MAXIMUM PRINCIPLE AND PROPAGATION FOR INTRINSICLY REGULAR SOLUTIONS OF DIFFERENTIAL INEQUALITIES STRUCTURED ON VECTOR FIELDS

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Abstract. We prove suitable versions of the weak maximum principle and of the maximum propagation for solutions u of a differential inequality $\mathcal{H}u \geq 0$. Here $\mathcal{H} = \sum_{i,j} a_{i,j}(z) Z_i Z_j + Z_0$ is a differential operator structured on the vector fields Z_j's, whereas u belongs to an appropriate intrinsic class of regularity modelled on the Z_j's.

1. Introduction

The aim of this paper is to study the following differential inequality

$$\mathcal{H}u(z) = \textstyle\sum_{i,j=1}^{m} a_{i,j}(z)\, Z_i Z_j u(z) + Z_0 u(z) \geq 0 \qquad z \in \Omega \subset \mathbb{R}^N, \tag{1.1}$$

where $(a_{i,j})_{i,j}$ is positive semi-definite, $Z_0, Z_1, \dots, Z_m$ are locally Lipschitz-continuous vector fields on the open set Ω and u is a real valued function which need not be differentiable. Indeed, the differentiation of u in inequality (1.1) is meant only along the integral curves of the fields Z_j's, as we shall explain below. In particular, we are here interested in weak maximum principles and maximum propagation for functions u in an intrinsic class of regularity modelled on the fields Z_j's, say $u \in \Gamma^2(\Omega)$, a somewhat minimal regularity class for which inequality (1.1) makes sense. Function spaces with a regularity modelled on vector fields have been widely used in the PDE's literature, since many related problems are not only more naturally posed in this context, but may not be solvable elsewhere. For instance, one cannot expect more than the Γ^2-regularity for a fundamental solution of an operator like $\mathcal{H}$ above, even in the case of smooth Z_j's. Hence, if one needs comparison results or maximum principles for $\mathcal{H}$, it is more natural to state them in the appropriate intrinsic class of regularity. In particular, our result in this paper is a step towards deriving an Harnack inequality for operators like (1.1) above on stratified Lie groups (see [4, 5]).

Classical results on weak and strong maximum principles have been proved by Amano [1], Bony [6], Friedman [9], Hill [13], Hopf [14], Nirenberg [20], Redheffer [24], Stroock&Varadhan [26]; see also the following monographs, containing related topics: Caffarelli&Cabré [7], Gilbarg&Trudinger [12], Oleinik&Radkevic [22], Protter&Weinberger, [23], Sperb [25], Taira [27]. The literature on maximum principles is nowadays still extremely rich and applications are given e.g., in the interpretation of Markov processes, in the study of the diffusion of particles, in the propagation of singularities (see [27] and the references therein). All the above papers deal with the case of functions of class C^2.

In order to state our main result here (Theorem 1.2 below) we first fix some definition: If $Z \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ (such a Z is identified with the differential operator) and $z_0 \in \Omega$, we

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say that u has Lie-derivative along Z at z_0, if $u \circ \gamma$ is differentiable at 0, where γ is the integral curve of Z such that $\gamma(0) = z_0$. Our intrinsic regularity class for the operator $\mathcal{H}$ is the following function space $\Gamma^2(\Omega)$.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $Z_0, Z_1, \dots, Z_m \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Then, we write $u \in \Gamma^2(\Omega)$ if $u : \Omega \to \mathbb{R}$ is a continuous function with continuous Lie-derivatives along $Z_1, \dots, Z_m$ up to second order and a continuous Lie-derivative along Z_0 (up to first order).

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $Z_1, \dots, Z_m \in C^1(\Omega, \mathbb{R}^N)$ and $Z_0 \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Suppose the matrix $A(z) = (a_{i,j}(z))_{i,j=1}^m$ is symmetric and positive semidefinite for any $z \in \Omega$. We have the following results.*

(Γ^2-Weak Maximum Principle). *If Ω is bounded and there exists $w \in \Gamma^2(\Omega)$ such that $\mathcal{H}w < 0$ and $w > 0$ in Ω, then $\mathcal{H}$ satisfies the Γ^2-(WMP) on Ω, i.e., for every function $u \in \Gamma^2(\Omega)$ satisfying $\mathcal{H}u \geq 0$ in Ω and $\limsup u \leq 0$ on $\partial\Omega$, there holds $u \leq 0$ in Ω.*

(Γ^2-Maximum Propagation). *Suppose $A(z)$ has continuous entries and is positive-definite at any point, and suppose $\mathcal{H}$ locally satisfies the Γ^2-(WMP). Then, for every function $u \in \Gamma^2(\Omega)$ satisfying $\mathcal{H}u \geq 0$, $u \leq 0$ in Ω, the set $F = \{z : u(z) = 0\}$ contains (the closure of) the following sets:*

1. *The points connected to any $z \in F$ by trajectories of the fields $Z_1, \dots, Z_m$ (backward or forward in time);*
2. *The points connected to any $z \in F$ by trajectories of principal vector fields w.r.t. the operator $\mathcal{H}$ (backward or forward in time);*
3. *The points connected to any $z \in F$ (backward or forward in time) by solutions to the controlled system $\dot\gamma(t) = \sum_{j=1}^m \alpha_j(t)\, Z_j(\gamma(t))$, for some bounded α_j's;*
4. *The points on the trajectories of the drift Z_0 starting from any $z \in F$, forward in time.*

The above (4) will be proved under an additional structure hypothesis on $\mathcal{H}$. For more precise statements, see Theorems 3.2, 4.3 and Proposition 4.5. We remark that, even if we follow classical lines in proving maximum principles, some non-trivial preliminary results about intrinsic differentiability along integral curves are needed. We provide them in Section 2.

To end the introduction, we would like to point out some explicit PDE's to which our results apply. First of all, we consider operators of the type $\mathcal{L} = \sum_{i,j=1}^m a_{i,j}(x,t)\, X_i X_j - \partial_t$, where (following Folland) the X_i's generate a stratified Lie group $\mathbb{G}$. The Γ^2-weak maximum principle in Theorem 1.2 is used in [4] in order to prove long time estimates for the fundamental solution for $\mathcal{L}$. In that context, the intrinsic regularity class Γ^2 is the right setting for the solutions to the Cauchy problem related to $\mathcal{L}$: these solutions can be represented as convolutions with the fundamental solution for $\mathcal{L}$, which in general is only of class Γ^2. Moreover, we will use the Γ^2-maximum propagation in a forthcoming paper [5], concerning with the Harnack inequality for $\mathcal{L}$.

Operators like the above $\mathcal{L}$ naturally intervene as linearizations of fully nonlinear operators, and this is the main motivation for our study. For example, the Levi Monge-Ampère operator in $\mathbb{C}^{n+1}$ can be written as $\sum_{i,j=1}^{2n} a_{i,j}(Z^2 u)\, Z_i Z_j u$: in the significant case of strictly Levi-convex functions u, the relevant linearized operator has a continuous positive definite matrix $(a_{i,j})_{i,j}$ and suitable C^1 vector fields Z_j's in $\mathbb{R}^{2n+1}$ (see Lascialfari&Montanari [18]). More generally, many operators arising in the study of the geometric theory of several complex variables and in curvature problems can be written in this way (see Montanari&Lanconelli [19], where a comparison principle for C^2 solutions of the Levi equation is also proved).

Another example of operators to which our results can be applied is given by the Kolmogorov-Fokker-Planck operators $\mathcal{K} = \sum_{i,j} a_{i,j}(x,y,t)\, \partial_{x_i}\partial_{x_j} - \langle x, \nabla_y \rangle - \partial_t$ and by a class of ultraparabolic operators studied in [17], generalizing $\mathcal{K}$. These operators arise, e.g., in the gas kinetic theory, in diffusion processes and in mathematical finance, see [16] for a survey on these topics. Finally, the above weak and strong maximum principles are related to an intrinsic definition of Γ^2-viscosity solution for linearized operators in some subelliptic contexts, where the differentiability along variable directions seems to be more natural to deal with (see remarks at the end of Section 3).

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2. Some results on differentiation along integral curves

The aim of this section is to prove some results about functions which are required to be differentiable only along integral curves of a locally Lipschitz vector field. The main results are contained in Lemma 2.1 and Propositions 2.2, 2.3 and 2.4 below.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $Z \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. For a fixed $z_0 \in \Omega$ we say that $u : \Omega \to \mathbb{R}$ has *Lie-derivative along* Z *at* z_0 if there exists

$$Zu(z_0) := \lim_{t\to 0} \tfrac{1}{t}(u(\gamma(t)) - u(\gamma(0))) \quad \text{where } \gamma \text{ solves:} \quad \dot\gamma(t) = Z(\gamma(t)),\ \ \gamma(0) = z_0.$$

We say that u has continuous Lie-derivative along Z in Ω if the map $z \mapsto Zu(z)$ is continuous in Ω. If $Z_1, Z_2 \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, if u has Lie-derivative along Z_1 in Ω and if $Z_1 u$ has Lie derivative along Z_2 in z_0, then we say that u has second-order Lie derivative $Z_2 Z_1 u(z_0) = (Z_2(Z_1 u))(z_0)$.

Let $z_0 \in \Omega$ be fixed and let $\gamma : (-\sigma, \sigma) \to \Omega$ be the solution to the Cauchy problem $\dot\gamma(t) = Z(\gamma(t))$, $\gamma(0) = z_0$. Then, our definition says that $Zu(z_0)$ exists iff the function $u(\gamma(\cdot))$ is differentiable at 0. We explicitly remark that, if in addition u admits Lie-derivative along Z in the whole Ω, then $u(\gamma(\cdot))$ is differentiable in the whole $(-\sigma, \sigma)$ and it holds

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(u(\gamma(t))) = (Zu)(\gamma(t)) \qquad \forall\, t \in (-\sigma, \sigma). \tag{2.1}$$

Indeed, fixed $t_* \in (-\sigma, \sigma)$ the solution μ to $\dot\mu(t) = Z(\mu(t))$, $\mu(0) = \gamma(t_*)$, by uniqueness is given by $\mu(t) = \gamma(t_* + t)$ for small t. We now prove the following lemma.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $Z \in C^1(\Omega, \mathbb{R}^N)$. If u is continuous in Ω together with its Lie-derivative Zu, then Zu is also a derivative in the weak sense of distributions. More precisely, if $Z = \sum_{j=1}^N a_j(z)\,\partial_j$, then*

$$-\textstyle\sum_{j=1}^N \int u\, \partial_j(a_j\, \varphi) = \int \varphi\, Zu, \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{2.2}$$

Proof. We fix $\varphi \in C_0^\infty(\Omega)$. We have to prove that

$$-\textstyle\int u\, Z\varphi = \int \varphi\, Zu + \int \varphi\, u\, \mathrm{div}(Z). \tag{2.3}$$

For any fixed $x \in \Omega$, we denote by $\gamma_Z(t, x)$ the solution to the Cauchy problem $\dot\gamma(t) = Z(\gamma(t))$, $\gamma(0) = x$. By definition of Lie-derivative and by dominated convergence we have

$$-\textstyle\int u(x)\, Z\varphi(x)\, \mathrm{d}x = \lim\limits_{t\to 0} \big\{ -\int \frac{1}{t}\, u(x)\, \varphi(\gamma_Z(t, x))\, \mathrm{d}x + \frac{1}{t} \int u(x)\, \varphi(x)\, \mathrm{d}x \big\}.$$

We set $K := \mathrm{supp}(\varphi)$ and denote by O an open neighborhood of K compactly contained in Ω. Then, there exists a positive $\varepsilon = \varepsilon(O, Z)$ such that $\gamma_{-Z}(t, x)$ and $\gamma_Z(t, x)$ are well posed for every $t \in [-\varepsilon, \varepsilon]$ and every $x \in O$. Moreover, since $Z \in C^1(\Omega, \mathbb{R}^N)$, by standard results of dependence on the initial values, then $\gamma_Z(t, x)$ has a C^1 dependence on $(t, x) \in [-\varepsilon, \varepsilon] \times O$. Since $\gamma_Z(t, \gamma_{-Z}(t, x)) = x$ whenever the first side is defined, then (for every fixed $t \in [-\varepsilon, \varepsilon]$, for a smaller $\varepsilon > 0$ if necessary) the map $O \ni x \mapsto \gamma_Z(t, x)$ is a diffeomorphism of class C^1 with inverse map given by $y \mapsto \gamma_{-Z}(t, y)$. We claim that

$$\det\big(\mathcal{J}_{\gamma_{-Z}(t,\cdot)}(y)\big) = 1 - t\, J(t, y) \tag{2.4}$$

(here $\mathcal{J}$ denotes the Jacobian matrix) with

$$\sup_{y\in K,\, t\in[-\varepsilon,\varepsilon]} |J(t, y)| < \infty, \quad J(t, y) \xrightarrow{t\to 0} \mathrm{div}(Z)(y). \tag{2.5}$$

Similar arguments on a decomposition analogous to (2.4) have been used in [8]. If (2.4) and (2.5) hold, then the proof is complete. Indeed, by the substitution $x = \gamma_{-Z}(t,y)$ we have (recalling that $\gamma_{-Z}(t,y) = \gamma_Z(-t,y)$)

$$\begin{aligned}
&-\textstyle\int \frac{1}{t}\, u(x)\,\varphi(\gamma_Z(t,x))\,\mathrm{d}x + \frac{1}{t}\int u(x)\,\varphi(x)\,\mathrm{d}x \\
&\qquad = -\textstyle\int \frac{1}{t}\, u(\gamma_{-Z}(t,y))\,\varphi(y)\,(1 - t\,J(t,y))\,\mathrm{d}y + \frac{1}{t}\int u(y)\,\varphi(y)\,\mathrm{d}y \\
&\qquad = -\textstyle\int \varphi(y)\,\frac{1}{t}(u(\gamma_Z(-t,y)) - u(y))\,\mathrm{d}y + \int u(\gamma_{-Z}(t,y))\,\varphi(y)\,J(t,y)\,\mathrm{d}y \\
&\qquad \xrightarrow{t\to 0} \textstyle\int \varphi(y)\,Zu(y)\,\mathrm{d}y + \int u(y)\,\varphi(y)\,\mathrm{div}(Z)(y)\,\mathrm{d}y,
\end{aligned}$$

and this is the right hand-side of (2.3). Here we used dominated convergence, since it holds $-(u(\gamma_Z(-t,y)) - u(y))/t = (Zu)(\gamma_Z(-t^*,y))$ for some $t^* \in [-\varepsilon,\varepsilon]$ (see also (2.1)) and since Zu is continuous in Ω.

We now prove (2.4) and (2.5). Obviously, we have $\gamma_{-Z}(t,y) = y - \int_0^t Z(\gamma_{-Z}(s,y))\,\mathrm{d}s$. By differentiating with respect to y and by the mean value theorem, we obtain

$$\frac{\partial\,\gamma_{-Z}(t,y)}{\partial y} = \mathbb{I}_N - \int_0^t \mathcal{J}_Z(\gamma_{-Z}(s,y))\,\frac{\partial\,\gamma_{-Z}(s,y)}{\partial y}\,\mathrm{d}s = \mathbb{I}_N - t\,\mathcal{J}_Z(\gamma_{-Z}(t^*,y))\,\frac{\partial\,\gamma_{-Z}(t^*,y)}{\partial y},$$

for a suitable t^* in the segment joining 0 and t. We recall that, if A is a $N\times N$ matrix, then $\det(\mathbb{I}_N - t\,A) = 1 - t\,\mathrm{trace}(A) + \mathcal{O}(t^2)$, as $t\to 0$ (where $\mathcal{O}(t^2)$ is t^2 times a polynomial in t and in the entries of A). Consequently we obtain

$$\det\Big(\frac{\partial\,\gamma_{-Z}(t,y)}{\partial y}\Big) = 1 - t\left(\mathrm{trace}\left(\mathcal{J}_Z(\gamma_{-Z}(t^*,y))\,\frac{\partial\,\gamma_{-Z}(t^*,y)}{\partial y}\right) + \mathcal{O}(t)\right).$$

We then define $J(t,y)$ satisfying (2.4) in the obvious way. In particular, we get

$$J(t,y) \xrightarrow{t\to 0} \mathrm{trace}\left(\mathcal{J}_Z(\gamma_{-Z}(0,y))\,\frac{\partial\,\gamma_{-Z}(0,y)}{\partial y}\right) = \mathrm{trace}\big(\mathcal{J}_Z(y)\big) = \mathrm{div}(Z)(y),$$

which gives the second assertion in (2.5). The estimate of $J(t,y)$ in (2.5) follows from the C^1 regularity of γ_{-Z} and of Z. This ends the proof. ■

In the sequel, we make use of classical Friedrichs mollifiers. We fix $J \in C_0^\infty(\mathbb{R}^N)$, $0 \le J \le 1$, $\mathrm{supp}(J) \subset D(0,1)$, $\int J = 1$. For $\varepsilon > 0$ we set $J_\varepsilon = \varepsilon^{-N} J((\cdot)/\varepsilon)$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ we let $u_\varepsilon(z) = \int u(\zeta)\,J_\varepsilon(z-\zeta)\,\mathrm{d}\zeta = \int u(z - \varepsilon\,\zeta)\,J(\zeta)\,\mathrm{d}\zeta$, defined for any $z \in \Omega$ such that $\mathrm{dist}(z, \mathbb{R}^N \setminus \Omega) > \varepsilon$. With Lemma 2.1 at hand, we can prove the following result.

Proposition 2.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $Z \in C^1(\Omega,\mathbb{R}^N)$. If u is continuous in Ω together with its Lie-derivative Zu, then we have*

$$Z(u_\varepsilon) \rightrightarrows Zu \text{ as } \varepsilon \to 0^+, \qquad \text{uniformly on compact subsets of } \Omega.$$

Proof. The proof follows the ideas in [10]. Such ideas have also been used in [8, 11] in proving L^p-approximations. For our aims, we need uniform approximation properties. We provide a self-contained proof for the sake of completeness. We fix a compact subset K of Ω. Since Zu is continuous, $(Zu)_\varepsilon \xrightarrow{\varepsilon\to 0} Zu$ uniformly on K, whence it is enough to prove that

$$Z(u_\varepsilon) - (Zu)_\varepsilon \xrightarrow{\varepsilon\to 0} 0 \qquad \text{uniformly on } K. \tag{2.6}$$

If $Z = \sum_{j=1}^N a_j(x)\,\partial_{x_j}$, for every $x \in K$ we have

$$\begin{aligned}
Z(u_\varepsilon)(x) - (Zu)_\varepsilon(x) &= \textstyle\int u(y) \sum_j a_j(x)\,\partial_{x_j}(J_\varepsilon(x-y))\,\mathrm{d}y - \int Zu(y)\,J_\varepsilon(x-y)\,\mathrm{d}y \\
&= -\textstyle\int u(y)\sum_j a_j(x)\,\partial_{y_j}(J_\varepsilon(x-y))\,\mathrm{d}y + \int u(y)\,\sum_j \partial_{y_j}(a_j(y)\,J_\varepsilon(x-y))\,\mathrm{d}y \\
&= \textstyle\int u(y)\Big\{\sum_j \partial_{y_j}\Big((a_j(y) - a_j(x))\,J_\varepsilon(x-y)\Big)\Big\}\,\mathrm{d}y.
\end{aligned}$$

In the second equality we have used Lemma 2.1. We denote by $k_\varepsilon(x,y)$ the kernel in curly brackets in the last integral above. We claim that this kernel has the following properties

$$k_\varepsilon(x,y)=0 \quad \text{whenever } |x-y|>\varepsilon, \tag{2.7}$$

$$\textstyle\int k_\varepsilon(x,y)\,\mathrm{d}y=0 \quad \text{for all } x\in K, \tag{2.8}$$

$$\textstyle\int |k_\varepsilon(x,y)|\,\mathrm{d}y\le c(K), \quad \text{for all } x\in K, \tag{2.9}$$

with $c(K)$ independent of ε. Indeed (2.7) and (2.8) are immediate. Moreover, chosen a suitable compact set K_0, $K\subset K_0\subset\Omega$, for every small $\varepsilon>0$ and every $x\in K$ we have

$$\begin{aligned}\textstyle\int |k_\varepsilon(x,y)|\,\mathrm{d}y &\le \textstyle\int\sum_j |a_j(y)-a_j(x)|\,\big|\partial_{y_j}\big(J_\varepsilon(x-y)\big)\big|\,\mathrm{d}y+\int J_\varepsilon(x-y)\,\big|\mathrm{div}(Z)(y)\big|\,\mathrm{d}y\\ &\le \textstyle\int_{D(x,\varepsilon)}\sum_j \max_{K_0}|\nabla a_j|\,|x-y|\,\varepsilon^{-N-1}\,\|\nabla J\|_\infty\,\mathrm{d}y+\max_{K_0}|\mathrm{div}(Z)|\int_{D(x,\varepsilon)}J_\varepsilon(x-y)\,\mathrm{d}y\\ &\le c\textstyle\int_{D(x,\varepsilon)}\varepsilon^{-N}\,\mathrm{d}y+\max_{K_0}|\mathrm{div}(Z)|\le c.\end{aligned}$$

We now complete the proof of (2.6). Using (2.7), (2.8) and (2.9), for every $x\in K$ we obtain

$$\begin{aligned}&|Z(u_\varepsilon)(x)-(Zu)_\varepsilon(x)|=\big|\textstyle\int u(y)\,k_\varepsilon(x,y)\,\mathrm{d}y\big|=\big|\int(u(y)-u(x))\,k_\varepsilon(x,y)\,\mathrm{d}y\big|\\ &\quad\le \sup_{|x-y|\le\varepsilon}|u(y)-u(x)|\textstyle\int|k_\varepsilon(x,y)|\,\mathrm{d}y\le c(K)\,\sup_{z\in K,|z-y|\le\varepsilon}|u(y)-u(z)|\xrightarrow{\varepsilon\to 0}0.\end{aligned}$$

■

The following proposition, though asserting a not unexpected result, is non trivial. It is indeed consequence of Lemma 2.1 and Proposition 2.2 and it will play a crucial rôle in the proof of the weak maximum principle in the next section.

Proposition 2.3. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set and let $Z_1,Z_2\in C^1(\Omega,\mathbb{R}^N)$. If $u\in C(\Omega,\mathbb{R})$ has continuous Lie-derivatives along Z_1 and Z_2 in Ω, then (for any $\alpha_1,\alpha_2\in\mathbb{R}$) u has Lie derivative along $\alpha_1Z_1+\alpha_2Z_2$ and it holds $(\alpha_1Z_1+\alpha_2Z_2)u=\alpha_1Z_1u+\alpha_2Z_2u$.*

Proof. We fix $x\in\Omega$ and consider the integral curve $\gamma(t)$ of $\alpha_1Z_1+\alpha_2Z_2$ passing through x at $t=0$. Since u_ε is smooth, by the mean value theorem we obviously have

$$\tfrac{1}{t}(u_\varepsilon(\gamma(t))-u_\varepsilon(x))=\big(\alpha_1Z_1(u_\varepsilon)+\alpha_2Z_2(u_\varepsilon)\big)(\gamma(\tau_{\varepsilon,t})),$$

for a suitable $\tau_{\varepsilon,t}$ in the segment joining 0 and t. By taking a sub-family of $\{u_\varepsilon\}_\varepsilon$ if necessary, we can suppose that $\tau_{\varepsilon,t}$ converges (as $\varepsilon\to 0$) to a suitable $\tau_{0,t}$ between 0 and t. As a consequence, letting ε go to 0 in the above identity, by means of Proposition 2.2 we obtain $\frac{1}{t}(u(\gamma(t))-u(x))=\big(\alpha_1Z_1u+\alpha_2Z_2u\big)(\gamma(\tau_{0,t}))$. Finally, we infer the existence of $(\alpha_1Z_1+\alpha_2Z_2)u(x)$ and its equality to $\alpha_1Z_1u(x)+\alpha_2Z_2u(x)$ letting t go to zero (here again we use the continuity of Z_1u and Z_2u). ■

Proposition 2.4. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set and let $Z_1,\dots,Z_m\in C^1(\Omega,\mathbb{R}^N)$. Let $v\in C(\Omega,\mathbb{R})$ have continuous Lie-derivatives along $Z_1,\dots,Z_m$ up to second order. Then, at any local maximum point $z_0\in\Omega$ for v, we have*

$$\begin{cases}Z_jv(z_0)=0 \quad j=1,\dots,m\\ (Z_iZ_jv(z_0))_{i,j\le m}\le 0 \quad \textit{i.e., } \sum_{i,j=1}^m Z_iZ_jv(z_0)\,\xi_i\xi_j\le 0 \quad \forall\,\xi\in\mathbb{R}^m.\end{cases}\tag{2.10}$$

We remark that the matrix $(Z_iZ_jv(z_0))_{i,j}$ need not be symmetric.

Proof. The first assertion in (2.10) is obvious, recalling the definition of Lie-derivative. Fix now $\xi\in\mathbb{R}^m$ and consider the solution to the Cauchy problem $\dot\gamma(t)=\sum_{i=1}^m\xi_i\,Z_i(\gamma(t))$, $\gamma(0)=z_0$. We set $\Phi(t)=v(\gamma(t))$, for $t\in(-\sigma,\sigma)$, $\sigma>0$ small enough. By applying Proposition 2.3 (and identity (2.1)) we obtain

$$\Phi\in C^1(-\sigma,\sigma),\qquad \dot\Phi(t)=\textstyle\sum_{j=1}^m\xi_j(Z_jv)(\gamma(t)).$$

Another application of Proposition 2.3 gives $\Phi\in C^2(-\sigma,\sigma)$ and $\ddot\Phi(t)=\sum_{i,j}\xi_i\xi_j\,(Z_iZ_jv)(\gamma(t))$. Finally, since 0 is a local maximum point of Φ, we have $0\ge\ddot\Phi(0)=\sum_{i,j=1}^m\xi_i\xi_j\,Z_iZ_jv(z_0)$. ■

3. The weak maximum principle

In this section, Ω will denote a bounded open subset of $\mathbb{R}^N$ and $Z_1, \dots, Z_m$ will be vector fields in $C^1(\Omega, \mathbb{R}^N)$ whereas $Z_0 \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover $\Gamma^2(\Omega)$ will denote the intrinsic regularity class w.r.t. these fields introduced in Definition 1.1. We suppose the matrix $(a_{i,j}(z))_{i,j=1}^m$ is symmetric and positive semi-definite for any $z \in \Omega$. We then consider the differential operator

$$\mathcal{H} = \textstyle\sum_{i,j=1}^m a_{i,j}(z)\, Z_i Z_j + Z_0.$$

In the following results and in the next section we shall make use of the following definition.

Definition 3.1. We say that the differential operator $\mathcal{H}$ satisfies the Γ^2-Weak Maximum Principle (in short, Γ^2-(WMP)) in the bounded set Ω if, for every function $u \in \Gamma^2(\Omega)$ satisfying

$$\begin{cases} \mathcal{H}u(z) \geq 0 & \text{for every } z \in \Omega, \\ \limsup_{z \to \zeta} u(z) \leq 0 & \text{for every } \zeta \in \partial\Omega, \end{cases} \tag{3.1}$$

there holds $u \leq 0$ in Ω.

We prove the *weak maximum principle* for $\mathcal{H}$ in the intrinsic regularity class $\Gamma^2(\Omega)$.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $Z_1, \dots, Z_m \in C^1(\Omega, \mathbb{R}^N)$ and $Z_0 \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Suppose the matrix $(a_{i,j}(z))_{i,j=1}^m$ is symmetric and positive semi-definite for any $z \in \Omega$. If there exists $w \in \Gamma^2(\Omega)$ such that $\mathcal{H}w < 0$ and $w > 0$ in Ω, then $\mathcal{H}$ satisfies the Γ^2-(WMP) on Ω.*

Proof. The scheme of the proof is classical; the new difficulty is due to the "weak regularity" of u, namely $u \in \Gamma^2(\Omega)$. The crucial point is then the application of the results in Section 2. For the reader's convenience, we provide all the details. Let $u \in \Gamma^2(\Omega)$ satisfy (3.1) and let w be as above. The theorem is proved if we show that,

$$\text{for every } \varepsilon > 0 \text{ we have} \quad u - \varepsilon\, w \leq 0 \quad \text{in } \Omega. \tag{3.2}$$

We set $v = u - \varepsilon\, w$. Then (3.2) will follow if we prove the following claim:

$$v \text{ has no maximum points in } \Omega. \tag{3.3}$$

Indeed, let $\overline{z} \in \overline{\Omega}$ be such that $\sup_{V \cap \Omega} v = \sup_\Omega v$ for every neighborhood V of $\overline{z}$. If (3.3) holds, then $\overline{z} \in \partial\Omega$, otherwise the continuous function v would have a maximum point in Ω. Since $\overline{z} \in \partial\Omega$, we have $\sup_\Omega v = \lim_{r \to 0^+} \sup_{B(\overline{z},r) \cap \Omega} v = \limsup_{z \to \overline{z}} v(z) \leq 0$. The last inequality holds since $v = u - \varepsilon\, w \leq u$, $\overline{z} \in \partial\Omega$ and u satisfies the boundary condition in (3.1). This gives $v \leq 0$ in Ω, i.e., (3.2) holds. We are only left with the proof of the claimed (3.3). Suppose by contradiction that there exists a maximum point $z_0 \in \Omega$ for v. Then, obviously $Z_0 v(z_0) = 0$, by the definition of Lie derivative. Moreover by means of Proposition 2.4,

$$0 < -\varepsilon \mathcal{H}w(z_0) \leq \mathcal{H}u(z_0) - \varepsilon \mathcal{H}w(z_0) = \mathcal{H}v(z_0) = \textstyle\sum_{i,j=1}^m a_{i,j}(z_0)\, Z_i Z_j v(z_0) \leq 0.$$

The last inequality is derived in the following way. Let $(b_{i,j})_{i,j \leq m}$ be a symmetric square root of the symmetric positive semi-definite matrix $(a_{i,j}(z_0))_{i,j \leq m}$. Then

$$\textstyle\sum_{i,j=1}^m a_{i,j}(z_0)\, Z_i Z_j v(z_0) = \sum_{k=1}^m \sum_{i,j=1}^m b_{i,k} b_{j,k}\, Z_i Z_j v(z_0) \leq 0$$

by (2.10). We explicitly remark that the matrix $(Z_i Z_j v(z_0))_{i,j}$ need not be symmetric. ■

The following expression in Cartesian derivatives of $\mathcal{H}$ will be useful in the next section.

Remark 3.3. We agree to denote by $\mathbf{Z}(z)$ the $N \times m$ matrix whose h-th column is given by $Z_h I(z)$, the column vector of the component functions of the field $Z_h = \sum_{k=1}^N (Z_h I)_k(z)\, \partial_{z_k}$.

The following one is the expression of the operator $\mathcal{H}=\sum_{i,j=1}^{m} a_{i,j}(z)\, Z_iZ_j+Z_0$, when $\mathcal{H}$ acts on C^2 functions

$$\mathcal{H}=\textstyle\sum_{h,k=1}^{N}\widetilde{a}_{h,k}(z)\,\partial_{z_h}\partial_{z_k}+\sum_{k=1}^{N}\widetilde{b}_k(z)\,\partial_{z_k},\qquad \text{where} \tag{3.4}$$

$$\widetilde{A}(z)=\mathbf{Z}(z)\,A(z)\,\big(\mathbf{Z}(z)\big)^T,\quad \widetilde{b}_k(z)=\textstyle\sum_{i,j=1}^{m}a_{i,j}(z)\,Z_i(Z_jI)_k(z)+(Z_0I)_k(z) \tag{3.5}$$

$(A(z)=\big(a_{i,j}(z)\big)_{i,j\le m}$, $\widetilde{A}(z)=\big(\widetilde{a}_{h,k}(z)\big)_{h,k\le N})$. With the above notation, for every $\xi\in\mathbb{R}^N$ we have $\big\langle\widetilde{A}(z)\xi,\xi\big\rangle=\big\langle A(z)(\mathbf{Z}(z))^T\xi,(\mathbf{Z}(z))^T\xi\big\rangle$. We remark that the h-th component of $(\mathbf{Z}(z))^T\xi$ is $\langle Z_hI(z),\xi\rangle$. Hence, if $\lambda_z\ge 0$ is the smallest eigenvalue of the positive semi-definite matrix $A(z)$, we derive the following inequality that will be used in the next section:

$$\big\langle\widetilde{A}(z)\,\xi,\xi\big\rangle\ge\lambda_z\textstyle\sum_{h=1}^{m}\langle Z_hI(z),\xi\rangle^2,\quad \forall\,\xi\in\mathbb{R}^N. \tag{3.6}$$

Example 3.4. Let $Z_1,\dots,Z_m\in C^1(\Omega,\mathbb{R}^N)$ and $Z_0\in \mathrm{Lip}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. Suppose the symmetric matrix $A=\big(a_{i,j}\big)_{i,j\le m}$ is locally bounded in Ω and A is locally uniformly positive definite, i.e., for every $z_0\in\Omega$ there exist a neighborhood W_0 of z_0 and $\lambda_0>0$ such that

$$\big\langle A(z)\,\xi,\xi\big\rangle\ge\lambda_0\,|\xi|^2\qquad\forall\,\xi\in\mathbb{R}^m,\quad\forall\,z\in W_0. \tag{3.7}$$

Suppose finally that the system of vector fields $Z_1,\dots,Z_m$ is non-totally-degenerate at any point of Ω, i.e., for every $z_0\in\Omega$ there exists $i_0\in\{1,\dots,m\}$ such that $Z_{i_0}(z_0)\ne0$. Then we claim that for every $z\in\Omega$ there exists a function w such that $w>0$ and $\mathcal{H}w<0$ in a suitable neighborhood of z. Consequently, by Theorem 3.2, the operator $\mathcal{H}=\sum_{i,j=1}^{m}a_{i,j}(z)\,Z_iZ_j+Z_0$ *locally* satisfies the Γ^2-(WMP), i.e., for every $z\in\Omega$, there exists a neighborhood V of z such that $\mathcal{H}$ satisfies the Γ^2-(WMP) in every open bounded subset of V.

Let us prove the claim. Let $z_0\in\Omega$ be fixed. We choose $i_0\in\{1,\dots,m\}$ such that $Z_{i_0}(z_0)\ne0$ and we consider a bounded neighborhood W_0 of z_0 and a scalar $\lambda_0>0$ as in (3.7). For the sake of brevity we write $Z_{i_0}(z_0)=(\xi_1,\dots,\xi_N)^T=\xi$. We define the function $w(z)=M-\exp\big(\mu\sum_{j=1}^{N}\xi_j\,z_j\big)$, where $M>0$ and $\mu>0$ are constants which will be determined in the sequel. A straightforward computation which makes use of the Cartesian expression (3.4) of $\mathcal{H}$ (we use all the notation in Remark 3.3) proves that, for every $z\in W_0$, $\mathcal{H}w(z)$ is equal to the negative factor $-\mu^2\,e^{\mu\langle\xi,z\rangle}$ times the expression (see also (3.7))

$$\begin{aligned}\langle A(z)(\mathbf{Z}(z))^T\xi,(\mathbf{Z}(z))^T\xi\rangle+\tfrac{1}{\mu}\textstyle\sum_{k=1}^{N}\widetilde{b}_k(z)\,\xi_k&\ge\lambda_0\,\big|(\mathbf{Z}(z))^T\xi\big|^2+\tfrac{1}{\mu}\textstyle\sum_{k=1}^{N}\widetilde{b}_k(z)\,\xi_k\\&\ge\lambda_0\,\langle Z_{i_0}I(z),\xi\rangle^2+\tfrac{1}{\mu}\textstyle\sum_{k=1}^{N}\widetilde{b}_k(z)\,\xi_k.\end{aligned}$$

We now consider a closed ball $\overline{D(z_0,2\rho)}\subset W_0$ and we take $z\in D(z_0,\rho)=D$. Since $\langle Z_{i_0}I(z_0),\xi\rangle^2=|Z_{i_0}I(z_0)|^4>0$, we can suppose ρ is so small that $\inf_{z\in D}\langle Z_{i_0}I(z),\xi\rangle^2>0$. We then choose μ such that $\mu\ge\big(2\sup_{z\in D}\big|\sum_{k=1}^{N}\widetilde{b}_k(z)\,\xi_k\big|\big)/\big(\lambda_0\inf_{z\in D}\langle Z_{i_0}I(z),\xi\rangle^2\big)$ (we remark that $\widetilde{b}_k$ is locally bounded, since we are assuming that so is A). With this choice of μ we get

$$\mathcal{H}w(z)\le-\tfrac12\lambda_0\,\mu^2\,e^{\mu\langle\xi,z\rangle}\langle Z_{i_0}I(z),\xi\rangle^2<0\qquad\forall\,z\in D.$$

Once ρ and μ are chosen, we take M large enough, so that also $w(z)>0$ in D. ■

We would like to end this section with a definition of *viscosity solution* with intrinsicly regular test functions. We consider on a bounded open set $\Omega\subset\mathbb{R}^N$ the operator $\mathcal{H}=\sum_{j=1}^{N}a_{i,j}(z)\,Z_iZ_j+Z_0$, where $Z_1,\dots,Z_m\in C^1(\Omega,\mathbb{R}^N)$, $Z_0\in\mathrm{Lip}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $a_{i,j}$ is a symmetric positive semi-definite real valued matrix. We say that an u.s.c. function u on Ω is a Γ^2-*viscosity (sub-)solution* to the differential inequality $\mathcal{H}u\ge0$ if, for every $\phi\in\Gamma^2(\Omega)$ and $z\in\Omega$ such that $u-\phi$ has a local maximum point at z, we have $\mathcal{H}\phi(z)\ge0$. It is easy to see that a weak maximum principle for Γ^2-viscosity sub-solutions holds if there exists a positive barrier function $w\in\Gamma^2(\Omega)$ such that $\mathcal{H}w<0$. Indeed, one shows that $u-\phi=u-\varepsilon w$ can not have interior maximum points. Moreover, from the results of Section 3, the following fact holds: *Any* Γ^2 *function satisfying (point-wise)* $\mathcal{H}u(z) \geq 0$*, is also a* Γ^2*-viscosity sub-solution.* Indeed, by the results in Proposition 2.4, if $\phi \in \Gamma^2(\Omega)$, $z \in \Omega$ and $u-\phi$ has a local maximum at z, then we have $-\mathcal{H}\phi(z) = \mathcal{H}(u-\phi)(z) - \mathcal{H}u(z) = \sum_{i,j} a_{i,j}(z)\, Z_i Z_j (u-\phi)(z) - \mathcal{H}u(z) \leq -\mathcal{H}u(z) \leq 0$. For related results on maximum principles in the classical viscosity context, we would like to point out the recent papers by Bardi&Da Lio [2] and by Kawohl&Kutev [15] (see also the references therein).

4. The Maximum Propagation

First, we introduce some notation and definition which will be used throughout the section. $D(x,r)$ will always denote the (Euclidean) ball of radius r centered at $x \in \mathbb{R}^N$. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let F be an arbitrary subset of Ω relatively closed in Ω. Suppose $\partial F \cap \Omega \neq \emptyset$ and let $z_0 \in \partial F \cap \Omega$ be fixed. A vector $\nu \in \mathbb{R}^N \setminus \{0\}$ will be said *externally orthogonal to* F *at* z_0 *(relatively to* Ω*)*, if the following condition is satisfied: $\overline{D(z_0 + \nu, |\nu|)} \subseteq \big(\Omega \setminus F\big) \cup \{z_0\}$. In the sequel we shall briefly write $\nu \perp F$ *ext. at* z_0. Finally, we let

$$F^* := \{z_0 \in \partial F \cap \Omega \,|\, \text{there exists } \nu \perp F \text{ ext. at } z_0\}.$$

If Ω is connected and F is a relatively closed proper subset of Ω, it can be seen that $F^* \neq \emptyset$.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $F \subseteq \Omega$ be relatively closed in Ω. Let $X \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. We say that F *is positively* X*-invariant* if, for every $\gamma : [0,T] \to \Omega$ integral curve of X with $\gamma(0) \in F$, then $\gamma(t) \in F$ for every $t \in [0,T]$. We say that F *is* X*-invariant* if X is both positively X-invariant and positively $(-X)$-invariant.

The following result is essentially contained in [6] (see also [27]).

Theorem 4.2 (Bony). *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $X \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *and let* $F \subseteq \Omega$ *be relatively closed in* Ω*. Then* F *is positively* X*-invariant iff*

$$\langle X(z_0), \nu \rangle \leq 0 \qquad \text{for any } z_0 \in F^* \text{ and any } \nu \perp F \text{ ext. at } z_0. \tag{4.1}$$

Hence, F *is* X*-invariant iff* $\langle X(z_0), \nu\rangle = 0$ *for any* $z_0 \in F^*$ *and any* $\nu \perp F$ *ext. at* z_0*.*

We are now ready to provide the main results of this section, the Maximum Propagation for the operator $\mathcal{H}$. First we fix some hypotheses. We consider the operator

$$\mathcal{H} = \textstyle\sum_{i,j=1}^m a_{i,j}(z)\, Z_i Z_j + Z_0 \quad \text{on the open set } \Omega \subseteq \mathbb{R}^N,$$

where, following Definitions 1.1 and 3.1, the following properties hold:

$$\begin{cases} Z_0 \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \quad Z_1, \dots, Z_m \in C^1(\Omega, \mathbb{R}^N), \\ a_{i,j} = a_{j,i} \in C(\Omega, \mathbb{R}), \quad \big(a_{i,j}(z)\big)_{i,j \leq m} \textit{ is positive-definite for any fixed } z \in \Omega, \\ \textit{for every fixed } z \in \Omega\textit{, there exists a neighborhood } V \textit{ of } z \textit{ such that} \\ \mathcal{H} \textit{ satisfies the } \Gamma^2\text{-(WMP)} \textit{ in every open subset of } V. \end{cases} \tag{4.2}$$

In the following result, we agree to call *principal vector field w.r.t.* $\mathcal{H}$ any $\mathrm{Lip}_{\mathrm{loc}}$ vector field Y such that, for any $z \in \Omega$ there exists $\lambda(z) > 0$ with $\langle \widetilde{A}(z)\xi, \xi\rangle \geq \lambda(z)\, \langle Y(z), \xi\rangle^2$ for every $\xi \in \mathbb{R}^N$ (here $\widetilde{A}$ is the $N \times N$ matrix in Remark 3.3).

Theorem 4.3. *Suppose the operator* $\mathcal{H}$ *satisfies the above hypotheses* (4.2)*. Then, for every function* $u \in \Gamma^2(\Omega)$ *satisfying*

$$\mathcal{H}u(z) \geq 0, \quad u(z) \leq 0 \quad \forall\, z \in \Omega,$$

the set $F = \{z \in \Omega | u(z) = 0\}$ *(if non-empty) is invariant with respect to all the vector fields* $Z_1, \dots, Z_m$ *and to all the principal vector fields w.r.t.* $\mathcal{H}$*. Moreover, if* γ *is a solution to*

$$\dot\gamma(t) = \textstyle\sum_{j=1}^m \alpha_j(t)\, Z_j(\gamma(t)), \quad \gamma(0) \in F \tag{4.3}$$

(for some bounded α_j's) then $\gamma(t) \in F$ for every t.

Proof. The propagation of F along solutions to (4.3) follows from the propagation along $Z_1, \dots, Z_m$, by means of a suitable adaptation of the arguments in [6] in the proof of Theorem 4.2. Moreover, it is easy to see that any principal vector field Y w.r.t. $\mathcal{H}$ can be pointwise written as $Y(z) = \sum_{j=1}^{m} \beta_j(z)\, Z_j(z)$; hence the invariance of F along Y directly follows from the invariance along $Z_1, \dots, Z_m$ and from Theorem 4.2. We are then left to prove the propagation along $Z_1, \dots, Z_m$. Let us fix u, F and $Z \in \{Z_1, \dots, Z_m\}$, as above. By Theorem 4.2, it suffices to show that $\langle Z(z^0), \nu\rangle = 0$ for any $z^0 \in F^*$ and any $\nu \perp F$ ext. at z^0. Moreover, we may suppose that Ω is connected. If $F = \Omega$, there is nothing to prove. Otherwise, being $F \neq \emptyset$, there exist $z^0 \in F^*$ and ν externally orthogonal to F at z^0. Suppose by contradiction that

$$\langle Z(z^0), \nu\rangle \neq 0. \tag{4.4}$$

For $\lambda > 0$, we consider the function $h_\lambda(z) = \exp(-\lambda|z - (z^0 + \nu)|^2) - \exp(-\lambda|\nu|^2)$. Hence, with the notation of Remark 3.3, we have

$$\mathcal{H}h_\lambda(z^0) = 4\lambda^2 \exp(-\lambda|\nu|^2)\big(\textstyle\sum_{i,j=1}^{N} \widetilde{a}_{i,j}(z^0)\, \nu_i\nu_j + \mathcal{O}(\frac{1}{\lambda})\big) \quad \text{as } \lambda \to \infty. \tag{4.5}$$

Now, by (3.6), if $\lambda_{z^0} > 0$ is the least eigenvalue of $(a_{i,j}(z_0))_{i,j}$, then $\sum_{i,j=1}^{N} \widetilde{a}_{i,j}(z^0)\, \nu_i\nu_j \geq \lambda_{z^0}\langle Z(z^0), \nu\rangle^2$ and this last term is strictly positive, thanks to our assumption (4.4). Consequently, from the continuity of the coefficients of $\mathcal{H}$ and from (4.5), we infer the existence of a large $\lambda > 0$ and of a small $\rho > 0$ such that $h = h_\lambda$ satisfies

$$\mathcal{H}h > 0 \quad \text{in } D(z^0, \rho). \tag{4.6}$$

We may also suppose that ρ is chosen so small that $\mathcal{H}$ satisfies the Γ^2-(WMP) in every open subset of $D(z^0, \rho)$ (see the hypotheses (4.2)). Let now $U = D(z^0, \rho) \cap D(z^0 + \nu, |\nu|)$ and set $B_1 = \overline{D}(z^0, \rho) \cap \partial D(z^0 + \nu, |\nu|)$, $B_2 = \overline{D}(z^0 + \nu, |\nu|) \cap \partial D(z^0, \rho)$, so that $\partial U = B_1 \cup B_2$. We choose ε such that

$$0 < \varepsilon < (-\max\nolimits_{B_2} u)/(\max\nolimits_{B_2} h). \tag{4.7}$$

This is possible since $u < 0$ in B_2 by $\overline{D}(z^0 + \nu, |\nu|) \setminus \{z^0\} \subseteq \Omega \setminus F$. With this choice of ε we set $u_\varepsilon = u + \varepsilon\, h$. The following facts hold (recall (4.6)):

$$u_\varepsilon \in \Gamma^2(\Omega), \quad \mathcal{H}u_\varepsilon = \mathcal{H}u + \varepsilon\mathcal{H}h \geq \varepsilon\mathcal{H}h > 0 \quad \text{in } U; \qquad u_\varepsilon \leq 0 \quad \text{in } \partial U. \tag{4.8}$$

To see the second assertion note that if $\zeta \in B_1$ then $u_\varepsilon(\zeta) = u(\zeta) \leq 0$, if $\zeta \in B_2$ we have $u_\varepsilon(\zeta) \leq \max_{B_2} u_\varepsilon \leq \max_{B_2} u + \varepsilon \max_{B_2} h < 0$ (by the definition (4.7) of ε). From (4.8) and since $\mathcal{H}$ satisfies the Γ^2-(WMP) in U, we infer that

$$u_\varepsilon = u + \varepsilon\, h \leq 0 \quad \text{in } U. \tag{4.9}$$

Let us now recall (4.4) and let γ be the curve so defined: if $\langle Z(z^0), \nu\rangle > 0$ then $\gamma(t)$ is the integral curve $\gamma_+(t)$ of the vector field $+Z$ passing through z^0 at $t = 0$; if otherwise $\langle Z(z^0), \nu\rangle < 0$ then $\gamma(t)$ is the integral curve $\gamma_-(t)$ of the vector field $-Z$ passing through z^0 at $t = 0$. We claim that, with this choice of γ, we have

$$\gamma(t) \in U \quad \text{for any small positive } t. \tag{4.10}$$

Indeed, we have $\gamma_\pm(t) = z^0 \pm tZ(z^0) + \mathcal{O}(t^2)$ as $t \to 0$. Consequently

$$\begin{aligned} \big|\gamma_\pm(t) - (z^0 + \nu)\big|^2 &= \big| - \nu \pm t\, Z(z^0) + \mathcal{O}(t^2)\big|^2 \\ &= |\nu|^2 \mp 2t\, \big\langle \nu, Z(z^0)\big\rangle + \mathcal{O}(t^2) \quad \text{as } t \to 0. \end{aligned} \tag{4.11}$$

Recalling the definition of γ (4.10) follows. Now (4.9) together with (4.10) give $u(\gamma(t)) + \varepsilon\, h(\gamma(t)) \leq 0$ for any small positive t. Since $u(z^0) = h(z^0) = 0$, we infer

$$\tfrac{1}{t}(u(\gamma(t)) - u(z^0)) \leq -\tfrac{\varepsilon}{t}(h(\gamma(t)) - h(z^0)) \quad \text{for any small } t > 0. \tag{4.12}$$

We let $t \to 0^+$ in both sides of (4.12). Since $u \in \Gamma^2(\Omega)$ and by the definition of γ, the left-hand side goes to $\pm Zu(z^0)$ which is zero since z^0 is a maximum point for u. The right-hand side of (4.12) goes to $\mp\varepsilon\, Zh(z^0) = \mp 2\varepsilon\, \lambda \exp(-\lambda|\nu|^2)\, \langle \nu, Z(z^0)\rangle$. By the choice of γ this limit is strictly negative in any case. This gives a contradiction and completes the proof. ■

Within the proof of Theorem 4.3, we have proved the following Hopf-type Lemma along vector fields, which has an interest on its own (see e.g., [1, 2, 3, 14, 21, 23, 24, 27] for other Hopf-type Lemmas for C^2 or viscosity functions).

Lemma 4.4. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\mathcal{H}$ be as in* (4.2)*. Let O be an open subset of Ω and let $z_0 \in \partial O \cap \Omega$ be such that $\overline{D}(z_0 + \nu, |\nu|) \subset O \cup \{z_0\}$, for a suitable $\nu \in \mathbb{R}^N \setminus \{0\}$. Suppose $u \in \Gamma^2(O) \cap C(O \cup \{z_0\})$ is such that $u(z_0) = 0$ and $\mathcal{H}u \geq 0$, $u < 0$ in O. Let $j \in \{1, \dots, m\}$ be fixed. If Z_j enters in O at z_0, i.e., $\langle Z_j(z_0), \nu\rangle > 0$, then we have*

$$\limsup_{t\to 0^+} \tfrac{1}{t}\,(u(\gamma(t)) - u(z_0)) < 0,$$

where γ is the integral curve of Z_j passing through z_0 at $t = 0$.

We now briefly consider a propagation along the drift Z_0. We explicitly remark that this case of propagation is more involved to deal with and many deep techniques have been elaborated in the C^2 setting by various authors (see e.g., [1, 24, 27]). A complete study of the drift propagation in our intrinsic Γ^2 case is out of our aims here. However, as an example, we treat the following simple (but significant) case. This case actually covers the classes of operators in [4, 5].

We then suppose $\mathbb{R}^N$ is split in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with coordinates $z = (x, t)$ where $x \in \mathbb{R}^{N_1}$ and $t \in \mathbb{R}^{N_2}$ and we suppose that $Z_1, \dots, Z_m$ annihilate $t_1, \dots, t_{N_2}$, i.e., with the notation introduced in Example 3.3,

$$Z_h = \sum_{k=1}^{N_1} (Z_h I)_k(x,t)\, \partial_{x_k} \qquad \text{for all } h = 1, \dots, m. \tag{4.13}$$

We explicitly remark that Z_0 may operate in all the variables x, t.

Proposition 4.5. *Suppose the operator $\mathcal{H} = \sum_{i,j=1}^m a_{i,j}(z)\, Z_i Z_j + Z_0$ satisfies hypotheses* (4.2) *with $Z_1, \dots, Z_m$ as in* (4.13)*. Let $u \in \Gamma^2(\Omega)$ be a function satisfying $\mathcal{H}u(z) \geq 0$, $u(z) \leq 0$ for all $z \in \Omega$, and let $F = \{z \in \Omega | u(z) = 0\}$ be non-empty. We make the following assumption:*

$$\begin{aligned}&\text{for every } z \in \Omega, \text{ there exists a neighborhood } W \text{ of } z \text{ such that,}\\ &\text{whenever } (\overline{x}, \overline{t}) \in F \cap W, \text{ then } W \cap \{t = \overline{t}\} \text{ lies entirely in } F.\end{aligned} \tag{4.14}$$

Then F is positively invariant with respect to Z_0.

Broadly speaking, in (4.14) we are assuming that the maximum of u locally propagates along the x-coordinates, for any fixed t-coordinate. A sufficient condition for (4.14) to hold is the following Hörmander condition: $Z_1, \dots, Z_m$ are smooth and

$$\text{rank}\big(\text{Lie}\{Z_1, \dots, Z_m\}\big)(x,t) = N_1 \qquad \forall\ (x,t) \in \Omega.$$

As a consequence of Theorem 4.3 and Proposition 4.5 (and using Chow&Hörmander theorem) we have propagation of maxima along $Z_1, \dots, Z_m$ and positively along Z_0 if, for example, the matrix $A = (a_{i,j}(z))_{i,j\leq m}$ is symmetric, positive definite and continuous in Ω and moreover $Z_1, \dots, Z_m$ satisfy (4.13) and fulfill Hörmander's condition. The operators studied in [4, 5] are contained in this class.

Proof. (of Proposition 4.5.) By Theorem 4.2, it suffices to show that $\langle Z_0(z^0), \nu\rangle \leq 0$ for any $z^0 \in F^*$ and any $\nu \perp F$ ext. at z^0. Let $z^0 = (x^0, t^0)$ and $\nu = (\xi, \tau)$ be as above. By hypothesis (4.14), it must be $\nu = (0, \tau)$. Suppose now by contradiction that $\langle Z_0(z^0), \nu\rangle > 0$. We fix small $\sigma > 0$ and $r > 0$ to be determined in the sequel and we consider the function

$$h(x,t) = \exp(-q(x,t)) - \exp(-r^2|\tau|^2) \quad \text{where} \quad q(x,t) = \sigma^2\,|x - x^0|^2 + |t - (t^0 + r\tau)|^2.$$

We also introduce the ellipsoid $\mathcal{E}$ as the set of (x,t) such that $\sigma^2|x-x^0|^2+|t-(t^0+r\tau)|^2<r^2|\tau|^2$. We choose $r|\tau|/\sigma \ll 1$ and $r|\tau| \ll 1$ so that $\mathcal{E}$ is sufficiently small in order to lie in the neighborhoods V and W for z^0 as in hypotheses (4.2) and (4.14) respectively. Hence, by (3.4) and (3.5), $\mathcal{H}h(z^0)$ equals $2e^{-r^2|\tau|^2}$ times the following factor

$$-\sigma^2\mathrm{trace}(\widetilde{A}(z^0)) + r\textstyle\sum_{i=1}^{N_2}(Z_0I)_{N_1+i}(z^0)\,\tau_i = -\sigma^2\mathrm{trace}(\widetilde{A}(z^0)) + r\big\langle Z_0(z^0),\nu\big\rangle.$$

Here, we have used that $\nu=(0,\tau)$ and the fact that, for every $i=1,\dots,N_2$ (see (3.5))

$$\widetilde{b}_{i+N_1} = \textstyle\sum_{h,k=1}^{m} a_{h,k}\,Z_h(Z_kI)_{i+N_1} + (Z_0I)_{i+N_1} = (Z_0I)_{i+N_1}$$

since Z_k operates only in $x\in\mathbb{R}^{N_1}$. Now we choose r/σ^2 large enough, namely $r/\sigma^2 > \mathrm{trace}(\widetilde{A}(z^0))/\langle Z_0(z^0),\nu\rangle$ (all the conditions on σ and r can be realized by taking $r=\sigma\sqrt{\sigma}$ and $\sigma\ll 1$). With these choices of σ, r, we have $\mathcal{H}h(z^0)>0$ and we infer the existence of a positive $\rho\ll 1$ such that $\mathcal{H}h>0$ in $D(z^0,\rho)$. We explicitly remark that $\overline{\mathcal{E}}\subseteq(\Omega\setminus F)\cup\{z^0\}$. Indeed, suppose by contradiction that $z^0=(x^0,t^0)\neq(\overline{x},\overline{t})\in\overline{\mathcal{E}}\cap F$. Then, by hypothesis (4.14), we have $(x^0,\overline{t})\in F$ which is impossible since $(x^0,\overline{t})\in D(z^0+r\nu,r|\nu|)\subseteq D(z^0+\nu,|\nu|)\subseteq\Omega\setminus F$. Let $U=D(z^0,\rho)\cap\mathcal{E}$ and set $B_1=\overline{D}(z^0,\rho)\cap\partial\mathcal{E}$, $B_2=\overline{\mathcal{E}}\cap\partial D(z^0,\rho)$ so that $\partial U=B_1\cup B_2$. We choose ε such that (4.7) holds. Arguing exactly as in the proof of Theorem 4.3, we prove that $u_\varepsilon=u+\varepsilon\,h\le 0$ in U. Let $\gamma(s)$ be the integral curve of the vector field Z_0 passing through z^0 at $s=0$. We claim $\gamma(s)\in U$ for any small positive s. Indeed, since $D(z^0+r\nu,r|\nu|)\subseteq\mathcal{E}$, it is enough to prove that $\gamma(s)\in D(z^0+r\nu,r|\nu|)$ for any small positive s. This follows by arguing as in (4.11) (recalling that we are assuming $\langle Z_0(z^0),\nu\rangle>0$). Now, for what has been proved above, $u(\gamma(s))+\varepsilon\,h(\gamma(s))\le 0$ for any small positive s. Since $u(z^0)=h(z^0)=0$, we infer $\frac{1}{s}(u(\gamma(s))-u(z^0))\le-\frac{\varepsilon}{s}(h(\gamma(s))-h(z^0))$ for any small $s>0$. We let $s\to 0^+$. Since $u\in\Gamma^2(\Omega)$ and by the definition of γ, the left-hand side goes to $Z_0u(z^0)$ which is zero since z^0 is a maximum point for u. The right-hand side goes to $-\varepsilon\,Z_0h(z^0)=-2r\varepsilon\,\exp(-r^2|\tau|^2)\,\langle Z_0(z^0),\nu\rangle<0$. This gives a contradiction. ■

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