R\}.$$

Therefore, using (5.11) and starting from $(\tilde{\mathcal{P}})_{R,\beta_1}$, we finally obtain (5.10). Collecting (5.9) and (5.10), we immediately infer that for every $\beta \in (0, 1)$ there exist positive

constants R, c such that

$$u \leq cG_{\beta,0} \quad \text{in } A_R. \tag{5.12}$$

We now prove that

$$u \leq cG_{1,1} \quad \text{in } A_R, \tag{5.13}$$

for suitable positive constants R, c . Choosing $\beta = \frac{Q-2}{Q+2}$ in (5.12) and recalling that $m \geq 4$, we get (also using the given computation of $-\Delta_{\mathbb{G}}G_{\beta,\gamma}$)

$$\begin{aligned} -\Delta_{\mathbb{G}}u &= u^{\frac{Q+2}{Q-2}} \leq c(G_{\frac{Q-2}{Q+2},0})^{\frac{Q+2}{Q-2}} = c(r^4 + \varrho^2)^{-\frac{Q+2}{4}} \sin s \\ &\leq c \frac{\sin s}{(r^4 + \varrho^2)^{\frac{Q-2}{4}}} \leq c \frac{r \sin s}{(r^4 + \varrho^2)^{\frac{Q-2}{4}}} \left(\frac{m-3}{r^4} + 1 \right) \leq -c\Delta_{\mathbb{G}}G_{1,1} \quad \text{in } A_R, \end{aligned}$$

if R is large enough. Moreover, the choice $\beta = \frac{1}{2}$ in (5.12) gives

$$u \leq cG_{\frac{1}{2},0} \leq cs^{\frac{1}{2}}\Gamma = c\frac{s}{r}\Gamma \leq cG_{1,1} \quad \text{in } \{0 < s = r^2 < 1, r^4 + \varrho^2 > R^4\}.$$

Recalling again (3.8) and (5.8), we obtain that $u \leq cG_{1,1}$ in $\partial A_R \cup \{\infty\}$. Therefore, by the weak maximum principle, (5.13) holds.

From (5.10), (5.11) and (5.13), it now follows that $(\mathcal{P})_{R,1}$ holds if R is large enough. As a consequence we have $u \leq cs\Gamma$ in D_R . Collecting (5.8), (5.13) and this last estimate, we finally obtain (5.5). \square

Recalling that u is smooth up to the boundary of Π away from the characteristic set $\{s = r = 0\}$ and that u vanishes on $\partial\Pi$, from Theorem 5.2 we immediately get the following important estimate.

Corollary 5.3. *We have $|\partial_s u| \leq M \min\{1, \Gamma\}$ in $\{s = 0, r \neq 0\}$.*

As in Section 4, we henceforth shall denote by w the convolution of Γ with the function $f = u^{2^* - 1}$. Moreover, $v = w - u$ will denote the $\Delta_{\mathbb{G}}$ -harmonic part of u . We recall that v is a $\Delta_{\mathbb{G}}$ -harmonic function in Π . In the sequel, we shall use the fact that, since $\Delta_{\mathbb{G}}$ and ∂_s commute, also $\partial_s v$ is $\Delta_{\mathbb{G}}$ -harmonic in Π . Since the estimates of w are known from Propositions 3.3 and 3.6, we turn our attention on v . Our aim is to apply Proposition 3.7 for v . In order to do so, we first need the following result concerning with the geometry of Π .

Lemma 5.4. For every $(x, t) \in \Pi$, let $R(x, t) > 0$ be the radius defined by

$$R(x, t) = \frac{4\langle b, t \rangle}{\sqrt{n}|x| + \sqrt{n|x|^2 + 4\langle b, t \rangle}}. \tag{5.14}$$

Then we have $B_d((x, t), R(x, t)) \subset \Pi$ for every $(x, t) \in \Pi$. Moreover, following the notation in (5.2) and (5.4), we have $R(x, t) \geq \tilde{R}(r, s)$, where

$$\tilde{R}(r, s) = \begin{cases} c_n\sqrt{s} & \text{in } \Pi_1, \\ c_n s/r & \text{in } \Pi_2, \\ c_n s/r & \text{in } \Pi_3, \\ c_n & \text{in } \Pi_4, \end{cases}$$

where $c_n = 4/(\sqrt{n} + \sqrt{n+4})$ is a constant only depending on n .

Proof. Let $(x, t) \in \Pi$ be fixed. We only have to prove that $B_d((x, t), R(x, t)) \Subset \Pi$, since the rest of the assertion is easily checked from definition (5.4) of the Π_i 's and (5.14). By (2.5), the generic point of $B_d((x, t), R)$ has coordinates

$$\left(\begin{array}{c} x_j + \xi_j, \quad j = 1, \dots, m \\ t_j + \tau_j + \frac{1}{2}\langle x, U^{(j)}\xi \rangle, \quad j = 1, \dots, n \end{array} \right), \quad \text{where } (\xi, \tau) \in B_d((0, 0), R).$$

We have to prove that, when $R = R(x, t)$ and for any (ξ, τ) such that $|\xi|^4 + 16|\tau|^2 < R^4$, we have

$$\sum_{j=1}^n b_j(t_j + \tau_j + \frac{1}{2}\langle x, U^{(j)}\xi \rangle) > 0. \tag{5.15}$$

Recalling that $|b| = 1$, the $U^{(j)}$'s are orthogonal matrices, $|\xi| < R$ and $|\tau| < R^2/4$, we have

$$|\langle b, \tau \rangle| + \frac{1}{2} \left| \sum_{j=1}^n b_j \langle x, U^{(j)}\xi \rangle \right| \leq |\tau| + \frac{\sqrt{n}}{2}|x||\xi| < \frac{1}{4}R^2 + \frac{\sqrt{n}}{2}|x|R.$$

As a consequence, (5.15) holds whenever $\frac{1}{4}R^2 + \frac{\sqrt{n}}{2}|x|R \leq \langle b, t \rangle$, which is certainly satisfied if $R \leq -\sqrt{n}|x| + \sqrt{n|x|^2 + 4\langle b, t \rangle} = R(x, t)$. \square

Form the above proof it also follows that, if $R(x, t)$ is as in (5.14), we have

$$B_d((x, t), R(x, t)) \subset \{(\xi, \tau) \in \mathbb{G} : |\langle b, \tau \rangle| < 2\langle b, t \rangle\}, \quad \forall (x, t) \in \Pi. \tag{5.16}$$

With this geometric lemma and Theorem 5.2 at hand, we are able to prove some first useful estimates for u .

Proposition 5.5. *With the notation in (5.2) and (5.4), for a suitable constant $M > 0$ we have*

$$|\partial_s u| \leq M \frac{1}{\sqrt{s}} \min\{1, \Gamma\} \quad \text{in } \Pi_1, \tag{5.17}$$

$$|\partial_s u| \leq M \frac{r}{s} \min\{1, \Gamma\} \quad \text{in } \Pi_2, \tag{5.18}$$

$$|\partial_s u| \leq M \frac{r^2}{s^2} \Gamma \quad \text{in } \Pi_3, \tag{5.19}$$

$$|\partial_s u| \leq M \Gamma \quad \text{in } \Pi_4, \tag{5.20}$$

$$|\nabla_{\mathbb{G}} u| \leq M \min\{1, \Gamma\} \quad \text{in } \Pi_1 \cup \Pi_2. \tag{5.21}$$

Proof. In the sequel, we shall write $1 \wedge \Gamma = \min\{1, \Gamma\}$. We fix $z_0 \in \Pi_1 \cup \Pi_2$ and we let (x_0, s_0, t'_0) be the coordinates of z_0 w.r.t. the basis introduced at the beginning of this section. If $v = w - u$ is the $\Delta_{\mathbb{G}}$ -harmonic part of u , we set $v_0 = v - v(x_0, 0, t'_0)$ (following the above notation for z_0). Let R be the radius defined in (5.14) and set $R_0 = R(z_0)$. It is $cs/r \leq R \leq c\sqrt{s}$ on $\Pi_1 \cup \Pi_2$, for a suitable constant $c > 0$. For any $z \in B_d(z_0, R_0)$, we have

$$\begin{aligned} |v_0(z)| &= |w(z) - u(z) - w(x_0, 0, t'_0)| \\ &\leq |u(z)| + |w(z) - w(z_0)| + |w(z_0) - w(x_0, 0, t'_0)| \\ &\leq cs(1 \wedge \Gamma)(z) + cd(z, z_0) \sup_{B_d(z_0, cd(z, z_0))} |\nabla_{\mathbb{G}} w| + |s_0| \sup_{0 \leq s^* \leq s_0} |\partial_s w(x_0, s^*, t'_0)| \\ &\leq cs(1 \wedge \Gamma)(z) + cR_0(1 \wedge \Gamma)(z_0) + cs_0(1 \wedge \Gamma)(z_0) \\ &\leq c(s_0 + R_0)(1 \wedge \Gamma)(z_0). \end{aligned}$$

Here we used the following facts: u vanishes on $(x_0, 0, t'_0) \in \partial\Pi$; $u \leq cs(1 \wedge \Gamma)$, see Theorem 5.2; $d(z, z_0) \leq R_0 \leq \sqrt{4}$; the estimates for $|\nabla_{\mathbb{G}} w|$ and $|\partial_s w|$ in Proposition 3.6; $s \leq 2s_0$ being $z \in B_d(z_0, R_0)$ (see (5.16)). This gives

$$|v_0(z)| \leq c(s_0 + R_0)(1 \wedge \Gamma)(z_0), \quad \forall z \in B_d(z_0, R_0), \quad \forall z_0 \in \Pi_1 \cup \Pi_2. \tag{5.22}$$

From (5.22) and Proposition 3.7, we first derive

$$\begin{aligned} |\partial_s v|(z_0) &= |\partial_s v_0|(z_0) \leq cR_0^{-2} \sup_{B_d(z_0, R_0)} |v_0| \\ &\leq \tilde{c} \frac{s_0 + R_0}{R_0^2} (1 \wedge \Gamma)(z_0), \quad \forall z_0 \in \Pi_1 \cup \Pi_2. \end{aligned} \tag{5.23}$$

If in (5.23) we take $z_0 \in \Pi_2$ and we recall that $R_0 \geq cs_0/r_0$ in Π_2 (see Lemma 5.4) we get $|\partial_s v|(z_0) \leq cr_0/s_0(1 \wedge \Gamma)(z_0)$. In particular, from (3.13) and $s_0 \leq r_0^2 \leq r_0$ in Π_2 , we obtain $|\partial_s u|(z_0) \leq |\partial_s w|(z_0) + |\partial_s v|(z_0) \leq c(1 \wedge \Gamma)(z_0) + cr_0/s_0(1 \wedge \Gamma)(z_0)$, for every $z_0 \in \Pi_2$. This gives (5.18). Otherwise, if in (5.23) we take $z_0 \in \Pi_1$ and we recall that $R_0 \geq c\sqrt{s_0}$ in Π_1 (see Lemma 5.4), we get $|\partial_s v|(z_0) \leq c/\sqrt{s_0}(1 \wedge \Gamma)(z_0)$ and (5.17) follows arguing as above. Furthermore (5.19) and (5.20) directly follow from Propositions 3.3, 3.6 and 3.7, Theorem 3.4 and Lemma 5.4. Finally, if $z_0 \in \Pi_1 \cup \Pi_2$, from (5.22) and again from Proposition 3.7, we have

$$|\nabla_{\mathbb{G}} v|(z_0) = |\nabla_{\mathbb{G}} v_0|(z_0) \leq cR_0^{-1} \sup_{B_d(z_0, R_0)} |v_0| \leq c \frac{s_0 + R_0}{R_0} (1 \wedge \Gamma)(z_0) \leq \tilde{c}(1 \wedge \Gamma)(z_0),$$

since in $\Pi_1 \cup \Pi_2$ we have $s_0 \leq cR_0$. Taking into account estimate (3.13) of $|\nabla_{\mathbb{G}} w|$, (5.21) follows. This completes the proof. \square

We now take up the proof of Theorem 5.1, by proving Lemmas 5.6, 5.7 and 5.8.

Lemma 5.6. *There exists a constant $M > 0$ such that $|\partial_s v| \leq M\Gamma$ in the set $\{r = 2, 0 \leq s \leq 1\}$.*

Proof. Let us fix $\beta \in (0, 1)$, $\delta > 2$ and define the barrier function

$$G = s^\beta ((r - 1)(3 - r))^{-\delta(\beta+2)} (\varrho_0^{\frac{3}{2}} ((2\varrho_0 - \varrho)(\varrho - 1))^{-1})^{\beta+2}$$

in the set (for large $\varrho_0 > 0$ and small $s_0 \in (0, 1]$)

$$\Omega_{\varrho_0, s_0} = \{1 < r < 3, 1 < \varrho < 2\varrho_0, 0 < s < \min\{s_0, (r - 1)^\delta (3 - r)^\delta, \varrho_0^{-\frac{3}{2}} (2\varrho_0 - \varrho)(\varrho - 1)\}\}.$$

Making use of (5.3), by direct computations (that we omit for the sake of brevity) it is not difficult to prove that

$$A_{\mathbb{G}} G \leq 0 \quad \text{in } \Omega_{\varrho_0, s_0}, \tag{5.24}$$

if ϱ_0 is large enough and s_0 is sufficiently small. We now want to estimate $\partial_s v$ in $\partial\Omega_{\varrho_0, s_0}$. Since we are far from the characteristic set of Π , we can use Corollary 5.3 and Proposition 3.6 (for $Z = \partial_s$) and obtain $|\partial_s v| \leq c\Gamma$ in $\partial\Omega_{\varrho_0, s_0} \cap \{s = 0\}$. It is easy

to see that, if ϱ_0 is large enough, we have

$$\varrho_0^{\frac{3}{2}}((2\varrho_0 - \varrho)(\varrho - 1))^{-1} \geq \varrho^{-1} \quad \forall \varrho \in (1, 2\varrho_0). \tag{5.25}$$

From (5.19) and (5.25), we infer that

$$\begin{aligned} |\partial_s v| &\leq c s^{-2} \Gamma \leq c s^{-2} \varrho^{\frac{2-Q}{2}} \leq c s^{-2} \varrho^{-2-\beta} \leq c s^{-2} (\varrho_0^{\frac{3}{2}}((2\varrho_0 - \varrho)(\varrho - 1))^{-1})^{\beta+2} \\ &= c G \quad \text{in } \partial\Omega_{\varrho_0, s_0} \cap \{s = (r - 1)^\delta (3 - r)^\delta\}, \end{aligned}$$

$$|\partial_s v| \leq c s_0^{-2} \Gamma \leq c_{s_0} \varrho^{-2-\beta} \leq c_{s_0} G \quad \text{in } \partial\Omega_{\varrho_0, s_0} \cap \{s = s_0\},$$

$$|\partial_s v| \leq c s^{-2} \Gamma \leq c s^{-2} \leq c G \quad \text{in } \partial\Omega_{\varrho_0, s_0} \cap \{s = \varrho_0^{-\frac{3}{2}}(2\varrho_0 - \varrho)(\varrho - 1)\}$$

(we recall that $Q \geq 8$ since we are assuming that \mathbb{G} is not a Heisenberg group \mathbb{H}^k). Therefore we have proved that $|\partial_s v| \leq c_{s_0}(G + \Gamma)$ in $\partial\Omega_{\varrho_0, s_0}$. Since ∂_s and $\Delta_{\mathbb{G}}$ commute and (5.24) holds, we also have

$$-\Delta_{\mathbb{G}}(\partial_s v) = 0 \leq -\Delta_{\mathbb{G}}(G + \Gamma) \quad \text{in } \Omega_{\varrho_0, s_0}.$$

Thus, the weak maximum principle yields

$$|\partial_s v| \leq c_{s_0}(G + \Gamma) \quad \text{in } \Omega_{\varrho_0, s_0} \tag{5.26}$$

for large ϱ_0 and small s_0 (we remark that c_{s_0} does not depend on ϱ_0). Finally, the choice $\varrho = \varrho_0$ in (5.26) gives the following estimate:

$$\begin{aligned} |\partial_s v| &\leq c_{s_0} \left(\Gamma + s^\beta ((r - 1)(3 - r))^{-\delta(\beta+2)} \varrho^{-\frac{\beta+2}{2}} \right) \\ &\text{in } \{1 < r < 3, 0 < s < \min\{s_0, (r - 1)^\delta (3 - r)^\delta\}\} \text{ and for } \varrho \geq 1. \end{aligned} \tag{5.27}$$

We shall use (5.27) in order to prove estimate (5.26). To this end, we introduce the barrier function

$$F = s^\beta ((r - 1)(3 - r))^{-\delta'(\beta+2)} \varrho^{\frac{2-Q}{2}}$$

in the following set:

$$\Omega = \{1 < r < 3, 0 < s < \min\{s_0, (r - 1)^{\delta'} (3 - r)^{\delta'}\}, \sigma\varrho\},$$

where s_0 is small, $\delta' = \delta(1 + \frac{2}{\beta})$ and $\sigma > 0$ is a suitable constant that can be chosen small enough to ensure that (we omit the computations for the sake of brevity)

$$\Delta_{\mathbb{G}} F \leq 0 \quad \text{in } \Omega.$$

Arguing as above, one can prove that $|\partial_s v| \leq c_{s_0}(F + \Gamma)$ in $\partial\Omega$. Moreover, $\partial_s v$ vanishes at infinity in Ω by means of (5.27) (and by the definition of δ'). Since $-A_{\mathbb{G}}(\partial_s v) = 0 \leq -A_{\mathbb{G}}(F + \Gamma)$, from the weak maximum principle it follows that

$$|\partial_s v| \leq c_{s_0}(F + \Gamma) \quad \text{in } \Omega.$$

As a consequence, we get

$$|\partial_s v| \leq c_{s_0}\Gamma \quad \text{in } \{r = 2, 0 < s < s_0\} \text{ and for } \varrho \gg 1.$$

To end the proof, it is sufficient to recall that $\partial_s v$ is continuous up to the boundary of Π away from the characteristic set $\{s = 0 = r\}$ and finally to use (5.19) in the set $\{r = 2, s_0 \leq s \leq 1\}$. \square

Lemma 5.7. *For every $\varepsilon > 0$ there exists a positive constant M such that $|\partial_s v| \leq M\Gamma$ in the set $\Pi \setminus \{r < \varepsilon, s < \varepsilon\}$.*

Proof. We shall prove the lemma in the case $\varepsilon = 2$. The same proof works for every ε , observing that one can adapt the proof of Lemma 5.6 in order to obtain the estimate in the set $\{r = \varepsilon, 0 \leq s \leq 1\}$ (with M depending on ε). Let us fix $\beta \in (0, 1)$ and a small $\sigma > 0$. We define the barrier function

$$\Psi_R = s^\beta ((2R)^2 - (r^4 + \varrho^2)^{\frac{1}{2}})^{-2-\beta}$$

in the set (for large $R > 0$)

$$\Omega_R = \{r > 2, 0 < s < \sigma \min\{r, (2R)^2 - (r^4 + \varrho^2)^{\frac{1}{2}}\}\}.$$

By direct computations (which we omit) one can show that, if σ is chosen small enough,

$$A_{\mathbb{G}}\Psi_R \leq 0 \quad \text{in } \Omega_R, \text{ for every } R \gg 1. \tag{5.28}$$

From Corollary 5.3 and Proposition 3.6, it follows that $|\partial_s v| \leq c\Gamma$ in $\partial\Omega_R \cap \{s = 0\}$. Recalling (5.20), we also have $|\partial_s v| \leq c\Gamma$ in $\partial\Omega_R \cap \{s = \sigma r\}$. Moreover, Lemma 5.6 ensures that $|\partial_s v| \leq c\Gamma$ in $\partial\Omega_R \cap \{r = 2\}$. Finally, from (5.19) it follows that

$$|\partial_s v| \leq cr^2 s^{-2}\Gamma \leq cs^{-2} = c\Psi_R \quad \text{in } \partial\Omega_R \cap \{s = \sigma((2R)^2 - (r^4 + \varrho^2)^{\frac{1}{2}})\}.$$

Therefore, we have $|\partial_s v| \leq c(\Psi_R + \Gamma)$ in $\partial\Omega_R$. On the other hand, (5.28) gives

$$-A_{\mathbb{G}}(\partial_s v) = 0 \leq -A_{\mathbb{G}}(\Psi_R + \Gamma) \quad \text{in } \Omega_R.$$

By the weak maximum principle, we infer that

$$|\partial_s v| \leq c(\Psi_R + \Gamma) \quad \text{in } \Omega_R, \text{ for every } R \gg 1$$

(with c not depending on R). As a consequence, we obtain

$$|\partial_s v| \leq c(\Gamma + s^\beta(r^4 + q^2)^{\frac{2+\beta}{2}}) \leq c(\Gamma + (r^4 + q^2)^{\frac{4+\beta}{4}})$$

$$\text{in } \Omega_R \cap \{r^4 + q^2 = R^4\} = \{r^4 + q^2 = R^4, r > 2, 0 < s < \sigma r\}, \text{ for every } R \gg 1.$$

This proves that $\partial_s v$ vanishes at infinity in the set $A = \{r > 2, 0 < s < \sigma r\}$. Since we also have $|\partial_s v| \leq c\Gamma$ in ∂A (by Corollary 5.3, Proposition 3.6, (5.20) and Lemma 5.6) and $-\Delta_{\mathbb{G}}(\partial_s v) = 0 = -\Delta_{\mathbb{G}}\Gamma$ in A , we get from the weak maximum principle

$$|\partial_s v| \leq c\Gamma \quad \text{in } \partial A.$$

In order to end the proof, it is now sufficient to recall that (5.20) also holds. \square

Lemma 5.8. *There exists a positive constant M such that $|\partial_s v| \leq M\tilde{d}^{-1} \min\{1, \Gamma\}$ in $\{r < 1, 0 < s < 1\}$. We recall that \tilde{d} is defined in (5.2).*

Proof. Let us fix $\beta \in (0, 1)$ and a small $\sigma > 0$. We define the barrier function

$$\Phi_\varepsilon = \varepsilon^{1-\beta} s^\beta (r - \varepsilon)^{-2-\beta} (1 + q^2)^{\frac{2-Q}{4}} + \varepsilon^{-1} \Gamma$$

in the set $A_\varepsilon = \{\varepsilon < r < 1, 0 < s < \sigma r(r - \varepsilon)\}$ (for $\varepsilon \in (0, \frac{1}{2})$). A direct computation (which we omit) based on (5.3) gives

$$\Delta_{\mathbb{G}}\Phi_\varepsilon \leq 0 \quad \text{in } A_\varepsilon$$

(if σ is chosen small enough). We now want to prove that

$$|\partial_s v| \leq c\Phi_\varepsilon \quad \text{in } \partial A_\varepsilon \cup \{\infty\}, \text{ for every } \varepsilon \in (0, \frac{1}{2}) \tag{5.29}$$

(with c not depending on ε). In the sets $\partial A_\varepsilon \cap \{s = 0\}$, $\partial A_\varepsilon \cap \{r = 1\}$ and at infinity, the desired estimate follows from Corollary 5.3, Lemma 5.6 (see also the remark at the beginning of the proof of Lemma 5.7) and Lemma 5.7, respectively. Moreover, by Proposition 3.6 and (5.18), we have

$$\begin{aligned} |\partial_s v| &\leq |\partial_s w| + |\partial_s u| \leq c(1 \wedge \Gamma + rs^{-1} 1 \wedge \Gamma) \leq c(r - \varepsilon)^{-1} 1 \wedge \Gamma \\ &\leq c\Phi_\varepsilon \quad \text{in } \partial A_\varepsilon \cap \{s = \sigma r(r - \varepsilon)\}. \end{aligned}$$

In the last inequality, we have used the facts that $(r - \varepsilon)^{-1} \leq c\varepsilon^{1-\beta} s^\beta (r - \varepsilon)^{-2-\beta}$ in $\partial A_\varepsilon \cap \{s = \sigma r(r - \varepsilon), \varepsilon \leq r \leq 2\varepsilon\}$ and $(r - \varepsilon)^{-1} \leq \varepsilon^{-1}$ in $\partial A_\varepsilon \cap \{s = \sigma r(r - \varepsilon), 2\varepsilon \leq r \leq 1\}$. Thus (5.29) is proved. Recalling that $-\Delta_{\mathbb{G}}(\partial_s v) = 0 \leq -\Delta_{\mathbb{G}}\Phi_\varepsilon$ in A_ε , by the weak maximum principle we infer

$$|\partial_s v| \leq c\Phi_\varepsilon \quad \text{in } A_\varepsilon, \text{ for every } \varepsilon \in (0, \frac{1}{2}). \tag{5.30}$$

Now, the choice $r = 2\varepsilon$ in (5.30) gives

$$|\partial_s v| \leq c(r^{-1-2\beta} s^\beta \mathbf{1} \wedge \Gamma + r^{-1} \Gamma) \leq cr^{-1} \Gamma \quad \text{in } \{r < 1, 0 < s < \sigma r^2 / 2\}.$$

Recalling that also (5.17) holds, we get $|\partial_s v| \leq c\tilde{d}^{-1} \Gamma$ in the set $\{r < 1, 0 < s < 1\}$.

Arguing as above, using the barrier function

$$\Phi_\varepsilon = \varepsilon^{1-\beta} s^\beta (r - \varepsilon)^{-2-\beta} (1 + \varrho^2)^{\frac{2-\varrho}{4}} + \varepsilon^{-1},$$

one also obtains the estimate $|\partial_s v| \leq c\tilde{d}^{-1}$ in $\{r < 1, 0 < s < 1\}$. This concludes the proof. \square

6. Pohozaev identities and proof of the main theorem

Apart from Lemma 6.1 and Theorem 6.2, where \mathbb{G} denotes an arbitrary homogeneous Carnot group, throughout this section \mathbb{G} will be a fixed H-type group. We shall make use of the notation introduced in Section 2 for H-type groups (see in particular (2.5)). Moreover, u will denote any given solution of the boundary value problem (1.1) in a half-space $\Pi \subset \mathbb{G}$. The aim of this section is to prove our main result Theorem 1.1. The proof will be split in two parts: Theorem 6.3 is devoted to the case of non-characteristic half-spaces, whereas Theorem 6.4 is devoted to the case of characteristic ones. The proofs mainly rely on the integral identity of Rellich–Pohozaev type proved in Theorem 6.2 and on the well-behaved estimates of the derivatives of u , proved in Theorems 4.4 and 5.1 (in the non-characteristic and in the characteristic case, respectively). We point out that Pohozaev-type identities have been proved in the previous papers [26] in the setting of the Heisenberg group and in [27] for general Carnot groups.

We fix some notation. If $\Omega \subset \mathbb{R}^N$ is a domain with boundary regular enough, we denote by ν the outer unit normal to $\partial\Omega$ and by $d\sigma$ the Hausdorff $(N - 1)$ -dimensional measure on $\partial\Omega$. The following result is contained in [27, Theorem 3.1]; however we give the proof for the reader’s convenience.

Lemma 6.1. *Let \mathbb{G} be an arbitrary homogeneous Carnot group and let $\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2$. Let Ω be a bounded domain in \mathbb{G} , regular for the divergence theorem. Finally, let Z be a vector field of class $C^1(\mathbb{G})$. Then, for every $\varphi \in C^2(\overline{\Omega})$ we have*

$$\begin{aligned} & 2 \int_{\partial\Omega} \sum_{i=1}^m X_i \varphi \langle X_i, \nu \rangle Z \varphi \, d\sigma - \int_{\partial\Omega} \langle Z, \nu \rangle |\nabla_{\mathbb{G}} \varphi|^2 \, d\sigma \\ & = 2 \int_{\Omega} \sum_{i=1}^m X_i \varphi [X_i, Z] \varphi + 2 \int_{\Omega} \Delta_{\mathbb{G}} \varphi Z \varphi - \int_{\Omega} |\nabla_{\mathbb{G}} \varphi|^2 \operatorname{div}(Z). \end{aligned} \quad (6.1)$$

Proof. By applying two times the divergence theorem (and recalling that $X_i^* = -X_i$) we obtain

$$\begin{aligned} \int_{\partial\Omega} |\nabla_{\mathbb{G}}\varphi|^2 \langle Z, \nu \rangle \, d\sigma &= \int_{\Omega} \operatorname{div}(|\nabla_{\mathbb{G}}\varphi|^2 Z) = \int_{\Omega} |\nabla_{\mathbb{G}}\varphi|^2 \operatorname{div}(Z) + \int_{\Omega} Z(|\nabla_{\mathbb{G}}\varphi|^2) \\ &= \int_{\Omega} |\nabla_{\mathbb{G}}\varphi|^2 \operatorname{div}(Z) + 2 \int_{\Omega} \langle Z(\nabla_{\mathbb{G}}\varphi), \nabla_{\mathbb{G}}\varphi \rangle \\ &= \int_{\Omega} |\nabla_{\mathbb{G}}\varphi|^2 \operatorname{div}(Z) + 2 \int_{\Omega} \sum_{i=1}^m [Z, X_i]\varphi X_i\varphi \\ &\quad + 2 \int_{\Omega} \sum_{i=1}^m X_i(Z\varphi) X_i\varphi \\ &= \int_{\Omega} |\nabla_{\mathbb{G}}\varphi|^2 \operatorname{div}(Z) + 2 \int_{\Omega} \sum_{i=1}^m [Z, X_i]\varphi X_i\varphi \\ &\quad + 2 \int_{\partial\Omega} \sum_{i=1}^m Z\varphi X_i\varphi \langle X_i, \nu \rangle \, d\sigma - 2 \int_{\Omega} Z\varphi \Delta_{\mathbb{G}}\varphi. \end{aligned}$$

This ends the proof. \square

We now specify the integral identity (6.1) when Z is given by the so-called generator of the translations. Let \circ be the group law on the Carnot group \mathbb{G} and fix $z_0 \in \mathbb{G}$. We denote by Z^{z_0} the following vector field on \mathbb{G} ,

$$Z^{z_0}(z) = \left. \frac{d}{dh} \right|_{h=0} ((hz_0) \circ z). \tag{6.2}$$

For example, if \mathbb{G} is a H-type group and $z_0 = (x^0, t^0)$, then $Z^{z_0}(x, t)$ is given by (see (2.5))

$$Z^{z_0}(x, t) = \left(\begin{array}{c} x_j^0, \quad j = 1, \dots, m \\ t_j^0 + \frac{1}{2} \langle x^0, U^{(j)}x \rangle, \quad j = 1, \dots, n \end{array} \right). \tag{6.3}$$

We know that, for every $i = 1, \dots, m$, $X_i(z) = (d/dh)_{h=0}(z \circ (he_i))$, where e_i is the i th versor of the canonical basis of \mathbb{R}^N . From this fact and from the associativity of \circ , it follows that the bracket $[X_i, Z^{z_0}]$ vanishes identically. Indeed, for every smooth function f , one has

$$\begin{aligned} [X_i, Z^{z_0}]f(z) &= X_i(Z^{z_0}f)(z) - Z^{z_0}(X_i f)(z) \\ &= \left. \frac{d}{dh} \right|_{h=0} ((Z^{z_0}f)(z \circ (he_i))) - \left. \frac{d}{ds} \right|_{s=0} ((X_i f)((sz_0) \circ z)) \\ &= \left. \frac{d}{dh} \right|_{h=0} \left(\left. \frac{d}{ds} \right|_{s=0} (f(((sz_0) \circ (z \circ (he_i)))))) \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{d}{ds}\Big|_{s=0} \left(\frac{d}{dh}\Big|_{h=0} (f(((sz_0) \circ z) \circ (he_i))) \right) \\
 & = \left(\frac{d}{dh}\Big|_{h=0} \frac{d}{ds}\Big|_{s=0} \right) (f((sz_0) \circ z \circ (he_i)) - f((sz_0) \circ z \circ (he_i))) = 0.
 \end{aligned}$$

On the other hand, the divergence of the vector field Z^{z_0} vanishes identically. Indeed (following the notation in (2.1)) we recall that \circ has the form $z \circ \zeta = ((z \circ \zeta)^{(1)}, \dots, (z \circ \zeta)^{(r)})$, where

$$(z \circ \zeta)^{(1)} = z^{(1)} + \zeta^{(1)}, \quad (z \circ \zeta)^{(j)} = z^{(j)} + \zeta^{(j)} + Q^{(j)}(z, \zeta), \quad 2 \leq j \leq r,$$

$Q^{(j)}$ being a function with values in \mathbb{R}^{N_j} and whose components are mixed polynomials in z and ζ such that $Q^{(j)}(\delta_\lambda z, \delta_\lambda \zeta) = \lambda^j Q^{(j)}(z, \zeta)$. We then recognize that the components of $Z^{z_0}(z)$ in the j th layer have the following form

$$(Z^{z_0}(z))^{(j)} = z_0^{(j)} + \langle z_0^{(1)}, q_1^{(j)}(z) \rangle + \dots + \langle z_0^{(j-1)}, q_{j-1}^{(j)}(z) \rangle,$$

where (for any $i = 1, \dots, j - 1$) $q_i^{(j)}$ is a function with values in \mathbb{R}^{N_i} whose components are polynomials δ_λ -homogeneous of degree $j - i$. In particular $(Z^{z_0}(z))^{(j)}$ does not depend on z_j , whence $\text{div}(Z^{z_0}) = 0$. From Lemma 6.1 and the above remarks, the next result immediately follows.

Theorem 6.2. *Let \mathbb{G} be a homogeneous Carnot group and let $\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2$. Let Ω be a bounded domain in \mathbb{G} , regular for the divergence theorem. Finally, for a fixed $z_0 \in \mathbb{G}$, let Z^{z_0} be the vector field defined in (6.2). Then, for every $\varphi \in C^2(\overline{\Omega})$ we have*

$$2 \int_{\partial\Omega} \sum_{i=1}^m X_i \varphi \langle X_i, \nu \rangle Z^{z_0} \varphi \, d\sigma - \int_{\partial\Omega} \langle Z^{z_0}, \nu \rangle |\nabla_{\mathbb{G}} \varphi|^2 \, d\sigma = 2 \int_{\Omega} \Delta_{\mathbb{G}} \varphi Z^{z_0} \varphi. \quad (6.4)$$

With the integral identity (6.4) at hand, we can now turn to the proof of Theorem 1.1.

Theorem 6.3. *Let $\Pi^{(1)}$ be a non-characteristic half-space of the H-type group \mathbb{G} . Then the Dirichlet problem (1.1) has no solution in $\Pi^{(1)}$.*

Proof. Any non-characteristic half-space of \mathbb{G} has form (4.1). It is non-restrictive to suppose $|a| = 1$. For the sake of brevity, we denote $\Pi^{(1)}$ simply by Π . Let u be any non-negative weak solution of $\Delta_{\mathbb{G}} u = -u^{2^* - 1}$, $u \in S_0^1(\Pi)$. We explicitly remark that the second statement in Theorem 3.1 holds and, since Π is non-characteristic for $\Delta_{\mathbb{G}}$, then u is also smooth up to the boundary of Π : this follows by a suitable adaptation to H-type groups of the Kohn and Nirenberg Theorem (see [37], see also [31] for the case of the Heisenberg groups). We fix $R \gg 1$ and we set $B_R = B_d(0, R)$. With the

notation in (4.1), we apply formula (6.4) with $z_0 = (-a, 0)$, $\Omega = \Pi \cap B_R$ and $\varphi = u$. We first consider the right-hand term in (6.4):

$$\begin{aligned} -2 \int_{\Pi \cap B_R} u^{2^\star-1} Z^{z_0} u &= -2 \int_{\Pi \cap B_R} \operatorname{div} \left(\frac{u^{2^\star}}{2^\star} Z^{z_0} \right) \\ &= -2 \int_{\Pi \cap \partial B_R} \frac{u^{2^\star}}{2^\star} \langle Z^{z_0}, \nu \rangle \, d\sigma. \end{aligned} \tag{6.5}$$

Here we used the divergence theorem, jointly with $\operatorname{div}(Z^{z_0}) \equiv 0$ and $u \equiv 0$ on $\partial\Pi$. We now turn our attention to the left-hand side of (6.4): since u is smooth up to $\partial\Pi$, $u \equiv 0$ on $\partial\Pi$ and $u \geq 0$ in Π , then we have $|\nabla u| \nu = -\nabla u$ on $\partial\Pi$. This gives

$$\begin{aligned} \sum_{i=1}^m X_i u \langle X_i, \nu \rangle Z^{z_0} u &= \sum_{i=1}^m X_i u \left\langle X_i, \frac{-\nabla u}{|\nabla u|} \right\rangle \langle Z^{z_0}, -|\nabla u| \nu \rangle \\ &= \langle Z^{z_0}, \nu \rangle |\nabla_{\mathbb{G}} u|^2 \quad \text{on } \partial\Pi. \end{aligned} \tag{6.6}$$

Since moreover, $\nu = z_0$ on $\partial\Pi$, from (6.3) we get $\langle Z^{z_0}, \nu \rangle = 1$ on $\partial\Pi$. Collecting together (6.5) and (6.6), (6.4) becomes

$$\begin{aligned} \int_{B_R \cap \partial\Pi} |\nabla_{\mathbb{G}} u|^2 \, d\sigma &= \int_{\Pi \cap \partial B_R} \left(\langle Z^{z_0}, \nu \rangle \left(|\nabla_{\mathbb{G}} u|^2 - 2 \frac{u^{2^\star}}{2^\star} \right) \right. \\ &\quad \left. - 2 \sum_{i=1}^m X_i u \langle X_i, \nu \rangle Z^{z_0} u \right) \, d\sigma. \end{aligned} \tag{6.7}$$

The rest of the proof consists in showing that there exists a sequence of radii $R_k \uparrow \infty$ such that, when R is replaced by R_k , the right-hand side of (6.7) goes to zero as $k \rightarrow \infty$. Indeed, from (6.7) this will give $\nabla_{\mathbb{G}} u = 0$ on $\partial\Pi$; as a consequence, by the unique continuation result in [27, Corollary 10.7], we shall obtain $u \equiv 0$ in Π .

We now turn to the estimate of the right-hand side of (6.7). On ∂B_R we have $\nu = \nabla d / |\nabla d|$. Hence, from (6.3) and $z_0 = (-a, 0)$, we get

$$|\langle Z^{z_0}, \nu \rangle| = \frac{1}{|\nabla d|} \left| \sum_{j=1}^m a_j \partial_{x_j} d + \frac{1}{2} \sum_{j=1}^n \langle a, U^{(j)} x \rangle \partial_{t_j} d \right| \leq \frac{c}{|\nabla d|} \quad \text{in } \Pi \cap \partial B_R. \tag{6.8}$$

Indeed, the functions $\partial_{x_j} d$ and $\langle a, U^{(j)} x \rangle \partial_{t_j} d$ are bounded since they are smooth away from the origin and homogeneous of degree zero w.r.t. the dilations of \mathbb{G} . Moreover, we have

$$\begin{aligned} |Z^{z_0} u| &= \left| \sum_{j=1}^m a_j \partial_{x_j} u + \frac{1}{2} \sum_{j=1}^n \langle a, U^{(j)} x \rangle \partial_{t_j} u \right| \\ &\leq c(|\nabla_{\mathbb{G}} u| + d |\nabla_t u|) \leq c(|\nabla_{\mathbb{G}} u| + d^{3-Q}). \end{aligned} \tag{6.9}$$

Here we used the following facts: since the group law in \mathbb{G} is given by (2.5), we have $\nabla_{\mathbb{G}} = (X_1, \dots, X_m)$ with $X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^n \sum_{i=1}^m U_{i,j}^{(k)} x_i \partial_{t_k}$; the functions x_i and $\langle a, U^{(j)} x \rangle$ are bounded by cd , since they are smooth and homogeneous of degree one w.r.t. the dilations of \mathbb{G} ; $|\nabla_t u| \leq cd^{2-Q}$ follows from Theorem 4.4, recalling that any partial derivative ∂_{t_j} is a second layer derivative and u is smooth up to the boundary of Π . Finally, again from $v = \nabla d / |\nabla d|$ on ∂B_R and since $|\nabla_{\mathbb{G}} d|$ is bounded, one has

$$|X_i u \langle X_i, v \rangle| = \frac{|X_i u X_i d|}{|\nabla d|} \leq \frac{c |\nabla_{\mathbb{G}} u|}{|\nabla d|} \quad \text{in } \Pi \cap \partial B_R. \tag{6.10}$$

Collecting together (6.8)–(6.10), the right-hand side of (6.7) is bounded by

$$\begin{aligned} & c \int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{G}} u|^2 + u^{2^*}}{|\nabla d|} d\sigma + c \int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{G}} u| d^{3-Q}}{|\nabla d|} d\sigma \\ & \leq c \int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{G}} u|^2 + u^{2^*}}{|\nabla d|} d\sigma + cR^{3-Q} \left(\int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} d\sigma \right)^{1/2} \\ & \quad \times \left(\int_{\Pi \cap \partial B_R} \frac{d\sigma}{|\nabla d|} \right)^{1/2}. \end{aligned} \tag{6.11}$$

By Federer’s co-area formula [20], we have

$$\begin{aligned} \int_{\partial B_R} |\nabla d|^{-1} d\sigma &= \frac{d}{dR} \int_0^R \left(\int_{\partial B_\rho} |\nabla d|^{-1} d\sigma \right) d\rho = \frac{d}{dR} \left(\int_{B_R} dz \right) = \frac{d}{dR} (cR^Q) \\ &= \tilde{c}R^{Q-1}. \end{aligned} \tag{6.12}$$

Moreover, arguing as in (6.12), the following simple fact holds:

$$g \in L^1(\mathbb{G}) \Rightarrow \left(\exists R_k \uparrow \infty : \int_{d=R_k} \frac{g d\sigma}{|\nabla d|} = o(1/R_k), \text{ as } k \uparrow \infty \right). \tag{6.13}$$

Now, we choose a sequence $R_k \uparrow \infty$ such that (6.13) holds when $g = |\nabla_{\mathbb{G}} u|^2 + u^{2^*}$ (this is a $L^1(\mathbb{G})$ function since $u \in S_0^1(\Pi)$). From (6.12), replacing R by R_k , the right-hand side in (6.11) is then bounded by $o(R_k^{-1}) + R_k^{(5-Q)/2} o(R_k^{-1/2}) \rightarrow 0$, as $k \rightarrow \infty$ (since $Q \geq 4$). This completes the proof of the theorem. \square

Although the proof of the next result follows the line of the previous one, the presence of characteristic points complicates the arguments.

Theorem 6.4. *Let $\Pi^{(2)}$ be a characteristic half-space of the H-type group \mathbb{G} . Then the Dirichlet problem (1.1) has no solution in $\Pi^{(2)}$.*

Proof. Any characteristic half-space of \mathbb{G} can be left-translated into a half-space of form (5.1). For the sake of brevity, we denote $\Pi^{(2)}$ simply by Π . We explicitly remark that the characteristic set of Π is (see Appendix A for the details)

$$\text{char}(\Pi) = \{(x, t) \in \mathbb{G} \mid x = 0, \langle b, t \rangle = 0\}.$$

Throughout the proof, we follow the notation introduced in (5.2). With this notation, we have $\Pi = \{s > 0\}$, $\text{char}(\Pi) = \{r = s = 0\}$. Let u be any non-negative weak solution of $\Delta_{\mathbb{G}}u = -u^{2^{\star}-1}$, $u \in S_0^1(\Pi)$. We explicitly remark that the second statement in Theorem 3.1 holds and, by a suitable version of the Kohn and Nirenberg Theorem for H-type groups, u is smooth up to $\partial\Pi$ away from the characteristic set. We now aim to apply the Pohozaev-type identity (6.4) on a suitable domain far from $\text{char}(\Pi)$.

For fixed $0 < \varepsilon \ll 1$, $k \in \mathbb{N}$, we introduce the set

$$\Pi_{\varepsilon}^k = \{s > 0, d < R_k, \tilde{d} > \varepsilon\},$$

where R_k is a divergent sequence of radii, to be chosen in the sequel. With the notation in (5.1), we apply formula (6.4) with $z_0 = (0, -b)$, $\Omega = \Pi_{\varepsilon}^k$ and $\varphi = u$. We explicitly remark that $Z^{z_0} = -(\partial/\partial_s)$. We introduce the following notation:

$$\partial(\Pi_{\varepsilon}^k) = F_1^{k,\varepsilon} \cup F_2^{k,\varepsilon} \cup F_3^{k,\varepsilon}, \quad \text{where} \quad \begin{cases} F_1^{k,\varepsilon} = \{s = 0, d \leq R_k, \tilde{d} \geq \varepsilon\}, \\ F_2^{k,\varepsilon} = \{s \geq 0, d = R_k, \tilde{d} \geq \varepsilon\}, \\ F_3^{k,\varepsilon} = \{s \geq 0, d \leq R_k, \tilde{d} = \varepsilon\}. \end{cases}$$

The notation $F_1^{\infty,\varepsilon}$ and $F_3^{\infty,\varepsilon}$ will also be allowed, with the obvious meaning. Moreover, we shall write $f \wedge g = \min\{f, g\}$. Arguing as in (6.5) and (6.6) the following analogue of identity (6.7) holds:

$$\begin{aligned} \int_{F_1^{k,\varepsilon}} |\nabla_{\mathbb{G}}u|^2 \, d\sigma &= \int_{F_2^{k,\varepsilon} \cup F_3^{k,\varepsilon}} \left(\langle Z^{z_0}, v \rangle \left(|\nabla_{\mathbb{G}}u|^2 - 2 \frac{u^{2^{\star}}}{2^{\star}} \right) \right. \\ &\quad \left. - 2 \sum_{i=1}^m X_i u \langle X_i, v \rangle Z^{z_0} u \right) d\sigma. \end{aligned} \tag{6.14}$$

We turn to estimate the right-hand side of (6.14). We begin with $\langle Z^{z_0}, v \rangle$: on $F_2^{k,\varepsilon}$ we have $v = \nabla d / |\nabla d|$, whence $\langle Z^{z_0}, v \rangle = |\partial_s d| / |\nabla d| = c s d^{-3} / |\nabla d| \leq c / (d |\nabla d|) = c / (R_k |\nabla d|)$. Analogously, on $F_3^{k,\varepsilon}$ it is $v = -\nabla \tilde{d} / |\nabla \tilde{d}|$, and we have $|\langle Z^{z_0}, v \rangle| \leq c / (\varepsilon |\nabla \tilde{d}|)$. This gives

$$|\langle Z^{z_0}, v \rangle| \leq \begin{cases} c / (R_k |\nabla d|) & \text{on } F_2^{k,\varepsilon}, \\ c / (\varepsilon |\nabla \tilde{d}|) & \text{on } F_3^{\infty,\varepsilon}. \end{cases} \tag{6.15}$$

Moreover, from the estimate of $\partial_s u$ in Theorem 5.1, we obtain

$$|Z^{z_0} u| = |\partial_s u| \leq \begin{cases} c \frac{1}{\varepsilon} R_k^{2-Q} & \text{on } F_2^{k,\varepsilon}, \\ c \frac{1}{\varepsilon} (1 + \varrho^2)^{(2-Q)/4} & \text{on } F_3^{\infty,\varepsilon}. \end{cases} \tag{6.16}$$

Here we used the fact that on $F_3^{\infty,\varepsilon}$ it is $1 \wedge d^{2-Q} = 1 \wedge (\tilde{d}^4 + 16\varrho^2)^{(2-Q)/4} = 1 \wedge (\varepsilon^4 + 16\varrho^2)^{(2-Q)/4} \leq c(1 + \varrho^2)^{(2-Q)/4}$. Finally, being $v = \nabla d / |\nabla d|$ on $F_2^{k,\varepsilon}$ and $v = -\nabla \tilde{d} / |\nabla \tilde{d}|$ on $F_3^{\infty,\varepsilon}$, we have

$$|X_i u \langle X_i, v \rangle| \leq \begin{cases} c |\nabla_{\mathbb{G}} u| / |\nabla d| & \text{on } F_2^{k,\varepsilon}, \\ c |\nabla_{\mathbb{G}} u| / |\nabla \tilde{d}| & \text{on } F_3^{\infty,\varepsilon}. \end{cases} \tag{6.17}$$

Here we used $|\nabla_{\mathbb{G}} d| \leq c$ and the following estimate (see (2.5)):

$$X_i \tilde{d} = \frac{r^2 x_i}{\tilde{d}^3} + 4 \sum_{h \leq n, j \leq m} U_{j,i}^{(h)} \frac{s x_j b_h}{\tilde{d}^3} \leq c \frac{r^3 + rs}{(r^4 + s^2)^{3/4}} \leq c.$$

We claim that the integral over $F_2^{k,\varepsilon}$ in the right-hand side of (6.14) goes to zero as $k \rightarrow \infty$, for every fixed $\varepsilon \in (0, 1)$. This will give

$$\begin{aligned} \int_{F_1^{\infty,\varepsilon}} |\nabla_{\mathbb{G}} u|^2 \, d\sigma &= \lim_{k \rightarrow \infty} \int_{F_1^{k,\varepsilon}} |\nabla_{\mathbb{G}} u|^2 \, d\sigma \leq \lim_{k \rightarrow \infty} \left(\left| \int_{F_2^{k,\varepsilon}} (\dots) \right| + \left| \int_{F_3^{k,\varepsilon}} (\dots) \right| \right) \\ &\leq \int_{F_3^{\infty,\varepsilon}} \left(|\langle Z^{z_0}, v \rangle| \left(|\nabla_{\mathbb{G}} u|^2 + 2 \frac{u^{2^*}}{2^*} \right) \right. \\ &\quad \left. + 2 \sum_{i=1}^m |X_i u \langle X_i, v \rangle Z^{z_0} u| \right) \, d\sigma. \end{aligned} \tag{6.18}$$

We now prove the above claim: collecting together (6.15)–(6.17) and arguing exactly as in the estimate of (6.11), the integral over $F_2^{k,\varepsilon}$ in the right-hand side of (6.14) is bounded by

$$\frac{c}{R_k} \int_{\{d=R_k\}} \frac{|\nabla_{\mathbb{G}} u|^2 + u^{2^*}}{|\nabla d|} \, d\sigma + \frac{c R_k^{2-Q}}{\varepsilon} \int_{\{d=R_k\}} \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \, d\sigma = \frac{c o(1)}{R_k^2} + \frac{o(1)}{\varepsilon R_k^{(Q-2)/2}} \xrightarrow{k \rightarrow \infty} 0.$$

This proves the claim. We now prove that, by a suitable choice of a sequence $\varepsilon_j \rightarrow 0^+$, the right-hand side of (6.18) goes to zero as $j \rightarrow \infty$. This will give

$$\int_{\partial \Pi} |\nabla_{\mathbb{G}} u|^2 \, d\sigma = \lim_{\varepsilon \rightarrow 0} \int_{F_1^{\infty,\varepsilon}} |\nabla_{\mathbb{G}} u|^2 \, d\sigma = 0.$$

By means of a unique continuation argument, as in the proof of Theorem 6.3, this will complete the proof. Collecting (6.15)–(6.17), the right-hand side of (6.18) is

bounded by

$$\begin{aligned} & \frac{c}{\varepsilon} \int_{F_3^{\infty,\varepsilon}} \frac{|\nabla_{\mathbb{G}} u|^2 + u^{2^*}}{|\nabla \tilde{d}|} d\sigma + \frac{c}{\varepsilon} \int_{F_3^{\infty,\varepsilon}} \frac{|\nabla_{\mathbb{G}} u|}{|\nabla \tilde{d}|} (1 + \varrho^2)^{\frac{2-Q}{4}} d\sigma \\ & \leq \frac{c}{\varepsilon} \int_{\tilde{d}=\varepsilon} \frac{(1 + \varrho^2)^{\frac{2-Q}{2}}}{|\nabla \tilde{d}|} d\sigma + \frac{c}{\varepsilon} \left(\int_{\tilde{d}=\varepsilon} \frac{(1 + \varrho^2)^{\frac{2-Q}{2}}}{|\nabla \tilde{d}|} d\sigma \right)^{1/2} \left(\int_{\tilde{d}=\varepsilon} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla \tilde{d}|} d\sigma \right)^{1/2}. \end{aligned} \tag{6.19}$$

Here we applied estimates (3.8) and (5.21) which give

$$|\nabla_{\mathbb{G}} u|^2 + u^{2^*} \leq c(1 \wedge \Gamma)^2 + c(1 \wedge \Gamma)^{2^*} \leq c(1 \wedge \Gamma)^2 \leq c(1 + \varrho^2)^{\frac{2-Q}{2}} \quad \text{on } \{\tilde{d} = \varepsilon\}.$$

Following the notation in (5.2), we have

$$\{(x, t) \in \mathbb{G} \mid \tilde{d} = \varepsilon\} = \{(x, s) \in \mathbb{R}^{m+1} \mid \tilde{d}(x, s) = \varepsilon\} \times \{t' \in \mathbb{R}^{n-1}\}.$$

As a consequence, denoting by H^j the Hausdorff j -dimensional measure, one has (arguing as in (6.12) and taking into account that $Q - n > 1$)

$$\begin{aligned} \int_{\tilde{d}=\varepsilon} \frac{(1 + \varrho^2)^{\frac{2-Q}{2}}}{|\nabla \tilde{d}|} d\sigma &= \int_{\tilde{d}(x,s)=\varepsilon} \frac{dH^m}{|\nabla \tilde{d}|} \cdot \int_{t' \in \mathbb{R}^{n-1}} (1 + |t'|^2)^{\frac{2-Q}{2}} dH^{n-1} \\ &= c\varepsilon^{m+1} \cdot \int_0^\infty (1 + \varrho^2)^{\frac{2-Q}{2}} \varrho^{n-2} d\varrho = \tilde{c}\varepsilon^{m+1}. \end{aligned} \tag{6.20}$$

Finally, we make use of (6.20) and of an analogous version of (6.13) when d is replaced by \tilde{d} and $g = |\nabla_{\mathbb{G}} u|^2$: we can thus derive that there exists a sequence of positive numbers $\{\varepsilon_j\}_j$ such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$ and such that (when $\varepsilon = \varepsilon_j$) the right-hand side of (6.19) is bounded by $c\varepsilon_j^m + \varepsilon_j^{(m-2)/2} o(1) \rightarrow 0$, as $j \rightarrow \infty$. This completes the proof. \square

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Appendix A. Remarks on H-type groups

The main aim of this section is to prove that (under a suitable system of coordinates) the group law of any H-type group has the somewhat explicit form (2.5). Moreover, we furnish several details on H-type groups which have been used throughout the paper.

Definition A.1. A H-type group is a Carnot group \mathbb{G} of step two with the following property: the Lie algebra \mathfrak{g} of \mathbb{G} is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that, if \mathfrak{z} is the center of \mathfrak{g} , then $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and moreover, for every fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \rightarrow \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle \quad \forall w \in \mathfrak{z}^\perp, \tag{A.1}$$

is an orthogonal map whenever $\langle z, z \rangle = 1$.

By means of the natural identification of a Carnot group with its Lie algebra, it is always non-restrictive to suppose to deal with groups endowed with a homogeneous structure. More explicitly, we can suppose that a system of Cartesian coordinates on \mathbb{R}^N is given with respect to which the family of dilations (2.1) forms a family of group automorphisms. We explicitly remark that, if $\mathfrak{G}_1 \oplus \dots \oplus \mathfrak{G}_r$ is the stratification of the algebra of a Carnot group (according to the classical definition), then, following the notation introduced in (2.1), we have

$$N_1 = \dim(\mathfrak{G}_1), \dots, N_r = \dim(\mathfrak{G}_r). \tag{A.2}$$

Moreover, it is non-restrictive to suppose that the group inverse x^{-1} is given by $-x$. As a consequence of these facts, simple arguments (which make use of the explicit coordinate expression of the group dilations) show that the group law of any step-two Carnot group has form (2.2).

Let now \mathbb{G} be a H-type group according to Definition A.1. We set $m = \dim(\mathfrak{z}^\perp)$ and $n = \dim(\mathfrak{z})$. Since \mathbb{G} has step two and since the stratification of the Lie algebra \mathfrak{g} is evidently $\mathfrak{z}^\perp \oplus \mathfrak{z}$, in the sequel we shall fix on \mathbb{G} a system of coordinates (x, t) and we shall suppose that the dilations on \mathbb{G} are (see (7.2)) $\delta_\lambda(x, t) = (\lambda x, \lambda^2 t)$ and that the group law has the form

$$(x, t) \circ (\xi, \tau) = \left(\begin{array}{l} x_j + \xi_j, \quad j = 1, \dots, m \\ t_j + \tau_j + \frac{1}{2} \langle x, U^{(j)} \xi \rangle, \quad j = 1, \dots, n \end{array} \right), \quad x, \xi \in \mathbb{R}^m, \quad t, \tau \in \mathbb{R}^n, \tag{A.3}$$

for suitable skew-symmetric matrices $U^{(j)}$'s. Our main aim is to give a necessary and sufficient condition on the $U^{(j)}$'s such that a composition law as in (7.3) defines on \mathbb{R}^{m+n} a H-type group. A complete answer is given by the following result.

Theorem A.2. \mathbb{G} is a H-type group if and only if \mathbb{G} is (isomorphic to) \mathbb{R}^{m+n} with the group law in (A.3) and the matrices $U^{(1)}, \dots, U^{(n)}$ have the following properties:

1. $U^{(j)}$ is a $m \times m$ skew-symmetric and orthogonal matrix, for every $j = 1, \dots, n$;
2. $U^{(i)} U^{(j)} + U^{(j)} U^{(i)} = 0$, for every $i, j \in \{1, \dots, n\}$ with $i \neq j$.

Remark A.3. Conditions (1) and (2) imply that $U^{(1)}, \dots, U^{(n)}$ are linearly independent.

Namely, as it will appear from the proof, if (1) and (2) holds, then $\sum_{s=1}^n z_s U^{(s)}$ is the product of $|z|$ times an orthogonal matrix. For example, the following three matrices

$$\begin{aligned}
 U^{(1)} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & U^{(2)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
 U^{(3)} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

satisfy conditions (1)–(2) above and hence, through (A.3), they define on \mathbb{R}^7 a H-type group whose center has dimension 3.

Proof. First of all, we prove the “only if” part of the theorem. Let \mathbb{G} be a H-type group: we make use of all the notation in Definition A.1. Let B_1, \dots, B_m and Z_1, \dots, Z_n be orthonormal bases for $\mathfrak{b} := \mathfrak{z}^\perp$ and \mathfrak{z} , respectively. The hypothesis $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{z}$ ensures that there exist scalars $U_{i,j}^{(s)}$ such that

$$[B_i, B_j] = \sum_{s=1}^n U_{i,j}^{(s)} Z_s, \quad \forall i, j \in \{1, \dots, m\}. \tag{A.4}$$

With the notation in (A.4), we define the following square matrices:

$$U^{(s)} := (U_{i,j}^{(s)})_{i,j \leq m}, \quad s = 1, \dots, n. \tag{A.5}$$

Since $[B_i, B_j] = -[B_j, B_i]$, $U^{(s)}$ is skew-symmetric. We now recall that the group (\mathbb{G}, \circ) is canonically isomorphic (via the exponential map) to its Lie algebra \mathfrak{g} (endowed with the Campbell–Hausdorff operation $(X, Y) \mapsto X + Y + \frac{1}{2}[X, Y]$). As a consequence, by means of the fixed basis $B_1, \dots, B_m, Z_1, \dots, Z_n$ on \mathfrak{g} , we can identify \mathbb{G} to $\mathfrak{g} \cong \mathbb{R}^{m+n}$ with the group law (A.3), where the matrices $U^{(j)}$ ’s are as in (A.5).

We fix $v = \sum_{i=1}^m v_i B_i$ and $z = \sum_{i=1}^n z_i Z_i$. We look for $x_1, \dots, x_m \in \mathbb{R}$ such that $J_z(v) = \sum_{i=1}^m x_i B_i$ satisfies condition (A.1) of Definition A.1, for every

$w = \sum_{i=1}^m w_i B_i$. Precisely

$$\begin{aligned} \sum_{j=1}^m x_j w_j &= \left\langle \sum_{i=1}^m x_i B_i, \sum_{i=1}^m w_i B_i \right\rangle = \langle J_z(v), w \rangle = \langle z, [v, w] \rangle \\ &= \left\langle \sum_{i=1}^n z_i Z_i, \sum_{i,j=1}^m v_i w_j [B_i, B_j] \right\rangle \\ &= \left\langle \sum_{i=1}^n z_i Z_i, \sum_{i,j=1}^m v_i w_j \sum_{s=1}^n U_{i,j}^{(s)} Z_s \right\rangle \\ &= \sum_{j=1}^m w_j \left(\sum_{s=1}^n \sum_{i=1}^m v_i U_{i,j}^{(s)} z_s \right). \end{aligned} \tag{A.6}$$

Since w is arbitrary, this gives $x_j = \sum_{s \leq n} \sum_{i \leq m} v_i U_{i,j}^{(s)} z_s$, whence

$$J_z : \mathfrak{b} \rightarrow \mathfrak{b}, \quad \sum_{j=1}^m v_j B_j \mapsto \sum_{j=1}^m \left(\sum_{s \leq n} \sum_{i \leq m} v_i U_{i,j}^{(s)} z_s \right) B_j.$$

As a consequence, w.r.t. the orthonormal basis B_1, \dots, B_m for \mathfrak{b} , the endomorphism J_z is represented by the following matrix (hereafter, the transpose of a matrix A is denoted by A^T)

$$\sum_{s=1}^n z_s \begin{pmatrix} U_{1,1}^{(s)} & \dots & U_{m,1}^{(s)} \\ \vdots & \ddots & \vdots \\ U_{1,m}^{(s)} & \dots & U_{m,m}^{(s)} \end{pmatrix} = \left(\sum_{s=1}^n z_s U^{(s)} \right)^T. \tag{A.7}$$

By Definition A.1, this matrix has to be orthogonal, whenever $\sum_{s=1}^n (z_s)^2 = 1$. In particular, this implies that every $U^{(s)}$ is orthogonal (whence condition (1) of the assertion is proved). We explicitly remark that since $U^{(s)}$ is both skew-symmetric and orthogonal, $U^{(s)}$ has no real eigenvalues, whence m is necessarily even. Moreover, $-U^{(s)} U^{(s)} = \mathbb{1}_m$ (the unit matrix of order m). If $\sum_{s=1}^n (z_s)^2 = 1$, the matrix $\sum_{s=1}^n z_s U^{(s)}$ is orthogonal if and only if

$$\begin{aligned} \mathbb{1}_m &= \left(\sum_{s=1}^n z_s U^{(s)} \right) \cdot \left(\sum_{s=1}^n z_s U^{(s)} \right)^T = - \sum_{r,s \leq n} z_r z_s U^{(r)} U^{(s)} \\ &= - \sum_{r \leq n} z_r^2 U^{(r)} U^{(r)} - \sum_{r,s \leq n, r \neq s} z_r z_s U^{(r)} U^{(s)} = \mathbb{1}_m - \sum_{r,s \leq n, r \neq s} z_r z_s U^{(r)} U^{(s)}. \end{aligned}$$

Therefore, we have

$$\sum_{r,s=1,r \neq s}^n z_r z_s U^{(r)} U^{(s)} = 0, \quad \forall z_1, \dots, z_n: \sum_{s=1}^n z_s^2 = 1. \tag{A.8}$$

If in (A.8) we take $z = (0, \dots, 1/\sqrt{2}, \dots, 1/\sqrt{2}, \dots, 0)$, we obtain

$$U^{(i)} U^{(j)} + U^{(j)} U^{(i)} = 0, \quad \text{for every } i, j \in \{1, \dots, n\} \text{ with } i \neq j, \tag{A.9}$$

which is condition (2) of the assertion. Since we also have

$$\sum_{r,s \leq n, r \neq s} z_r z_s U^{(r)} U^{(s)} = \sum_{r,s \leq n, r < s} z_r z_s (U^{(r)} U^{(s)} + U^{(s)} U^{(r)}),$$

(A.8) turns out to be equivalent to (A.9).

We now prove the “if” part of the theorem. Let $U^{(1)}, \dots, U^{(n)}$ be matrices satisfying conditions (1)–(2) of the assertion. Suppose \mathbb{R}^{m+n} is endowed with the composition law (A.3). It is immediately verified that \circ defines a Lie group nilpotent of step at most two in which the identity is the origin and the inverse of (x, t) is $(-x, -t)$. Moreover, $\delta_\lambda(x, t) = (\lambda x, \lambda^2 t)$ is a group of automorphisms. An easy computation shows that the vector field in the algebra \mathfrak{g} of $\mathbb{G} = (\mathbb{R}^{m+n}, \circ)$ that agrees at the origin with $\partial/\partial x_j$ ($j = 1, \dots, m$) is given by

$$X_j = (\partial/\partial x_j) + \frac{1}{2} \sum_{s=1}^n \left(\sum_{i=1}^m x_i U_{i,j}^{(s)} \right) (\partial/\partial t_s), \tag{A.10}$$

and that \mathfrak{g} is spanned by $X_1, \dots, X_m, \partial/\partial t_1, \dots, \partial/\partial t_n$. From (A.10) and the skew-symmetry of $U^{(s)}$ we obtain $[X_i, X_j] = \sum_{s=1}^n U_{i,j}^{(s)} (\partial/\partial t_s)$, for every $i, j \in \{1, \dots, m\}$. Now, since $U^{(1)}, \dots, U^{(n)}$ are linearly independent (see Remark A.3) the dimension of the vector space spanned by $(U_{i,j}^{(1)}, \dots, U_{i,j}^{(n)})$ as $i, j \in \{1, \dots, m\}$, equals n . As a consequence, \mathbb{G} is a homogeneous Carnot group. For every $s = 1, \dots, n$, we set $Z_s = \partial/\partial t_s$.

We claim that \mathfrak{z} , the center of \mathfrak{g} , is spanned by Z_1, \dots, Z_n . Indeed, we suppose by contradiction that (for suitable scalars α_i 's) $\sum_{i=1}^m \alpha_i X_i \in \mathfrak{z}$, i.e.,

$$0 = \left[\sum_{i=1}^m \alpha_i X_i, X_j \right] = \sum_{s=1}^n \sum_{i=1}^m \alpha_i U_{i,j}^{(s)} Z_s, \quad \text{for every } j \in \{1, \dots, m\}.$$

Since the Z_s 's are linearly independent, this means that $\sum_{i=1}^m \alpha_i U_{i,j}^{(s)} = 0$ for every $s \leq n$ and every $j \leq m$, i.e., $\alpha = (\alpha_1, \dots, \alpha_m)$ belongs to the kernel of the transpose matrix of $U^{(s)}$, for every $s \leq n$. Since every $U^{(s)}$ is orthogonal, this is possible only if $\alpha = 0$, which proves the claim.

Finally, let \langle, \rangle be the standard inner product on \mathfrak{g} w.r.t. the basis $X_1, \dots, X_m, Z_1, \dots, Z_n$. For what has been proved above, we have $\mathfrak{z}^\perp = \text{span}\{X_1, \dots, X_m\}$ and

$[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \text{span}\{Z_1, \dots, Z_n\} = \mathfrak{z}$. By means of a computation analogous to (7.6), it is easy to recognize that conditions (1) and (2) of the assertion ensure that (with the above choice of \langle, \rangle) \mathbb{G} is a H-type group, according to Definition A.1. This completes the proof. \square

From the above proof it is also evident that the following characterization holds for general step-two Carnot groups.

Remark A.4. The N -dimensional Carnot groups of step two and m generators are characterized by being (canonically isomorphic to) (\mathbb{R}^N, \circ) with the following Lie group law ($N = m + n$, $x^{(1)} \in \mathbb{R}^m$, $x^{(2)} \in \mathbb{R}^n$),

$$(x^{(1)}, x^{(2)}) \circ (y^{(1)}, y^{(2)}) = \left(\begin{array}{l} x_j^{(1)} + y_j^{(1)}, \quad j = 1, \dots, m \\ x_j^{(2)} + y_j^{(2)} + \frac{1}{2} \langle x^{(1)}, S^{(j)} y^{(1)} \rangle, \quad j = 1, \dots, n \end{array} \right),$$

where the $S^{(j)}$'s are $m \times m$ linearly independent skew-symmetric matrices.

From the explicitness of operation (A.3), we can derive some more properties of H-type groups. We first describe characteristic and non-characteristic half-spaces; then we give the explicit form of the canonical sub-Laplacian.

Remark A.5. The general half-space of the H-type group \mathbb{G} , $\Pi = \{(x, t) \in \mathbb{G} \mid \langle a, x \rangle + \langle b, t \rangle > c\}$ (here $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are fixed) possesses characteristic points (w.r.t. the canonical sub-Laplacian) if and only if $b \neq 0$. Any half-space with characteristic points can be left-translated into a half-space of the type $\{(x, t) \in \mathbb{G} \mid \langle b, t \rangle > 0\}$.

Indeed, a point $(x, t) \in \partial \Pi$ is characteristic if and only if for every $j \leq m$ we have $X_j(x, t) \in \{(\xi, \tau) \in \mathbb{R}^{m+n} \mid \langle a, \xi \rangle + \langle b, \tau \rangle = 0\}$. Since the canonical sub-Laplacian is $\sum_{j=1}^m X_j^2$ where X_j is given by (A.10), this is equivalent to $a_j + \frac{1}{2} \sum_{s=1}^n (\sum_{i=1}^m x_i U_{i,j}^{(s)}) b_s = 0$ for every $j \leq m$, or equivalently,

$$(Ch) \quad \begin{cases} \langle a, x \rangle + \langle b, t \rangle = c, \\ (\sum_{s=1}^n b_s U^{(s)})^T x = -2a. \end{cases}$$

If $b = 0$ then $a \neq 0$ (otherwise Π would not define a half-space) and the second equation in (Ch) has clearly no solution. Suppose now $b \neq 0$. Then, by (A.7), the matrix $(\sum_{s=1}^n b_s / |b| U^{(s)})^T$ is orthogonal hence non-singular. As a consequence, the second equation in (Ch) admits a solution $\bar{x} \in \mathbb{R}^m$. The characteristic set for Π is then given by

$$\{(x, t) \in \mathbb{G} \mid x = \bar{x}, \langle b, t \rangle = c - \langle a, \bar{x} \rangle\}.$$

Finally, let Π be as in the assertion with $b \neq 0$. If $(\xi, \tau) \in \mathbb{G}$ is fixed, we set $\Pi = (\xi, \tau) \circ \tilde{\Pi}$, i.e., $\tilde{\Pi}$ is the set of all points $(\tilde{x}, \tilde{t}) \in \mathbb{G}$ such that (see (A.3))

$$\sum_{j=1}^n b_j(\tilde{t}_j + \tau_j) + \sum_{j=1}^m a_j \xi_j + \sum_{j=1}^m \tilde{x}_j \left(a_j + \frac{1}{2} \sum_{s=1}^n b_s \sum_{i=1}^m \xi_i U_{i,j}^{(s)} \right) > c.$$

As we discussed above, there exists $\bar{\xi} \in \mathbb{R}^m$ such that $a_j + \frac{1}{2} \sum_{s=1}^n b_s \sum_{i=1}^m \bar{\xi}_i U_{i,j}^{(s)} = 0$ for every $j \leq m$. With such a choice of $\bar{\xi}$, again from $b \neq 0$, there also exists $\tau \in \mathbb{R}^n$ such that $\sum_{j=1}^n b_j \tau_j = c - \sum_{j=1}^m a_j \bar{\xi}_j$. In this way, the half-space $\tilde{\Pi}$ is obtained from Π by the left-translation by $(\bar{\xi}, \bar{\tau})^{-1}$ and we have $\tilde{\Pi} = \{(\tilde{x}, \tilde{t}) \in \mathbb{G} \mid \langle b, \tilde{t} \rangle > 0\}$. Remark A.5 is thus completely proved.

Remark A.6. The canonical sub-Laplacian on the H-type group \mathbb{G} is given by

$$\Delta_{\mathbb{G}} = \Delta_x + \frac{1}{4}|x|^2 \Delta_t + \sum_{s=1}^n \langle x, U^{(s)} \nabla_x \rangle \frac{\partial}{\partial t_s}, \tag{A.11}$$

where the $U^{(s)}$ are as in Theorem A.2 (here we used the notation $\Delta_x = \sum_{j=1}^m \left(\frac{\partial}{\partial x_j}\right)^2$, $\Delta_t = \sum_{s=1}^n \left(\frac{\partial}{\partial t_s}\right)^2$ and $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)^T$. Moreover, on functions $u(x, t) = \tilde{u}(|x|, t)$, $\Delta_{\mathbb{G}}$ has the form

$$\Delta_{\mathbb{G}} = \Delta_x + \frac{1}{4}|x|^2 \Delta_t = \left(\frac{\partial}{\partial r}\right)^2 + \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{1}{4} r^2 \Delta_t, \quad r = |x| \neq 0. \tag{A.12}$$

In particular, following the notation in (5.2), $\Delta_{\mathbb{G}}$ on functions $u(x, t) = \tilde{u}(r, s, \varrho)$ has the following form:

$$\Delta_{\mathbb{G}} = \left(\frac{\partial}{\partial r}\right)^2 + \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{1}{4} r^2 \left(\frac{\partial}{\partial s}\right)^2 + \frac{1}{4} r^2 \left(\left(\frac{\partial}{\partial \varrho}\right)^2 + \frac{n-2}{\varrho} \frac{\partial}{\partial \varrho} \right), \quad r \neq 0 \neq \varrho. \tag{A.13}$$

Indeed, since the canonical sub-Laplacian is $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$ where X_j is given by (A.10), one has

$$\begin{aligned} \Delta_{\mathbb{G}} = & \Delta_x + \frac{1}{4} \sum_{j=1}^m \sum_{s=1}^n \left(\sum_{i=1}^m x_i U_{i,j}^{(s)} \right)^2 \left(\frac{\partial}{\partial t_s}\right)^2 + \frac{1}{2} \sum_{j=1}^m \sum_{s=1}^n U_{j,j}^{(s)} \frac{\partial}{\partial t_s} \\ & + \frac{1}{4} \sum_{j,h,k=1}^m \sum_{r,s=1, r \neq s}^n x_h x_k U_{h,j}^{(r)} U_{k,j}^{(s)} \frac{\partial^2}{\partial t_r \partial t_s} + \sum_{i,j=1}^m \sum_{s=1}^n x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s} \end{aligned}$$

$$\begin{aligned}
 &= \Delta_x + \frac{1}{4} \sum_{s=1}^n |(U^{(s)})^T x|^2 \left(\frac{\partial}{\partial t_s}\right)^2 + \sum_{i,j=1}^m \sum_{s=1}^n x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s} \\
 &\quad + \frac{1}{2} \sum_{r,s=1,r < s}^n \langle (U^{(r)})^T x, (U^{(s)})^T x \rangle \frac{\partial^2}{\partial t_r \partial t_s} \\
 &= \Delta_x + \frac{1}{4} |x|^2 \Delta_t + \sum_{i,j=1}^m \sum_{s=1}^n x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s} + \frac{1}{2} \sum_{r,s=1,r < s}^n \langle U^{(r)} U^{(s)} x, x \rangle \frac{\partial^2}{\partial t_r \partial t_s} \\
 &= \Delta_x + \frac{1}{4} |x|^2 \Delta_t + \sum_{i,j=1}^m \sum_{s=1}^n x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s}.
 \end{aligned}$$

Here we used the following facts: $U_{j,j}^{(s)} = 0$ since $U^{(s)}$ is skew-symmetric; $|(U^{(s)})^T x| = |x|$ since $U^{(s)}$ is orthogonal; from (A.9) we have $\langle (U^{(r)})^T x, (U^{(s)})^T x \rangle = \langle U^{(r)} x, U^{(s)} x \rangle = \langle -U^{(s)} U^{(r)} x, x \rangle = \langle U^{(r)} U^{(s)} x, x \rangle$, for every $r \neq s$. Again from (A.9) it follows that $U^{(r)} U^{(s)}$ is skew-symmetric (for $r \neq s$), for $(U^{(r)} U^{(s)})^T = (-U^{(s)}) (-U^{(r)}) = U^{(s)} U^{(r)} = -U^{(r)} U^{(s)}$, whence $\langle U^{(r)} U^{(s)} x, x \rangle = 0$ for every $r \neq s$. This proves the first part of Remark A.6.

We now prove the second part of Remark A.6. On functions $u(|x|, t)$, the third differential summand in the right-hand side of (A.11) vanishes: indeed, we have

$$\left(\sum_{i,j=1}^m \sum_{s=1}^n x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s} \right) u(|x|, t) = \frac{1}{r} \sum_{s=1}^n \left(\sum_{i,j=1}^m x_i x_j U_{i,j}^{(s)} \right) \left(\frac{\partial^2 u}{\partial r \partial t_s} \right) (|x|, t) = 0,$$

since for every $s \leq n$, $\sum_{i,j=1}^m x_i x_j U_{i,j}^{(s)} = \langle U^{(s)} x, x \rangle = 0$, being $U^{(s)}$ skew-symmetric. Moreover, with the notation introduced at the beginning of Section 5, there exists an orthogonal matrix O of order n such that $t = O \cdot (s, t'_1, \dots, t'_{n-1})^T$. By the well-known properties of the classical Laplace operator, we hence have $\Delta_t = \left(\frac{\partial}{\partial s}\right)^2 + \Delta_{t'}$. Finally, (A.13) follows from the expression of the classical Laplace operator on radial functions.

Remark A.7. Let (\mathbb{G}, \circ) be a H-type group. Suppose that the center of the algebra of \mathbb{G} has dimension 1. Then \mathbb{G} is isomorphic to a Heisenberg group \mathbb{H}^k (here $k = \frac{1}{2}(\dim(\mathbb{G}) - 1)$).

Indeed, by Theorem 7.2 (and by the hypothesis $n = 1$) it is non-restrictive to suppose that \mathbb{G} is \mathbb{R}^{m+1} equipped with the group law $(x, t) \circ (\xi, \tau) = (x + \xi, t + \tau + \frac{1}{2} \langle x, U^{(1)} \xi \rangle)$. Let M be a $m \times m$ non-singular matrix and consider the following bijection:

$$\tilde{M} : \mathbb{G} \rightarrow \mathbb{R}^{m+1}, \quad (x, t) \mapsto (M x, t).$$

Clearly, if $\widetilde{\mathbb{G}}$ is \mathbb{R}^{m+1} equipped with the composition $(Mx, t) * (M\xi, \tau) := \widetilde{M}((x, t) \circ (\xi, \tau))$, then $\widetilde{M} : (\mathbb{G}, \circ) \rightarrow (\widetilde{\mathbb{G}}, *)$ is a Lie group isomorphism. It is then sufficient to show that there exists M such that $(\widetilde{\mathbb{G}}, *)$ is isomorphic to a Heisenberg group. In order to prove it, if $(\widetilde{x}, \widetilde{t}), (\widetilde{\xi}, \widetilde{\tau}) \in \widetilde{\mathbb{G}}$, we notice that

$$\begin{aligned} (\widetilde{x}, \widetilde{t}) * (\widetilde{\xi}, \widetilde{\tau}) &= \widetilde{M}((M^{-1}\widetilde{x}, \widetilde{t}) \circ (M^{-1}\widetilde{\xi}, \widetilde{\tau})) \\ &= \widetilde{M}(M^{-1}\widetilde{x} + M^{-1}\widetilde{\xi}, \widetilde{t} + \widetilde{\tau} + \frac{1}{2}\langle M^{-1}\widetilde{x}, U^{(1)}M^{-1}\widetilde{\xi} \rangle) \\ &= (\widetilde{x} + \widetilde{\xi}, \widetilde{t} + \widetilde{\tau} + \frac{1}{2}\langle \widetilde{x}, (M^{-1})^T U^{(1)} M^{-1}\widetilde{\xi} \rangle). \end{aligned}$$

It is known that (see e.g., [30, Corollary 2.5.14]) every skew-symmetric orthogonal matrix is congruent to a block diagonal matrix of the following type:

$$J = \text{diag} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Hence, if P is a non-singular matrix such that $P^T U^{(1)} P = J$ and if we take $M = P^{-1}$, Remark A.7 is proved.

Appendix B. L^p estimates of solutions

Throughout this section, \mathbb{G} will denote an arbitrary homogeneous Carnot group. Moreover, we shall always denote by u a solution of (1.1). Finally, Ω will be supposed to be an arbitrary (bounded or unbounded) open subset of \mathbb{G} . Our aim is to prove L^p summability properties of u . The main results are contained in Propositions 8.2, 8.3 and Theorem 8.4. Since we will use boot-strap and iteration techniques inspired to those of Brezis and Kato [12] and of Moser [29, Chapter 8], the proofs of Propositions 8.2 and 8.3 will only be sketched (we refer the reader to [39, Section 2] for more detailed proofs in the case of the Heisenberg group). On the other side, we furnish a more detailed proofs for Theorem 8.4: indeed, the proof of the L^p summability of u for p lower than 2^\star leads to novel and significant modifications of the standard arguments (see also [39, Lemma 2.3]). We shall need the following technical result which will be useful in constructing suitable test functions.

Lemma B.1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with $g(0) = 0$ and such that $g \in C^1$ out from a finite set $F \subset \mathbb{R}$. If $u \in S_0^1(\Omega)$ then $g(u) \in S_0^1(\Omega)$ and $\nabla_{\mathbb{G}}(g(u)) = \dot{g}(u)\nabla_{\mathbb{G}}u$.*

The proof of Lemma B.1 follows the line of its classical analogue, provided classical mollifiers are replaced by ad hoc mollifiers on \mathbb{G} . At the end of this section, we sketch a proof for the reader’s convenience.

Proposition B.2. *We have $u \in L^p(\Omega)$ for every $p \in [2^\star, \infty)$.*

Proof. If $M > 0$, let $\eta \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\eta \equiv 1$ in $[0, M]$ and $\eta \equiv 0$ in $[2M, \infty)$. We set $V = (1 - \eta(u))u^{4/(Q-2)}$, $g = \eta(u)u^{2^\star-1}$. Then u is the only weak solution of $-\Delta_{\mathbb{G}} u = Vu + g$ in Ω , $u \in S_0^1(\Omega)$. If $M = M(u, Q) \gg 1$, from (2.8) we have $\int_{\Omega} V\varphi^2 \leq 1/2 \|\nabla_{\mathbb{G}} \varphi\|_2^2$ for every $\varphi \in S_0^1(\Omega)$. If $k \in \mathbb{N}$, we set $V_k = \min\{V, k\}$ and we let $u_k \in S_0^1(\Omega)$ be the (unique) weak solution of $-\Delta_{\mathbb{G}} u_k = V_k u_k + g$ in Ω : we then have $\|u_k\|_{2^\star} \leq 2S_Q \|g\|_{2Q/(Q+2)}$. We claim that

$$\forall p \in [2^\star, \infty) \exists c_p > 0: \sup_k \|u_k\|_p \leq c_p. \tag{B.1}$$

Since, up to a subsequence, $u_k \rightarrow u$ weakly in $S_0^1(\Omega)$, this would prove (B.1). We now sketch the proof of the claim: it is enough to show that, if $\beta = 2^\star/2$, then (B.1) holds for βp , under the hypothesis that (B.1) holds for p . We fix $k \in \mathbb{N}$ and we set $v = u_k$, $v_m = \min\{v^+, m\}$, $\varphi_m = v_m^{p-1}$, $f_m = v_m^{p/2}$ ($m \in \mathbb{N}$). By Lemma B.1, we remark that $v_m, \varphi_m, f_m \in S_0^1(\Omega)$. Hence, if $\alpha_p = 4(p-1)/p^2$, we get

$$\alpha_p \int_{\Omega} |\nabla_{\mathbb{G}} f_m|^2 = \int_{\Omega} (V_k v \varphi_m + g \varphi_m) \leq \int_{\{v \leq m\}} V_k f_m^2 + k \int_{\{v > m\}} v^p + \int_{\Omega} g \varphi_m.$$

If $h = h(p, V, Q) \gg 1$, we then have (taking into account (2.8))

$$\int_{\Omega} V f_m^2 \leq h \int_{\{V \leq h\}} f_m^2 + \int_{\{V > h\}} V f_m^2 \leq h \|f_m\|_2^2 + \alpha_p/2 \|\nabla_{\mathbb{G}} f_m\|_2^2.$$

From the above inequalities, it easily follows that (for a suitable constant $c_p > 0$)

$$\alpha_p/(2S_Q) \|v^+\|_{\beta p}^p \leq h \|v^+\|_p^p + \|g\|_p \|v^+\|_p^{p-1} \leq h c_p^p + c_p^{p-1} \|g\|_{\infty}^{1-2^\star/p} \|g\|_{2^\star}^{2^\star/p} < \infty.$$

Arguing as above for v^- , (8.1) and hence the proposition are proved. \square

Proposition B.3. *We have $u \in L^\infty(\Omega)$.*

Proof. It is enough to prove that there exist $c > 0$ and a sequence $p_j \uparrow \infty$ such that $\sup_j \|u\|_{p_j} \leq c$. By Proposition B.2 (following the therein notation) we know that $g \in L^p$ for all $p \in [2Q/(Q+2), \infty]$, $V \in L^q$ for all $q \in [Q/2, \infty)$. Setting $r = \sqrt{\beta}$, we hence have

$$\begin{aligned} \alpha_p \int_{\Omega} |\nabla_{\mathbb{G}} f_m|^2 &\leq \|V\|_{r/(r-1)} \|v_m^p\|_r + k \int_{\{v > m\}} v^p + \|g\|_{rp/(rp-p+1)} \|v_m\|_{rp}^{p-1} \\ &\leq c \|v^+\|_{rp}^p + k \int_{\{v > m\}} v^p + c \|v^+\|_{rp}^{p-1}. \end{aligned}$$

Moreover, since $\|v_m\|_{\beta p}^p \leq S_Q \|\nabla_{\mathbb{G}} f_m\|_2^2$, letting $m \rightarrow \infty$ we obtain (being $c > 0$ independent of p)

$$\alpha_p S_Q^{-1} \|v^+\|_{\beta p}^p \leq c (\|v^+\|_{r p}^p + \|v^+\|_{r p}^{p-1}).$$

By a standard technique, set $H_p = \max\{1, \|v^+\|_p\}$ and $p_j = r^j 2^{\star}$, the above inequality gives

$$\|v^+\|_{p_j} \leq H_{p_j} \leq H_{r^2 2^{\star}} \prod_{n=1}^{j-2} (c r^n)^{1/(2^{\star} r^n)} \leq c.$$

Arguing as above for v^- , the assertion is proved. \square

Although the proof of the following theorem follows the lines of the previous ones, it presents new difficulties. Indeed, we need to choose suitable truncated potentials and particular test functions.

Theorem B.4. *We have $u \in L^p(\Omega)$ for every $p \in (2^{\star}/2, 2^{\star}]$.*

Proof. We fix $\varepsilon \in (0, 1/2)$ and we set $c_\varepsilon = (1 - 2\varepsilon)^{-2}$. Since $u \in L^{2^{\star}}(\Omega)$ there exists $M > 0$ such that $\int_{u < 2M} u^{2^{\star}} < (c_\varepsilon S_Q)^{-Q/2}$. Let $\eta \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\eta \equiv 1$ in $[0, M]$, $\eta \equiv 0$ in $[2M, \infty)$. We set $V = \eta(u)u^{4/(Q-2)}$, $g = (1 - \eta(u))u^{2^{\star}-1}$. Then $u \in S_0^1(\Omega)$ is the unique weak solution of $-\Delta_{\mathbb{G}} u = Vu + g$ in Ω . By Propositions B.2 and B.3, we have $V \in L^{Q/2} \cap L^\infty(\Omega)$, $g \in L^p(\Omega)$ for all $p \in [1, \infty)$. Moreover, by the choice of c_ε ,

$$\int_{\Omega} V \varphi^2 \leq \|V\|_{Q/2} \|\varphi\|_{2^{\star}}^2 < c_\varepsilon^{-1} \|\nabla_{\mathbb{G}} \varphi\|_2^2, \quad \forall \varphi \in S_0^1(\Omega). \tag{B.2}$$

We now choose (for any $k \in \mathbb{N}$) cut-off functions $\eta_k \in C^\infty(\mathbb{R}, [0, 1])$ such that $\eta_k \equiv 0$ in $[0, M/(k+1)]$, $\eta_k \equiv 1$ in $[M/k, \infty)$. We define $V_k = \eta_k(u)V$: notice that $V_k \in L^1 \cap L^\infty(\Omega)$ and $V_k \uparrow V$. By Lax-Milgram’s Theorem, we can see that there exists exactly one weak solution u_k of the Dirichlet problem

$$-\Delta_{\mathbb{G}} u_k = V_k u_k + g \quad \text{in } \Omega, \quad u_k \in S_0^1(\Omega). \tag{B.3}$$

Moreover, we have

$$\sup_k \|u_k\|_{S_0^1(\Omega)} \leq c(Q, u, \varepsilon) \tag{B.4}$$

and $u_k \rightarrow u$ weakly in $S_0^1(\Omega)$. We now want to prove that $\sup_k \|u_k\|_{2^{\star}(1-\varepsilon)} \leq \tilde{c}(Q, u, \varepsilon)$, which straightforwardly ends the proof. We fix $k \in \mathbb{N}$, and (for any $m \in \mathbb{N}$) we set

$v = u_k$ and $\varphi_m = v_m^{1-2\varepsilon}$, $f_m = v_m^{1-\varepsilon}$ where

$$v_m = m^{2\varepsilon/(1-2\varepsilon)}(v^+)^{1/(1-2\varepsilon)} \quad \text{if } v \leq 1/m \text{ and } v_m = v \text{ otherwise.}$$

We explicitly remark that (by Lemma B.1) $v_m, \phi_m, f_m \in S_0^1(\Omega)$, since $0 < 1 - 2\varepsilon < 1 - \varepsilon < 1$. Moreover, $v_m, \varphi_m, f_m \geq 0$ and $v_m \rightarrow v^+$ pointwise, as $m \rightarrow \infty$. Choosing φ_m as a test function in the weak formulation of (B.3) and setting $h_\varepsilon = (\frac{1-\varepsilon}{1-2\varepsilon})^2$, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla_{\mathbb{G}} f_m|^2 &= (1 - 2\varepsilon) h_\varepsilon \int_{v > 1/m} \langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \varphi_m \rangle \\ &\quad + h_\varepsilon \int_{0 < v \leq 1/m} m^{1-2\varepsilon} v^{1-2\varepsilon} \langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \varphi_m \rangle \\ &\leq h_\varepsilon \int_{\Omega} \langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \varphi_m \rangle = h_\varepsilon \int_{\Omega} (V_k v \varphi_m + g \varphi_m) \\ &= h_\varepsilon \left(\int_{v > 1/m} V_k f_m^2 + m^{2\varepsilon} \int_{0 < v \leq 1/m} V_k v^2 + \int_{v > 1/m} g v^{1-2\varepsilon} \right. \\ &\quad \left. + m^{2\varepsilon} \int_{0 < v \leq 1/m} g v \right) \\ &\leq h_\varepsilon (c_\varepsilon^{-1} \|\nabla_{\mathbb{G}} f_m\|_2^2 + m^{2\varepsilon-2} \|V_k\|_1 + \|g\|_{2^\star/(2^\star-1+2\varepsilon)} \|v\|_{2^\star}^{1-2\varepsilon} + m^{2\varepsilon-1} \|g\|_1). \end{aligned}$$

Hence $\int_{\Omega} |\nabla_{\mathbb{G}} f_m|^2 < c_1(Q, u, \varepsilon, k) m^{2\varepsilon-1} + c_2(Q, u, \varepsilon)$. Here we have used (2.8), (B.2), (B.4), the definitions of c_ε and h_ε , the L^p -properties of V, g, V_k , and the fact that $\langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \varphi_m \rangle \geq 0$ in Ω . Hence $\|v_m\|_{2^\star(1-\varepsilon)}^{2(1-\varepsilon)} = \|f_m\|_{2^\star}^2 \leq S_Q \|\nabla_{\mathbb{G}} f_m\|_2^2 \leq c_1 S_Q m^{2\varepsilon-1} + c_2 S_Q$ and (letting $m \rightarrow \infty$) Fatou’s Lemma yields $\|v^+\|_{2^\star(1-\varepsilon)} \leq (c_2 S_Q)^{1/(2-2\varepsilon)} =: \tilde{c}(Q, u, \varepsilon)$. Since a similar estimate can be proved for v^- , the proof is complete. \square

We now prove some more properties of u .

Proposition B.5. *u vanishes at infinity.*

Proof. We extend u to be zero outside Ω . Let d be any homogeneous norm on \mathbb{G} (consider e.g., the one in (2.3)): we aim to prove that $u(\xi) \rightarrow 0$ a.e. as $d(\xi) \rightarrow \infty$. This follows if we show that the following claim holds:

$$\exists \mathbf{c} = \mathbf{c}(Q, u) > 0 : \|u, L^\infty(B_d(\xi, 1))\| \leq \mathbf{c} \|u, L^{2^\star}(B_d(\xi, 2))\|, \quad \forall \xi \in \mathbb{G}.$$

Let us fix $1 \leq r_1 < r_2 \leq 2$ and choose $\alpha \in C^\infty(\mathbb{R}, [0, 1])$ such that $\alpha \equiv 1$ in $[0, r_1]$, $\alpha \equiv 0$ in $[r_2, \infty)$ and $|\dot{\alpha}| \leq 2/(r_2 - r_1)$. We now fix $\xi \in \mathbb{G}$ and $p > 2^\star$ and we set $\eta = \alpha(d(\xi^{-1} \cdot))$, $\varphi = \eta^2 u^{p-1}$, $\psi = \eta u^{p/2}$, $\beta = 2^\star/2$. By Lemma B.1, $\varphi, \psi \in S_0^1(\Omega)$. Finally, we define $I = \int \eta^2 u^{p-2} |\nabla_{\mathbb{G}} u|^2$. Choosing φ as a test function in the weak formulation

of $-\Delta_{\mathbb{G}}u = u^{2^{\star}-1}$, we get

$$\begin{aligned} \int u^{2^{\star}-1}\varphi &\geq (p-1) \int \eta^2 u^{p-2} \langle \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}u \rangle - I/2 - 2 \int |\nabla_{\mathbb{G}}\eta|^2 u^p \\ &\geq I/2 - 2 \int |\nabla_{\mathbb{G}}\eta|^2 u^p. \end{aligned}$$

Hence, since $\|\nabla_{\mathbb{G}}\eta\|_{\infty} = \|\dot{\alpha}(d)\nabla_{\mathbb{G}}d\| \leq \mathbf{c}/(r_2 - r_1)$ (being $\nabla_{\mathbb{G}}d$ homogeneous of degree 0 w.r.t. δ_{λ}) and taking into account the above inequality, we derive

$$\begin{aligned} \|u, L^{\beta p}(B_d(\xi, r_1))\|^p &\leq \|\psi\|_{2^{\star}}^2 \leq S_Q \|\nabla_{\mathbb{G}}\psi\|_2^2 \leq \mathbf{c}p^2 I + \mathbf{c}\|u^{p/2}\nabla_{\mathbb{G}}\eta\|_2^2 \\ &\leq \mathbf{c}p^2 \left(\int u^{2^{\star}-1}\varphi + \int |\nabla_{\mathbb{G}}\eta|^2 u^p \right) + \mathbf{c}\|u^{p/2}\nabla_{\mathbb{G}}\eta\|_2^2 \\ &\leq \mathbf{c}p^2 \|u\|_{\infty}^{2^{\star}-2} \int_{B_d(\xi, r_2)} \eta^2 u^p + \mathbf{c}/(r_2 - r_1)^2 \int_{B_d(\xi, r_2)} u^p \\ &\leq \mathbf{c}(Q, u)p^2/(r_2 - r_1)^2 \|u, L^p(B_d(\xi, r_2))\|^p. \end{aligned}$$

By standard iterative techniques, it is then possible to prove that (set $p_j = \beta^j 2^{\star}$ and $H_j = \|u, L^{p_j}(B_d(\xi, 1 + 2^{-j}))\|$) there exists a positive constant $\mathbf{c} = \mathbf{c}(Q, u)$ such that

$$H_j \leq H_0 \prod_{m=0}^{j-1} (\mathbf{c}^{\beta^m} (2\beta)^{\mathbf{c}m\beta^m}) \leq \mathbf{c}H_0.$$

Letting $j \rightarrow \infty$, we finally obtain $\|u, L^{2^{\star}}(B_d(\xi, 2))\| = H_0 \geq \mathbf{c}H_j \geq \mathbf{c}\|u, L^{p_j}(B_d(\xi, 1))\| \rightarrow \mathbf{c}\|u, L^{\infty}(B_d(\xi, 1))\|$. This proves the claim and hence the proposition. \square

We now want to provide some regularity properties of the non-negative weak solutions u of (1.1). To begin with, since we have $u \in L^p(\Omega)$ for all $p \in (2^{\star}/2, \infty]$, it can be proved (by means of a standard regularization technique, based on some results in [21] and in [44]) that $u \in C^{\infty}(\Omega)$. Let us now also suppose that Ω satisfies the following boundary regularity condition:

$$\exists \delta, r_0 > 0 : |B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \quad \forall \xi \in \partial\Omega, \quad \forall r \in (0, r_0). \tag{B.5}$$

Then (extending u to be zero outside Ω) u satisfies the following Hölder regularity property (w.r.t. any homogeneous norm d): there exist constants $\alpha \in (0, 1)$ and $M > 0$ such that

$$|u(\xi) - u(\xi')| \leq M d^{\alpha}(\xi, \xi') \quad \forall \xi, \xi' \in \mathbb{G},$$

i.e., following Folland [21] and Folland and Stein [22], u belongs to the Hölder space $\Gamma^{\alpha}(\mathbb{G})$. Indeed, from Proposition B.3 we derive that $u \in S_0^1(\Omega) \cap L^{\infty}(\Omega)$ and $\Delta_{\mathbb{G}}u \in L^{\infty}(\Omega)$. This enables us to apply an analogue of Moser’s iteration technique

(see e.g., [29, Chapter 8]), suitably adapted to the case of a general sub-Laplacian. In order to do so, we only have to replace the Euclidean distance with the quasi-distance d , to choose ad hoc cut-off functions modelled on d and to use a suitable version of the John and Nirenberg Theorem adapted to the homogeneous structure of \mathbb{G} (see [14]). In particular, we obtain that $u \in C(\overline{\Omega})$ and $u = 0$ in $\partial\Omega$. We summarize the above remarks in the following proposition.

Proposition B.6. *If Ω satisfies (B.5), then any weak solution u of (1.1) is a classical solution of*

$$\begin{cases} -\Delta_{\mathbb{G}} u = u^{\frac{Q+2}{Q-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u(\xi) \rightarrow 0 & \text{as } d(\xi) \rightarrow \infty. \end{cases}$$

Moreover, if we continue u on \mathbb{G} by setting $u = 0$ outside Ω , then there exists $\alpha \in (0, 1)$ such that $u \in \Gamma^\alpha(\mathbb{G})$.

We explicitly remark that every half-space on every Carnot group of step two satisfies the boundary regularity condition (B.5). Indeed, when $\xi = 0$, condition (B.5) trivially holds with $\delta = 1/2$, since the d -balls centered at the origin are symmetric with respect to the group inversion $x^{-1} = -x$ (see (2.2)) and they hence are split into two isometric parts by any half-space through 0. On the other hand, the case of a general ξ can be reduced to the previous one by means of a left translation by ξ^{-1} , which is an affine bijection preserving the Lebesgue measure and mapping half-spaces into half-spaces (see again (2.2)).

We finally close this section with the proof of Lemma B.1.

Proof of Lemma B.1. Let $\Psi \in C_0^\infty(\mathbb{G}, [0, 1])$ be a cut-off function supported in $B_d(0, 1)$, such that $\int_{\mathbb{G}} \Psi = 1$ and with the following property: $\Psi(x^{-1}) = \Psi(x)$. For $\varepsilon > 0$, we set $\Psi_\varepsilon(x) = \varepsilon^{-Q} \Psi(\delta_{\varepsilon^{-1}}(x))$ and, for an arbitrary $f \in L^1_{\text{loc}}(\mathbb{G})$, we define the ε -mollified of f as follows:

$$f_\varepsilon(x) := \int_{\Omega} \Psi_\varepsilon(y \circ x^{-1}) f(y) \, dy.$$

The mollifier has been defined in such a way that, when f is smooth, we have $\nabla_{\mathbb{G}}(f_\varepsilon) = (\nabla_{\mathbb{G}} f)_\varepsilon$. We now split the proof in six steps. From (I) to (V), we suppose that $u, \nabla_{\mathbb{G}} u \in L^1_{\text{loc}}(\Omega)$ and we prove that $\nabla_{\mathbb{G}}(g(u)) = \dot{g}(u) \nabla_{\mathbb{G}} u$. In (VI) we prove that $u \in S_0^1(\Omega)$ yields $g(u) \in S_0^1(\Omega)$. We explicitly remark that, from the invariance of Ψ under the group inversion, it is not difficult to recognize that $\nabla_{\mathbb{G}}(u_\varepsilon) = (\nabla_{\mathbb{G}} u)_\varepsilon$ in the sense of distributions.

(I) We first claim that for every $K \Subset \Omega$, $\sigma > 0$ and $r > 0$ such that $K^* := \overline{B}_d(0, r) \circ K \Subset \Omega$, there exists $v \in C_0^\infty(\Omega)$ such that $\|u - v, L^1(K)\| < \sigma$, $\|\nabla_{\mathbb{G}} u - \nabla_{\mathbb{G}} v, L^1(K)\| < \sigma$. Indeed, let $\psi \in C_0^\infty(\Omega)$ with $\psi = 1$ on K^* and set $\alpha = \psi u$. Fix $\varepsilon \in (0, r)$ so small that $B_d(0, \varepsilon) \circ \text{supp}(\psi) \subset \Omega$. Then $\alpha_\varepsilon \in C_0^\infty(\mathbb{G})$ and $\text{supp}(\alpha_\varepsilon) \Subset \Omega$. Moreover, $\lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = u$ and $\lim_{\varepsilon \rightarrow 0} \nabla_{\mathbb{G}}(\alpha_\varepsilon) = \lim_{\varepsilon \rightarrow 0} (\nabla_{\mathbb{G}} \alpha)_\varepsilon = \nabla_{\mathbb{G}} \alpha$ in $L^1(K)$. Since $\alpha = u$ about K , the claim follows taking $v = \alpha_\varepsilon$, for ε small.

(II) We then claim that there exist $w_1, w_2 \in L^1_{\text{loc}}(\Omega)$ and a sequence v_n in $C_0^\infty(\Omega)$ such that almost everywhere on Ω we have $\lim_{n \rightarrow \infty} v_n = u$, $\lim_{n \rightarrow \infty} \nabla_{\mathbb{G}} v_n = \nabla_{\mathbb{G}} u$ and $|v_n| \leq w_1$, $|\nabla_{\mathbb{G}} v_n| \leq w_2$. Indeed, let K_n be an increasing exhaustion of Ω with compact sets and $r_n \downarrow 0^+$ such that $\overline{B}_d(0, r_n) \circ K_n \Subset K_{n+1}$. From (I), there exist $v_n \in C_0^\infty(\Omega)$ such that $\|v_n - u, L^1(K_n)\|$, $\|\nabla_{\mathbb{G}} v_n - \nabla_{\mathbb{G}} u, L^1(K_n)\| < 2^{-n}$. It is then not difficult to recognize that, for $w_1 := \sup_n |v_n|$ and $w_2 := \sup_n |\nabla_{\mathbb{G}} v_n|$, the claim is proved.

(III) We first prove that $\nabla_{\mathbb{G}}(g(u)) = \dot{g}(u) \nabla_{\mathbb{G}} u$ when $g \in C^1(\mathbb{R})$ and $\|\dot{g}\|_\infty < \infty$. Let v_n, w_1, w_2 be as in (II). From an integration by parts, for every $\varphi \in C_0^\infty(\Omega)$ we have $-\int_{\Omega} \nabla_{\mathbb{G}} \varphi g(v_n) = \int_{\Omega} \varphi \dot{g}(v_n) \nabla_{\mathbb{G}} v_n$. The assertion then follows by dominated convergence, taking into account the dominations given by w_1, w_2 and proved in the second step.

(IV) We claim that the Lebesgue measure of $\{x \in \Omega : u(x) = 0, \nabla_{\mathbb{G}} u(x) \neq 0\}$ is zero. Indeed, let $\psi \in C^\infty(\mathbb{R}, [0, 1])$, $\psi = 1$ out from $[-2, +2]$ and vanishing on $[-1, +1]$ and set $g_n(s) := \int_0^s \psi(nt) dt$. From (III) it follows that for every $\varphi \in C_0^\infty(\Omega)$ we have $-\int_{\Omega} \nabla_{\mathbb{G}} \varphi g_n(u) = \int_{\Omega} \varphi \psi(nu) \nabla_{\mathbb{G}} u$. We end by dominated convergence: indeed, if $A := \{x \in \Omega : u(x) \neq 0\}$, we have $\varphi \psi(nu) \nabla_{\mathbb{G}} u \rightarrow \varphi \chi_A \nabla_{\mathbb{G}} u$ and $|\varphi \psi(nu) \nabla_{\mathbb{G}} u| \leq \|\varphi\|_\infty \chi_{\text{supp } \varphi} |\nabla_{\mathbb{G}} u| \in L^1(\Omega)$, whence $-\int_{\Omega} \varphi (\nabla_{\mathbb{G}} u - \chi_A \nabla_{\mathbb{G}} u) = 0$ and the claim is proved.

(V) If g is as in the assertion of the lemma, we prove that $\nabla_{\mathbb{G}}(g(u)) = \dot{g}(u) \nabla_{\mathbb{G}} u$. Let $C > 0$ be a Lipschitz constant for g and let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ coincide with \dot{g} where g is differentiable and such that $\|\gamma\|_\infty \leq C$. We trivially have $g(u), \gamma(u) \nabla_{\mathbb{G}} u \in L^1_{\text{loc}}(\Omega)$. Let γ_n be a sequence of classical mollified for γ and set $g_n(s) := \int_0^s \gamma_n(t) dt$. We have $|g_n(s)| \leq C|s|$ and $\lim_n g_n = g$. From (III) we derive that $-\int_{\Omega} \nabla_{\mathbb{G}} \varphi g_n(u) = \int_{\Omega} \varphi \dot{g}_n(u) \nabla_{\mathbb{G}} u$ for every test function φ . We now argue by dominated convergence: first we have $\nabla_{\mathbb{G}} \varphi g_n(u) \rightarrow \nabla_{\mathbb{G}} \varphi g(u)$, $|\nabla_{\mathbb{G}} \varphi g_n(u)| \leq C |\nabla_{\mathbb{G}} \varphi| |u| \in L^1(\Omega)$; moreover $|\varphi \dot{g}_n(u) \nabla_{\mathbb{G}} u| \leq C |\varphi| |\nabla_{\mathbb{G}} u| \in L^1(\Omega)$ and $\varphi \dot{g}_n(u) \nabla_{\mathbb{G}} u \rightarrow \varphi \gamma(u) \nabla_{\mathbb{G}} u$. This last assertion also follows from (IV), recalling that $\nabla_{\mathbb{G}} u(x) = 0$ for almost every x such that $u(x) \in F$.

(VI) We follow the notation in (V). If $u \in S_0^1(\Omega)$, there exist $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n \rightarrow u$ in $L^{2^*}(\Omega)$ and $\nabla_{\mathbb{G}} \varphi_n \rightarrow \nabla_{\mathbb{G}} u$ in $L^2(\Omega)$. It is also non-restrictive to suppose that these limits are valid pointwise a.e., and that there exist $w_1 \in L^{2^*}, w_2 \in L^2$ such that $|\varphi_n| \leq w_1$, $|\nabla_{\mathbb{G}} \varphi_n| \leq w_2$. Moreover, from (V) we have $\nabla_{\mathbb{G}}(g(u)) = \gamma(u) \nabla_{\mathbb{G}} u$. Let h_n be a sequence of classical mollified for g and set $g_n := h_n - h_n(0)$. In order to end the proof, it is enough to show that

$$C_0^\infty(\Omega) \ni g_n(\varphi_n) \rightarrow g(u) \text{ in } L^{2^*}(\Omega), \quad \nabla_{\mathbb{G}}(g_n(\varphi_n)) \rightarrow \nabla_{\mathbb{G}}(g(u)) \text{ in } L^2(\Omega).$$

To begin with, $g_n \in C^\infty(\mathbb{R})$, $\varphi_n \in C_0^\infty(\Omega)$ and $g_n(0) = 0$ give $g_n(\varphi_n) \in C_0^\infty(\Omega)$. Furthermore, we have $\|g_n(\varphi_n) - g(u)\|_{2^*}^2 \leq c \int_\Omega |g_n(\varphi_n) - g_n(u)|^{2^*} + c \int_\Omega |g_n(u) - g(u)|^{2^*} =: I_1 + I_2$. Now, the integrand in I_1 is bounded by $C^{2^*} |\varphi_n - u|^{2^*}$, whence I_1 vanishes as $n \rightarrow \infty$. We also have $I_2 \rightarrow 0$ by dominated convergence: $g_n(s) = h_n(s) - h_n(0) \rightarrow g(s) - g(0) = g(s)$; $|g_n(u) - g(u)|^{2^*} \leq c \|\gamma\|_\infty |u|^{2^*} \in L^1$. We now notice that $\nabla_{\mathbb{G}}(g_n(\varphi_n)) = \dot{h}_n(\varphi_n) \nabla_{\mathbb{G}} \varphi_n$. Now, for a.e. x such that $u(x) \in F$, (IV) gives $\nabla_{\mathbb{G}} u(x) = 0$. Hence, if $A := \{x \in \Omega : u(x) \in F\}$, for a.e. $x \in A$, we have $\dot{h}_n(\varphi_n(x)) \nabla_{\mathbb{G}} \varphi_n(x) \rightarrow 0 = \gamma(u(x)) \nabla_{\mathbb{G}} u(x)$, since \dot{h}_n is bounded and $\nabla_{\mathbb{G}} \varphi_n(x) \rightarrow \nabla_{\mathbb{G}} u(x) = 0$. Let otherwise $u(x) \in \mathbb{R} \setminus F$. Since F is finite, for $n \gg 1$, $\varphi_n(x)$ belongs to a compact neighborhood of $u(x)$ contained in $\mathbb{R} \setminus F$. Hence, $\dot{h}_n(\varphi_n(x)) \nabla_{\mathbb{G}} \varphi_n(x) \rightarrow \gamma(u(x)) \nabla_{\mathbb{G}} u(x)$ also a.e. outside A . Being $|\dot{h}_n(\varphi_n)|^2 |\nabla_{\mathbb{G}} \varphi_n|^2 \leq C^2 |w_2|^2 \in L^1$, by dominated convergence we have $\|\nabla_{\mathbb{G}}(g_n(\varphi_n)) - \nabla_{\mathbb{G}}(g(u))\|_2^2 = \int_\Omega |\dot{h}_n(\varphi_n) \nabla_{\mathbb{G}} \varphi_n - \gamma(u) \nabla_{\mathbb{G}} u|^2 \rightarrow 0$. This completes the proof of the lemma. \square

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