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Critical semilinear equations on the Heisenberg group: the effect of the topology of the domain[☆]

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1. Introduction

Let Δ_H be the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $q=2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. Precisely, if we denote (x,y,t) the elements of $\mathbb{H}^n=\mathbb{R}^{2n+1}$, with $x,y\in\mathbb{R}^n$, $t\in\mathbb{R}$, the operator Δ_H can be represented as a sum of squares of vector fields

$$\Delta_H=\sum_{j=1}^{n}(X_j^2+Y_j^2), \tag{1.1}$$

where $X_j=\partial/\partial x_j+2y_j\partial/\partial t$, $Y_j=\partial/\partial y_j-2x_j\partial/\partial t$.

We are concerned with the critical semilinear boundary value problem

$$\begin{aligned}&-\Delta_H u=u^{(q+2)/(q-2)} \quad \text{in } \Omega,\\ &u>0 \quad \text{in } \Omega,\\ &u=0 \quad \text{in } \partial\Omega,\end{aligned} \tag{1.2}$$

where Ω is a connected bounded domain of $\mathbb{H}^n$ with boundary regular enough. We prove that this problem has a solution if Ω has at least a nontrivial homology group (with $\mathbb{Z}_2$-coefficients) $H_d(\Omega)$ ($d\in\mathbb{N}$). This result, which is the Kohn Laplacian counterpart of a celebrated theorem by Bahri and Coron [2], completes a research started in the papers [16,28,36].

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After the pioneeristic works by Jerison and Lee [24–26] on the Yamabe problem for CR manifolds, several authors have investigated, with different techniques, on semilinear equations for the Kohn Laplacian. Existence and nonexistence results have been established in [22,4,16,9] via variational methods; in [11,31,39] via super and subsolutions methods; in [5–8,14] via blow-up techniques; in [28,29,36–38] via mean value formulas; in [10] via moving planes techniques. We point out that, in spite of the striking similarities of the operator Δ_H with the classical Laplacian, the presence of a degenerate direction gives rise to new difficulties in our context, principally related to the behavior of solutions near the boundary.

Since the exponent $(q+2)/(q-2)$ is the critical Sobolev exponent, the existence of a solution of (1.2) strictly depends on the geometry of Ω and of the operator Δ_H. If Ω is H-starshaped, Garofalo and Lanconelli proved in [22] that such problem has no solutions. In the same paper they gave a first example of solution to (1.2) on a non-contractible domain $\Omega=\{(x,y,t)|r<|x|^2+|y|^2<R,\ 0<t<T\}$, where r,R,T are positive constants. The existence of a solution can be investigated with the concentration compactness principle, exactly as in the critical case of the Laplace operator (see for example [35,30,3], see also [12]). This was first done in [16]: denoted I the functional naturally associated to Eq. (1.2), its nonnegative Palais–Smale sequences can be represented in terms of the solutions to the same problem on different open subsets D of $\mathbb{H}^n$, called sets at infinity (see Definition 3.1). If the structure of these sets is not known, the representation result shows compactness of (PS) sequences only at some low levels of the functional I and does not apply to the “non-perturbed” critical equation on general domains Ω.

Here we give a complete description of such sets D which can be the whole $\mathbb{H}^n$ or an open set of the form

$$D=\{(x,y,t)|a\cdot x+b\cdot y+ct+q(x,y)>0\},$$

with $a,b\in\mathbb{R}^n, c\in\mathbb{R}$ and q a quadratic form on $\mathbb{R}^{2n}$ (see Lemma 3.4). The problem of existence or nonexistence of solutions on these sets is, at the authors' knowledge, still open. On the other hand, if $\partial\Omega$ satisfies a suitable geometrical hypothesis at characteristic points analogous to the differentiability for the Heisenberg group (see Definition 3.3 of H-flat domains), then the sets at infinity D are only half-spaces or the whole $\mathbb{H}^n$. This geometrical hypothesis is analogous to the H-convexity condition introduced in [8] and seems to be the natural regularity to be required for a domain of $\mathbb{H}^n$ since it reflects the geometrical structure of the operator Δ_H. We also refer to [21] for a weak rectifiability condition at non-characteristic points. When D is the whole space, the problem at infinity (3.1) has been studied in [25]. When D is a half-space, the recent results in [28,36] ensure that problem (3.1) has no solutions. Hence, for an H-flat domain Ω we get a complete characterization of the compactness levels of the functional I (Theorem 3.5).

As it is well known this characterization is one of the most important tools in the proof of existence theorems for semilinear equations with critical growth. In the setting of the Laplace operator, Bahri and Coron [2] introduced a powerful topological argument which allows to establish the existence result if the domain Ω has a nontrival homology group. The same technique has been extended to any Riemannian Laplacian by Bahri and Brezis [1] (similar ideas are also developed in [34,15]). In particular they have pointed out that such technique is based on a few properties of the functional I: precisely on the structure of its (PS) sequences, on a parameterization theorem for a neighborhood of the set of critical points at infinity and on an estimate of I on suitable linear combinations of solutions to the problems at infinity.

Once these estimates are also established in our context, the existence theorem follows.

Theorem 1.1. *Let* Ω *be an H-flat connected bounded domain of* $\mathbb{H}^n$. *If there exists a positive integer* d *such that the homology group* (*with* $\mathbb{Z}_2$-*coefficients*) $H_d(\Omega)$ *is nontrivial*, *then problem* (1.2) *has a solution in* Ω.

As in the Euclidean case, the hypothesis on $H_d(\Omega)$ is only a sufficient condition for the existence of a solution; indeed we give here an example of a contractible domain Ω such that (1.2) has a solution (Theorem 3.9).

The paper is organized as follows. In Section 2 we fix the notation. In Section 3 we study the Palais–Smale sequences of the functional I. Sections 4 and 5 are devoted to the parameterization of the set of critical points at infinity and to the required estimates of the functional I, respectively. In Section 6 we conclude our main result.

2. Notation

The Heisenberg group $\mathbb{H}^n$ is the homogeneous Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ and whose group law is defined by

$$\tau_{\xi'}(\xi) = \xi'\xi = (x + x', y + y', t + t' + 2(x \cdot y' - x' \cdot y)),$$

for every $\xi = (x, y, t)$, $\xi' = (x', y', t') \in \mathbb{H}^n$ (here $\cdot$ denotes the inner product in $\mathbb{R}^n$).

The H-dilations are given by

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$$

for $\lambda > 0$. The Jacobian determinant of δ_λ is λ^{2n+2}, so that the homogeneous dimension of the group becomes

$$q = 2n + 2. \tag{2.1}$$

The homogeneous norm of the space is

$$d_0(x, y, t) = ((|x|^2 + |y|^2)^2 + t^2)^{1/4}$$

and the natural distance is accordingly defined

$$d(\xi, \xi_0) = d_0(\xi_0^{-1}\xi).$$

We shall denote by $B_d(\xi, r)$ the d-ball of center ξ and radius r. By the definition of d we have

$$\tau_\xi(B_d(0, r)) = B_d(\xi, r), \qquad \delta_r(B_d(0, 1)) = B_d(0, r).$$

Hence, if $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$, from (2.1) we deduce

$$|B_d(\xi,r)| = r^q|B_d(0,1)|.$$

As a consequence, for every $0 \le a < b$ and for every measurable function $f:[a,b] \to \mathbb{R}$, we have

$$\int_{B_d(0,b)\setminus B_d(0,a)} f(d_0(\xi))\,\mathrm{d}\xi = q|B_d(0,1)| \int_a^b f(r)r^{q-1}\,\mathrm{d}r$$

if at least one of the integrals exists.

The Kohn Laplacian on $\mathbb{H}^n$ is the second-order operator Δ_H defined in (1.1), which is invariant with respect to the left translations τ_ξ of $\mathbb{H}^n$ and homogeneous of degree two with respect to the dilations δ_λ. We call subelliptic gradient

$$D_H = (X,Y) = (X_1,\dots,X_n,Y_1,\dots,Y_n).$$

A remarkable property of the Kohn Laplacian is that a fundamental solution of $-\Delta_H$ with pole at zero is given by

$$\Gamma(\xi) = \frac{c_q}{d_0(\xi)^{q-2}}, \tag{2.2}$$

where c_q is a suitable positive constant (see [19]).

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality

$$\|\varphi\|_{q^*}^2 \le c\|D_H\varphi\|_2^2 \ \forall\varphi \in C_0^\infty(\mathbb{H}^n),$$

where

$$q^* := \frac{2q}{q-2} \tag{2.3}$$

and c is a positive constant. This inequality ensures in particular that for every domain Ω the function

$$\|\varphi\| = \|D_H\varphi\|_2$$

is a norm on $C_0^\infty(\Omega)$. We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to this norm; $S_0^1(\Omega)$ becomes a Hilbert space with the inner product

$$\langle u,v\rangle_{S_0^1} = \int_\Omega \langle D_Hu, D_Hv\rangle.$$

Thus there exists a natural orthogonal projection

$$P: S_0^1(\mathbb{H}^n) \to S_0^1(\Omega). \tag{2.4}$$

The exponent q^* in (2.3) is the critical Sobolev exponent for Δ_H since the embedding $S_0^1(\Omega) \hookrightarrow L^{q^*}(\Omega)$ is continuous but not compact even if Ω is bounded.

A variational solution of (1.2) can be found as a critical point of the functional

$$I: S_0^1(\Omega) \to \mathbb{R}, \qquad I(u) = \frac{1}{2}\int_\Omega |D_Hu|^2 - \frac{q-2}{2q}\int_\Omega |u|^{2q/(q-2)}. \tag{2.5}$$

Moreover, every variational solution is also a classical solution (see [20,22]). Here we look for solutions of (1.2) as critical points of I constrained on the manifold

$$M = \{u \in S_0^1(\Omega)\backslash\{0\} \mid dI(u)(u) = 0\}$$

(more precisely on $M^+ = \{u \in M \mid u \geq 0\}$).

Remark 2.1. If $u \in S_0^1(\Omega)\backslash\{0\}$ then u is a critical point of I iff u is a critical point of $I_{|_M}$. Moreover for every $u \neq 0$, if we set

$$\theta(u) = \left(\frac{\|u\|^2}{\|u\|_{q^*}^{q^*}}\right)^{1/(q^*-2)}$$

then $\theta(u)u \in M$ and we have

$$I(\theta(u)u) = \frac{1}{q}\left(\frac{\|u\|}{\|u\|_{q^*}}\right)^q. \tag{2.6}$$

3. Structure of (PS) sequences, H-flat domains

In this section we introduce the definition of H-flat domains and we study the behavior of the (PS) sequences of the functional $I_{|_M}$.

It is well known that the Palais–Smale condition is not satisfied by the functional I and that the loss of compactness is in general due to the solutions of the so-called problems at infinity.

Definition 3.1. If Ω is a smooth bounded domain of $\mathbb{H}^n$, we call set at infinity of problem (1.2) any open set D obtained as limit of a subsequence of the following sequence of sets

$$\Omega_k = \delta_{\lambda_k}(\tau_{\xi_k}^{-1}(\Omega)),$$

given any sequence (ξ_k) in Ω and any divergent sequence (λ_k) in R^+ (the structure of these sets at infinity will be studied in Lemma 3.4). We call problem at infinity related to (1.2) any problem

$$-\Delta_H \omega = \omega^{(q+2)/(q-2)}, \qquad \omega > 0, \qquad \omega \in S_0^1(D), \tag{3.1}$$

being D a set at infinity of (1.2). Moreover, we will denote by I_∞ the functional $I : S_0^1(D) \to \mathbb{R}$ whose critical points are the solutions of (3.1).

When $D = \mathbb{H}^n$, a solution to (3.1) is the following C^∞ function:

$$\omega_0(x, y, t) = \frac{C_0}{(t^2 + (1 + |x|^2 + |y|^2)^2)^{(q-2)/4}}, \tag{3.2}$$

where C_0 is a suitable positive constant. Moreover, *every* solution to (3.1) with $D = \mathbb{H}^n$ takes the form

$$\omega_{\lambda,\xi} = \lambda^{(q-2)/2}\omega_0 \circ \delta_\lambda \circ \tau_{\xi^{-1}} \tag{3.3}$$

for some $\lambda > 0$ and $\xi \in \mathbb{H}^n$ (this deep result is due to Jerison and Lee [25]).

Let us recall the representation theorem for Palais–Smale sequences proved in [16]:

Theorem C. *Let* Ω *be a smooth bounded domain of* $\mathbb{H}^n$ *and let* (u_k) *be a nonnegative sequence in* $S_0^1(\Omega)$ *such that*

$$I(u_k) \to \ell > 0 \quad \textit{and} \quad dI(u_k) \to 0 \qquad \textit{as } k \to +\infty.$$

Then there exist $u_0 \in S_0^1(\Omega)$, *m solutions* $\omega^i, \dots, \omega^m$ *of the problems at infinity* (3.1), *m sequences* $(\xi_{1k}), \dots, (\xi_{mk})$ *in* Ω *and m divergent sequences* $(\lambda_{1k}), \dots, (\lambda_{mk})$ *in* $\mathbb{R}^+$ $(m \in \mathbb{N} \cup \{0\})$ *such that*

(i) $u_k = u_0 + \sum_{i=1}^m P\omega^i_{\lambda_{ik}\xi_{ik}} + o(1)$ *as* $k \to +\infty$, *in* $S_0^1(\Omega)$,

where u_0 *is a solution of* $dI(u_0) = 0$, $\omega^i_{\lambda_{ik}\xi_{ik}}$ *are obtained by translating and dilating* ω^i *accordingly to* (3.3) *and P is defined in* (2.4).

Moreover, if we denote by $\omega^1, \dots, \omega^{m_1}$ *the solutions on the whole* $\mathbb{H}^n$ *and by* $\omega^{m_1+1}, \dots, \omega^m$ *the solutions on the other sets at infinity, we also have*

(ii) $I(u_k) = I(u_0) + m_1 I_\infty(\omega_0) + \sum_{i=m_1+1}^m I_\infty(\omega^i) + o(1)$, *as* $k \to +\infty$,

(iii) $\lambda_{ik} d(\xi_{ik}, \partial\Omega) \to +\infty$ *as* $k \to +\infty$, $\forall i = 1, \dots, m_1$,

(iv) $\lambda_{ik} d(\xi_{ik}, \partial\Omega)$ *has a finite limit as* $k \to +\infty$, $\forall i = m_1 + 1, \dots, m$.

The description of the sets at infinity D was not known. Actually it requires a careful study of the boundary of Ω. We first introduce some further notation and then give such description in Lemma 3.4.

Given a smooth function $\varphi : \mathbb{H}^n \to \mathbb{R}$ and a point $\xi_0 \in \mathbb{H}^n$, we denote by $q_H\varphi(\xi_0)$ the quadratic form associated to the Hessian matrix $D_H^2\varphi$ along the vector fields of the subelliptic gradient D_H, i.e.

$$(q_H\varphi(\xi_0))(z) = \sum_{i,j=1}^{2n} ((D_H)_i(D_H)_j\varphi)(\xi_0) z_i z_j \qquad \forall z \in \mathbb{R}^{2n},$$

where $(D_H)_i$ denotes the ith component of D_H. Let us note explicitly that $D_H^2\varphi(\xi_0)$ is not symmetrical in general, since $[X_j, Y_j] = -4\partial_t$.

Remark 3.2. A smooth function φ has the following Taylor expansion of first order with initial point ξ_0 in the direction of the vector fields of D_H:

$$\varphi(\xi) = \varphi(\xi_0) + u \cdot X\varphi(\xi_0) + v \cdot Y\varphi(\xi_0) + o(d(\xi, \xi_0)) \qquad \text{as } \xi \to \xi_0, \tag{3.4}$$

where (u, v, w) are the components of $\tau_{\xi_0}^{-1}(\xi)$ and $\cdot$ denotes the scalar product in $\mathbb{R}^n$. The second-order expansion is the following:

$$\begin{aligned} \varphi(\xi) = {} & \varphi(\xi_0) + u \cdot X\varphi(\xi_0) + v \cdot Y\varphi(\xi_0) + w\partial_t\varphi(\xi_0) \\ & + \tfrac{1}{2} q_H\varphi(\xi_0)(u, v) + o(d^2(\xi, \xi_0)). \end{aligned} \tag{3.5}$$

We explicitly remark that the limits in (3.4) and (3.5) are locally uniform in the variable ξ_0.

Definition 3.3. Let Ω be a smooth bounded domain of $\mathbb{H}^n$, let $\xi_0 \in \partial\Omega$ and let φ be a smooth function which describes the boundary of Ω in a neighborhood of ξ_0

(i.e. $\varphi : B_d(\xi_0,R) \to \mathbb{R}$, $\varphi(\xi) = 0$ iff $\xi \in \partial\Omega \cap B_d(\xi_0,R)$, $\varphi(\xi) > 0$ iff $\xi \in \Omega \cap B_d(\xi_0,R)$ and the Euclidean gradient of φ is always different from 0). The point ξ_0 is called characteristic if $D_H\varphi(\xi_0) = 0$. If ξ_0 is characteristic, we shall say that Ω is H-flat at ξ_0 if

$$q_H\varphi(\xi_0) = 0. \tag{3.6}$$

We shall call Ω H-flat if it is H-flat at any characteristic point of its boundary.

According to (3.4) and (3.5), the condition $q_H\varphi(\xi_0) = 0$ implies that the nontrivial Taylor polynomial of least order is a plane. Hence the boundary of an H-flat domain Ω has at every point a tangent plane, in the sense of the Heisenberg distance. This condition seems to be natural, since it is satisfied by the balls of the metric.

Lemma 3.4. *Let Ω be a smooth bounded domain of $\mathbb{H}^n$, let (ξ_k) be a sequence in Ω and let (λ_k) be a divergent sequence in $\mathbb{R}^+$. We set*

$$\Omega_k = \delta_{\lambda_k}(\tau_{\xi_k}^{-1}(\Omega)).$$

Then (taking a subsequence if necessary) $\Omega_k \to D$ as $k \to \infty$, where

(i) *if $\lambda_k d(\xi_k, \partial\Omega)$ is unbounded, then D is the whole space $\mathbb{H}^n$;*

(ii) *if $\lambda_k d(\xi_k, \partial\Omega)$ is bounded, then there exist a, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and a quadratic form q on $\mathbb{R}^{2n}$ such that $D = \{(x,y,t) | a\cdot x + b\cdot y + ct + q(x,y) > 0\}$ (up to a left translation);*

(iii) *if $\lambda_k d(\xi_k, \partial\Omega)$ is bounded and Ω is H-flat, then D is a half-space of $\mathbb{H}^n$.*

Proof. (i) We may assume $\lambda_k d(\xi_k, \partial\Omega) \to +\infty$. Since

$$B_d(0, \lambda_k d(\xi_k, \partial\Omega)) = (\delta_{\lambda_k} \circ \tau_{\xi_k}^{-1})(B_d(\xi_k, d(\xi_k, \partial\Omega))) \subseteq \Omega_k$$

we get $\Omega_k \to \mathbb{H}^n$ as $k \to \infty$.

(ii) and (iii) Since $\lambda_k d(\xi_k, \partial\Omega)$ is bounded, $\lambda_k \to +\infty$ and Ω is bounded, then the sequence ξ_k tends to a limit $\xi_0 \in \partial\Omega$. Let φ be a function that defines the boundary of Ω in a neighborhood $B_d(\xi_0, r_0)$ of ξ_0.

We now fix $\zeta = (x_\zeta, y_\zeta, t_\zeta) \in \mathbb{H}^n$. By definition

$$\zeta \in \Omega_k \Leftrightarrow \zeta_k \in \Omega,$$

where $\zeta_k := \xi_k(\delta_{\lambda_k^{-1}}\zeta)$. Note that we can assume $\zeta_k \in B_d(\xi_0, r_0)$ for every k sufficiently large. Indeed

$$\begin{aligned} d(\xi_0, \zeta_k) &\le d(\xi_0, \xi_k) + d(\xi_k, \xi_k(\delta_{\lambda_k^{-1}}\zeta)) \\ &= d(\xi_0, \xi_k) + d(0, \delta_{\lambda_k^{-1}}\zeta) = d(\xi_0, \xi_k) + \lambda_k^{-1} d(0, \zeta) \end{aligned}$$

and this last term tends to 0 as $k \to \infty$. Then by the choice of φ,

$$\zeta \in \Omega_k \Leftrightarrow \varphi(\zeta_k) > 0.$$

Let now $\bar{\xi}_k$ be a point on $\partial\Omega$ such that $d(\xi_k, \partial\Omega) = d(\xi_k, \bar{\xi}_k)$. If we consider the Taylor expansion of φ with initial point $\bar{\xi}_k$ stated in (3.4) we get

$$\varphi(\zeta_k) = u \cdot X\varphi(\bar{\xi}_k) + v \cdot Y\varphi(\bar{\xi}_k) + o(d(\zeta_k, \bar{\xi}_k)) \quad \text{as } k \to \infty,$$

where (u,v,w) are the components of $\tau_{\bar{\xi}_k}^{-1}(\zeta_k) = \bar{\xi}_k^{-1}\xi_k(\delta_{\lambda_k^{-1}}\zeta)$, given by

$$u = x_k - \bar{x}_k + \lambda_k^{-1}x_\zeta, \quad v = y_k - \bar{y}_k + \lambda_k^{-1}y_\zeta,$$

$$w = t_k - \bar{t}_k - 2(x_k\cdot\bar{y}_k - \bar{x}_k\cdot y_k) + 2\lambda_k^{-1}(x_\zeta\cdot(y_k-\bar{y}_k) - y_\zeta\cdot(x_k-\bar{x}_k)) + \lambda_k^{-2}t_\zeta.$$

Hence (for any large k)

$$\begin{aligned}\zeta\in\Omega_k &\Leftrightarrow \varphi(\zeta_k) > 0\\ &\Leftrightarrow 0 < \lambda_k\varphi(\zeta_k) = (\lambda_k(x_k-\bar{x}_k) + x_\zeta)\cdot X\varphi(\bar{\xi}_k)\\ &\qquad + (\lambda_k(y_k-\bar{y}_k) + y_\zeta)\cdot Y\varphi(\bar{\xi}_k) + o(1) \quad \text{as } k\to\infty.\end{aligned}$$

We first assume that ξ_0 is not characteristic. Moreover, since $\lambda_k d(\xi_k,\bar{\xi}_k)$ is bounded we can assume that $\lambda_k(x_k-\bar{x}_k)$ and $\lambda_k(y_k-\bar{y}_k)$ have limits c_1 and c_2, respectively. Hence

$$\zeta\in\Omega_k \Leftrightarrow 0 < (c_1+x_\zeta)\cdot X\varphi(\xi_0) + (c_2+y_\zeta)\cdot Y\varphi(\xi_0) + o(1) \quad \text{as } k\to\infty.$$

This proves that Ω_k tends to the half-space

$$D = \{(x,y,t) | (c_1+x)\cdot X\varphi(\xi_0) + (c_2+y)\cdot Y\varphi(\xi_0) > 0\}.$$

We now assume that ξ_0 is characteristic. In this case we have to use the second-order Taylor expansion recalled in (3.5):

$$\begin{aligned}\varphi(\zeta_k) &= ((x_k-\bar{x}_k) + \lambda_k^{-1}x_\zeta)\cdot X\varphi(\bar{\xi}_k) + ((y_k-\bar{y}_k) + \lambda_k^{-1}y_\zeta)\cdot Y\varphi(\bar{\xi}_k)\\ &\quad + (t_k - \bar{t}_k - 2(x_k\cdot\bar{y}_k - \bar{x}_k\cdot y_k) + 2\lambda_k^{-1}(x_\zeta\cdot(y_k-\bar{y}_k)\\ &\quad - y_\zeta\cdot(x_k-\bar{x}_k)) + \lambda_k^{-2}t_\zeta)\partial_t\varphi(\bar{\xi}_k) + \tfrac{1}{2}q_H\varphi(\bar{\xi}_k)(x_k-\bar{x}_k\\ &\quad + \lambda_k^{-1}x_\zeta, y_k-\bar{y}_k + \lambda_k^{-1}y_\zeta) + o(d(\zeta_k,\bar{\xi}_k)^2) \quad \text{as } k\to\infty.\end{aligned}$$

We get (for any large k)

$$\begin{aligned}\zeta\in\Omega_k &\Leftrightarrow \varphi(\zeta_k) > 0 \Leftrightarrow 0 < \lambda_k^2\varphi(\zeta_k)\\ &= (\lambda_k(x_k-\bar{x}_k) + x_\zeta)\cdot\lambda_k X\varphi(\bar{\xi}_k) + (\lambda_k(y_k-\bar{y}_k) + y_\zeta)\cdot\lambda_k Y\varphi(\bar{\xi}_k)\\ &\quad + (\lambda_k^2(t_k-\bar{t}_k - 2(x_k\cdot\bar{y}_k - \bar{x}_k\cdot y_k)) + 2(x_\zeta\cdot\lambda_k(y_k-\bar{y}_k)\\ &\quad - y_\zeta\cdot\lambda_k(x_k-\bar{x}_k)) + t_\zeta)\partial_t\varphi(\bar{\xi}_k) + \tfrac{1}{2}q_H\varphi(\bar{\xi}_k)(\lambda_k(x_k-\bar{x}_k)\\ &\quad + x_\zeta, \lambda_k(y_k-\bar{y}_k) + y_\zeta) + o(1) \quad \text{as } k\to\infty.\end{aligned}$$

As before, we can assume that the sequences $\lambda_k(x_k-\bar{x}_k)$, $\lambda_k(y_k-\bar{y}_k)$ and $\lambda_k^2(t_k - \bar{t}_k - 2(x_k\cdot\bar{y}_k - \bar{x}_k\cdot y_k))$ converge to c_1, c_2 and c_3, respectively. If $\varrho_k := |\lambda_k D_H\varphi(\bar{\xi}_k)|$ is unbounded, we can also assume that $\varrho_k\to+\infty$, $\varrho_k^{-1}\lambda_k D_H\varphi(\bar{\xi}_k)\to(\alpha,\beta)$ and we get (dividing all by ϱ_k)

$$\zeta\in\Omega_k \Leftrightarrow 0 < (c_1+x_\zeta)\cdot\alpha + (c_2+y_\zeta)\cdot\beta + o(1) \quad \text{as } k\to\infty.$$

Hence, in this case, Ω_k tends to a half-space. If ϱ_k is bounded, we can assume that $\lambda_k D_H\varphi(\bar{\xi}_k)$ has limit (α,β) and get that Ω_k tends to the set

$$\begin{aligned} D=&\{(x_\zeta,y_\zeta,t_\zeta)\colon (c_1+x_\zeta)\cdot\alpha+(c_2+y_\zeta)\cdot\beta \\ &+(c_3+2(x_\zeta\cdot c_2-y_\zeta\cdot c_1)+t_\zeta)\partial_t\varphi(\xi_0)+\tfrac{1}{2}q_H\varphi(\xi_0)(c_1+x_\zeta,c_2+y_\zeta)>0\} \\ =&\,\tau_c^{-1}(\{(x,y,t)\colon \alpha\cdot x+\beta\cdot y+t\partial_t\varphi(\xi_0)+\tfrac{1}{2}q_H\varphi(\xi_0)(x,y)>0\}). \end{aligned}$$

Moreover if Ω is H-flat, then $q_H\varphi(\xi_0)=0$ and D is again a half-space. □

The recent nonexistence results on half-spaces contained in [28] and [36] ensure that (3.1) has no solutions on a half-space. Then the representation theorem can be improved as follows:

Theorem 3.5. *Let Ω be an H-flat domain of $\mathbb{H}^n$ and let (u_k) be a nonnegative sequence in $S_0^1(\Omega)$ such that*

$$I(u_k)\to\ell>0 \quad \text{and} \quad dI(u_k)\to 0 \quad \text{as } k\to+\infty.$$

Then there exist $u_0\in S_0^1(\Omega)$, m sequences $(\xi_{1k}),\dots,(\xi_{mk})$ in Ω and m divergent sequences $(\lambda_{1k}),\dots,(\lambda_{mk})$ in $\mathbb{R}^+ (m\in\mathbb{N}\cup\{0\})$ such that

(i) *$u_k=u_0+\sum_{i=1}^m P\omega_{\lambda_{ik}\xi_{ik}}+o(1)$ as $k\to+\infty$, in $S_0^1(\Omega)$,*
 where u_0 is a solution of $dI(u_0)=0$ and $\omega_{\lambda_{ik}\xi_{ik}}$ are the well-known functions defined by (3.2) *and* (3.3),

(ii) *$I(u_k)=I(u_0)+mI_\infty(\omega_0)+o(1)$, as $k\to+\infty$,*

(iii) *$\lambda_{ik}d(\xi_{ik},\partial\Omega)\to+\infty$, as $k\to+\infty$,*

(iv) *$\frac{\lambda_{ik}}{\lambda_{jk}}+\frac{\lambda_{jk}}{\lambda_{ik}}+\lambda_{ik}\lambda_{jk}d^2(\xi_{ik},\xi_{jk})\to+\infty$, as $k\to+\infty$, $\forall i\neq j$.*

Proof. Since Ω is H-flat, the sets at infinity can be only half-spaces or the whole $\mathbb{H}^n$ (see Lemma 3.4). On the other hand the problem at infinity (3.1) has no solutions when D is a half-space by means of [28,36]. Hence (i)–(iii) follow directly from Theorem C. Assertion (iv) can be proved exactly as the analogous assertion in [12]. □

Proposition 3.6. *An analogous result holds for the constrained functional $I_{|_M}$. Indeed the following relations between Palais–Smale sequences of I and $I_{|_M}$ hold*:

(i) *If (u_k) is a sequence in $S_0^1(\Omega)$ such that $I(u_k)\to\lambda$ and $dI(u_k)\to 0$, then $\theta(u_k)\to 1$, $I(\theta(u_k)u_k)\to\lambda$ and $dI_{|_M}(\theta(u_k)u_k)\to 0$ as $k\to\infty$.*

(ii) If *(u_k) is a sequence in M such that $I(u_k)\to\lambda$ and $dI_{|_M}(u_k)\to 0$, then $dI(u_k)\to 0$ as $k\to\infty$.*

As a consequence we can give a complete characterization of the compactness levels of (u_k), exactly as in the Euclidean case.

Remark 3.7. In the hypothesis of Theorem 3.5, if $\ell\in\mathbb{R}^+\setminus\{mI_\infty(\omega_0)\}_{m\in\mathbb{N}}$, then there exists $u_0\in S_0^1(\Omega)\setminus\{0\}$ such that $dI(u_0)=0$, i.e. u_0 is a variational solution to the equation in (1.2). Moreover, if $\ell\in\,]I_\infty(\omega_0),2I_\infty(\omega_0)[$, then $u_k\to u_0$ strongly in $S_0^1(\Omega)$.

We remark that in Theorem 3.5, ℓ cannot belong to $]0, I_\infty(\omega_0)[$ since $dI(u_0) = 0$ and $u_0 \neq 0$ imply $I(u_0) > I_\infty(\omega_0)$.

We now want to make a few comments on the notion of H-flat domains. Roughly speaking, they are C^2-domains which are enough flat at characteristic points. Let us start with a property of the characteristic points:

Remark 3.8. If ξ_0 is a characteristic point of $\partial\Omega$ then the straight line through ξ_0 orthogonal and incident to the t-axis is tangent to $\partial\Omega$ at ξ_0. Indeed if $\xi_0 = (z_0, t_0) = (x_0, y_0, t_0)$ is characteristic and $\varphi = 0$ is a local equation of $\partial\Omega$ at ξ_0 then $D_H\varphi(\xi_0) = 0$ implies

$$\begin{aligned} 0 &= \langle D_H\varphi(\xi_0), z_0\rangle \\ &= \sum_{j=1}^{n} (x_{0,j}(\partial_{x_j}\varphi(\xi_0) + 2y_{0,j}\partial_t\varphi(\xi_0)) + y_{0,j}(\partial_{y_j}\varphi(\xi_0) - 2x_{0,j}\partial_t\varphi(\xi_0))) \\ &= \langle \nabla_z\varphi(\xi_0), z_0\rangle, \end{aligned}$$

i.e. $(z_0, 0) \perp \nabla\varphi(\xi_0)$ where $\nabla\varphi(\xi_0)$ generates the orthogonal space to $\partial\Omega$ at ξ_0. Here we have denoted by $\nabla\varphi(\xi_0) = (\nabla_z\varphi(\xi_0), \partial_t\varphi(\xi_0))$ the Euclidean gradient of φ.

This simple condition immediately provides some examples: the domain $\tilde{A}$ obtained by rounding off the edges of the annular region $A = \{(z,t) = (x,y,t) \in \mathbb{H}^n \mid 1 < |z| < 2, |t| < 1\}$, has no characteristic points. Hence it is H-flat. The d-balls $B_d(\xi, r)$ are remarkable examples of H-flat domains which exhibit characteristic points.

As an application of Theorem 3.5 we can prove the following result.

Theorem 3.9. *There exists a contractible bounded domain Ω of $\mathbb{H}^n$ on which problem* (1.2) *admits a solution.*

Proof. In the papers [17,18,32,33] analogous results are obtained, with different proofs, in the setting of the Laplace operator. Not all these proofs can be extended directly to our context. For example in adapting the proof of [18] a delicate question arises when we try to modify the domain near the characteristic points in order to make it H-flat and keep the required symmetry properties.

On the other hand, the proof in [33], for its generality, can be extended to our setting. It is based on a deformation argument and on the notion of Newtonian capacity. The topological argument can be repeated with no difficulties for the Heisenberg operator; the notion of capacity has been extended to the H-capacity in our context (see for example [27,23,13] for its properties). Then, by means of Theorem 3.5 and the arguments of [33], problem (1.2) has a solution on any contractible *H-flat* domain Ω obtained by removing a small H-capacity set from a non-contractible domain.

A simple example is given below. Let us set $\Omega_\varepsilon = (B_d(0,1) \backslash B_d(0, \frac{1}{2})) \backslash \{(z,t) = (x,y,t) \in \mathbb{H}^n | t \geq 0, |z| < \varepsilon\}$ for a small $\varepsilon > 0$ and define Ω by smoothing Ω_ε and keeping the symmetry in $|z|$. In light of Remark 3.8, it is immediate to recognize that Ω has exactly two characteristic points in $(0, -\frac{1}{4})$ and $(0, -1)$ and it is H-flat. $\square$

4. Parameterization of the set of quasi-critical points

In this section we give a parameterization of the set $V(m,\varepsilon)$ which contains the Palais Smale sequences of the functional I. By the representation given in Theorem 3.5, it is natural to introduce (for every $m \in \mathbb{N}$) the function

$$\begin{aligned} &\varrho = \varrho_m : (0,+\infty)^m \times \Omega^m \times (0,+\infty)^m \to S_0^1(\Omega), \\ &\varrho(\alpha,\xi,\lambda) = \sum_{i=1}^m \alpha_i P\omega_{\lambda_i,\xi_i}, \end{aligned} \tag{4.1}$$

where ω_{λ_i,ξ_i} are defined by (3.2) and (3.3) and P by (2.4). We also set, for every $\varepsilon > 0$,

$$\begin{aligned} B_\varepsilon = \Bigg\{ &(\alpha,\xi,\lambda) \mid \lambda_i d(\xi_i,\partial\Omega) > \varepsilon^{-1}, \frac{1}{2} < \alpha_i < 2 \ \forall i, \\ &\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j d^2(\xi_i,\xi_j) > \varepsilon^{-1} \ \forall i \neq j \Bigg\} \end{aligned} \tag{4.2}$$

and

$$V(m,\varepsilon) = \{u \in M^+ \colon \exists (e,\xi,\lambda) \in B_\varepsilon \text{ such that } \|u - \varrho(e,\xi,\lambda)\| \leq \varepsilon\}, \tag{4.3}$$

where M^+ is defined in Remark 2.1 and $e = (1,\dots,1) \in (0,+\infty)^m$.

Lemma 4.1. *For every $m \in \mathbb{N}$ there exists $\varepsilon_0 > 0$ such that for every $u \in V(m,\varepsilon_0)$ the problem*

$$\min\{\|u - \varrho(\alpha,\xi,\lambda)\| \colon (\alpha,\xi,\lambda) \in B_{4\varepsilon_0}\}$$

has a unique solution (up to permutations) and defines a continuous function

$$\Xi : V(m,\varepsilon_0) \to \Omega^m/\sigma_m,$$

where Ω^m/σ_m is the quotient of Ω^m with respect to the group of permutations of $\{1,\dots,m\}$.

Proof. The proof follows the lines of the analogous one in [2]. In particular the variational argument is the same and we omit it, but we focus on a technical estimate, which depends on the geometrical properties of the operator and requires some care in our setting.

The existence of a minimum can be proved as in [2]. In order to prove that it is unique we assume by contradiction that there exists a sequence (ε_k) in $\mathbb{R}^+$ such that $\varepsilon_k \to 0$ as $k \to +\infty$, and for every $k \in \mathbb{N}$ there exists $u_k \in V(m,\varepsilon_k)$ such that the function $(\alpha,\xi,\lambda) \mapsto \|u_k - \varrho(\alpha,\xi,\lambda)\|$ attains its minimum at two distinct points $(\alpha_k,\xi_k,\lambda_k)$ and $(\tilde\alpha_k,\tilde\xi_k,\tilde\lambda_k)$. One can prove that (up to permutations) $b_{ki} := \tilde\lambda_{ki}/\lambda_{ki} - 1 = o(1)$, $\eta_{ki} := \delta_{\tilde\lambda_{ki}}(\tilde\xi_{ki}^{-1}\xi_{ki}) = o(1)$, $\mu_{ki} := \alpha_{ki} - \tilde\alpha_{ki} = o(1)$, as $k \to +\infty$. For simplicity of notation, in the sequel we will omit the index k and we will denote by $o(1)$ a sequence which goes to zero as $k \to \infty$.

Arguing as in [2, (A26)], we get that

$$\mu_i = o(1)\sum_{j=1}^{m}(|b_j| + |\eta_j| + |\mu_j|) - C\alpha_i \int \omega_i^{(q+2)/(q-2)}(\omega_i - \tilde{\omega}_i), \tag{4.4}$$

where we have set $\omega_i = \omega_{\lambda_i,\xi_i}$, $\tilde{\omega}_i = \omega_{\tilde{\lambda}_i,\tilde{\xi}_i}$. Let us define $\tau_i = \int_{\mathbb{H}^n} \omega_i^{(q+2)/(q-2)}(\omega_i - \tilde{\omega}_i)$. Using the definition of ω_i, we get

$$\tau_i = \int_{\mathbb{H}^n} \lambda_i^{(q+2)/2}\omega_0^{(q+2)/(q-2)}(\delta_{\lambda_i}(\xi_i^{-1}\xi))$$
$$\times(\lambda_i^{(q-2)/2}\omega_0(\delta_{\lambda_i}(\xi_i^{-1}\xi)) - \tilde{\lambda}_i^{(q-2)/2}\omega_0(\delta_{\tilde{\lambda}_i}(\tilde{\xi}_i^{-1}\xi)))\,\mathrm{d}\xi$$

(if we set $\eta = \delta_{\lambda_i}(\xi_i^{-1}\xi)$)

$$= \int_{\mathbb{H}^n} \omega_0^{(q+2)/(q-2)}(\eta)(\omega_0(\eta) - (1+b_i)^{(q-2)/2}\omega_0(\eta_i\delta_{(1+b_i)}\eta))\,\mathrm{d}\eta.$$

Now, we take the Taylor expansion (in the usual sense) of order one of the function

$$\bar{\omega}(b_i,\eta_i)(\eta) = \omega_0(\eta) - (1+b_i)^{(q-2)/2}\omega_0(\eta_i\delta_{(1+b_i)}\eta)$$

with respect to the variables b_i and η_i in a neighborhood of 0. If we call $\eta_i = (x_i, y_i, t_i)$ and $\eta = (x, y, t)$, it is easy to prove that $\mathrm{d}\bar{\omega}/\mathrm{d}t_i(0,0)(\eta)$ is odd as a function of t, while ω_0 is even with respect to the same variable. Hence

$$\int_{\mathbb{H}^n} \omega_0^{(q+2)/(q-2)}(\eta)t_i\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t_i}(0,0)(\eta)\,\mathrm{d}\eta = 0.$$

Arguing in the same way with all the components of η_i, we get

$$\tau_i = \int_{\mathbb{H}^n} \omega_0^{(q+2)/(q-2)}(\eta)b_i\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}b_i}(0,0)(\eta)\,\mathrm{d}\eta + O(|b_i| + |\eta_i|)^2.$$

Now setting

$$Z = \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right) + 2t\frac{\partial}{\partial t} \tag{4.5}$$

we have

$$\tau_i = -b_i\int_{\mathbb{H}^n}(\omega_0^{(q+2)/(q-2)}Z\omega_0(\eta) + \frac{q-2}{2}\omega_0^{2q/(q-2)}(\eta))\,\mathrm{d}\eta + O(|b_i| + |\eta_i|)^2.$$

Let us estimate

$$I_R := \int_{B_d(0,R)}(\omega_0^{(q+2)/(q-2)}Z\omega_0(\eta) + \frac{q-2}{2}\omega_0^{2q/(q-2)}(\eta))\,\mathrm{d}\eta$$
$$= \frac{q-2}{2q}\int_{B_d(0,R)}(Z\omega_0^{2q/(q-2)}(\eta) + q\omega_0^{2q/(q-2)}(\eta))\,\mathrm{d}\eta$$

(if we denote by $\nu = \nabla d_0/|\nabla d_0|$ the outer unit normal to $B_d(0,R)$, by $\mathrm{d}H_{q-2}$ the Hausdorff $(q-2)$-dimensional measure and we notice that $q = \operatorname{div} Z$)

$$= \frac{q-2}{2q} \int_{\partial B_d(0,R)} \omega_0^{2q/(q-2)} Z \cdot \nu \, \mathrm{d}H_{q-2}$$

(since $Z \cdot \nu = Zd_0/|\nabla d_0| = d_0/|\nabla d_0|$ and setting for brevity

$g = ((q-2)/2q)\omega_0^{2q/(q-2)} (\in L^1))$

$$= R \int_{\partial B_d(0,R)} \frac{g}{|\nabla d_0|} \, \mathrm{d}H_{q-2}.$$

By Federer's coarea formula, since $g \in L^1(\mathbb{H}^n)$, we have

$$\int_0^{+\infty} \left(\int_{\partial B_d(0,R)} \frac{g}{|\nabla d_0|} \, \mathrm{d}H_{q-2} \right) \mathrm{d}R = \int_{\mathbb{H}^n} g < +\infty.$$

Hence, there exists a divergent sequence of radii $(R_k)_{k \in \mathbb{N}}$ such that

$$I_{R_k} \to 0 \qquad \text{as } k \to \infty.$$

This proves that

$$\tau_i = O(|b_i| + |\eta_i|)^2.$$

Recalling (4.4), we obtain

$$\mu_i = o(1) \sum_{j=1}^{m} (|b_j| + |\eta_j| + |\mu_j|).$$

With essentially the same arguments, it is possible to show that

$$|\eta_i| + |b_i| = o(1) \sum_{j=1}^{m} (|b_j| + |\eta_j| + |\mu_j|).$$

This implies that $\eta_i = b_i = \mu_i = 0$ for every $i = 1, \ldots, m$, which is a contradiction. Hence the minimum is unique and Ξ is well defined. With similar arguments one can prove that Ξ is also continuous. □

5. Estimates of the functional I

In this section, we give some estimates of the functional $I_{|M}$, on a suitable finite dimensional set. The main idea is to find an expression of the functional I in terms of the Green function G of Ω related to Δ_H. In what follows we denote by $H_\xi(\eta) = H(\xi,\eta) = \Gamma(\xi,\eta) - G(\xi,\eta)$ the regular part of G (we recall that $\Gamma(\xi,\eta) = c_q d(\xi,\eta)^{2-q}$, see (2.2)).

Proposition 5.1. *Let K be a compact subset of Ω and let $\Delta_{m-1} = \{\alpha \in [0,1]^m \mid \sum_{i=1}^m \alpha_i = 1\}$ be the standard simplex. Let $\alpha \in \Delta_{m-1}$, $\xi \in K^m$, $\lambda > 0$. We set $d = \min_{i \neq j} d(\xi_i, \xi_j)$, $\|\alpha\|^2 = \sum_{i=1}^m \alpha_i^2$ and $|\alpha|^{2q/(q-2)} = \sum_{i=1}^m \alpha_i^{2q/(q-2)}$. If $\lambda d \geq 1$, then we have*

$$\begin{aligned} &I(\theta(\varrho(\alpha,\xi,\lambda))\varrho(\alpha,\xi,\lambda)) \\ &\quad = I_\infty(\omega_0)\left(\frac{\|\alpha\|}{|\alpha|}\right)^q \left(1 + \frac{C}{\lambda^{q-2}}\left(\sum_{i=1}^m \left(2\left(\frac{\alpha_i}{|\alpha|}\right)^{2q/(q-2)} - \frac{\alpha_i^2}{\|\alpha\|^2}\right) H(\xi_i,\xi_i)\right.\right. \\ &\qquad \left.\left. - \sum_{i \neq j}\left(2\frac{\alpha_i^{(q+2)/(q-2)}\alpha_j}{|\alpha|^{2q/(q-2)}} - \frac{\alpha_i\alpha_j}{\|\alpha\|^2}\right) G(\xi_i,\xi_j)\right)\right) + O\left(\frac{1}{(\lambda d)^{q-1}}\right), \end{aligned}$$

where $\varrho(\alpha,\xi,\lambda) = \varrho(\alpha_1,\ldots,\alpha_m,\xi_1,\ldots,\xi_m,\lambda,\ldots,\lambda)$ has been defined in (4.1) *and θ in Remark* 2.1.

Proof. We only consider the part of the proof which needs to be handled with a little of care in the setting of the operator Δ_H (the rest of the proof is analogous to the one in [2]). Recalling (2.6), we only need to estimate $\|\sum_{j=1}^m \alpha_j P\omega_j\|$ and $\|\sum_{j=1}^m \alpha_j P\omega_j\|_{q^*}$, where $\omega_j = \omega_{\lambda,\xi_j}$. Setting $h_i = \omega_i - P\omega_i$, we have

$$\begin{aligned} &\int D_H P\omega_j \cdot D_H P\omega_i \\ &\quad = \int_\Omega D_H\omega_j \cdot D_H P\omega_i = \int_\Omega \omega_j^{(q+2)/(q-2)} P\omega_i = \int_\Omega \omega_j^{(q+2)/(q-2)}(\omega_i - h_i) \\ &\quad = \int_{\mathbb{H}^n} \omega_j^{(q+2)/(q-2)}\omega_i - \int_{\mathbb{H}^n\setminus\Omega} \omega_j^{(q+2)/(q-2)}\omega_i - \int_\Omega \omega_j^{(q+2)/(q-2)} h_i. \end{aligned} \tag{5.1}$$

Let us consider the case when $i = j$. The delicate point is to estimate the integral in the far right-hand side. For this purpose we notice that h_i is Δ_H-harmonic in Ω and compare it with H_{ξ_i}. Indeed, for every $\chi \in \partial\Omega$ we have

$$\left|h_i(\chi) - \frac{C_0}{\lambda^{(q-2)/2} d(\chi,\xi_i)^{q-2}}\right| \leq \frac{C}{\lambda^{(q+2)/2}},$$

where C_0 is the constant introduced in (3.2). Then, by the maximum principle for Δ_H, we get

$$\left|h_i(\chi) - \frac{C_0}{c_q\lambda^{(q-2)/2}} H_{\xi_i}(\chi)\right| \leq \frac{C}{\lambda^{(q+2)/2}} \quad \forall \chi \in \Omega. \tag{5.2}$$

The Taylor expansion of second order of the function H_{ξ_i} in the direction of the vector fields of D_H can be written (see (3.5))

$$H_{\xi_i}(\chi) = H_{\xi_i}(\xi_i) + (\tau_{\xi_i}^{-1}(\chi))_1 \cdot XH_{\xi_i}(\xi_i) + (\tau_{\xi_i}^{-1}(\chi))_2 \cdot YH_{\xi_i}(\xi_i) + O(d^2(\chi,\xi_i)).$$

Let $l = d(K, \partial\Omega)$. By the symmetry of $B_d(0, l)$ with respect to the first $2n$ variables, we get

$$\begin{aligned}
&\int_{B_d(\xi_i, l)} \omega_i^{(q+2)/(q-2)} H(\chi, \xi_i)\,\mathrm{d}\chi \\
&\quad = H(\xi_i, \xi_i) \int_{B_d\,(\xi_i, l)} \omega_i^{(q+2)/(q-2)}\,\mathrm{d}\chi + O\left(\int_{B_d(\xi_i, l)} \omega_i^{(q+2)/(q-2)} d(\chi, \xi_i)^2\,\mathrm{d}\chi\right) \\
&\quad = H(\xi_i, \xi_i) \int_{\mathbb{H}^n} \omega_i^{(q+2)/(q-2)} - H(\xi_i, \xi_i) \int_{\mathbb{H}^n \setminus B_d(\xi_i, l)} \omega_i^{(q+2)/(q-2)} \\
&\qquad + O\left(\int_{B_d(\xi_i, l)} \omega_i^{(q+2)/(q-2)} d(\chi, \xi_i)^2\,\mathrm{d}\chi\right)
\end{aligned}$$

(with some simple computations)

$$= C_2 \frac{H(\xi_i, \xi_i)}{\lambda^{(q-2)/2}} + O\left(\frac{1}{\lambda^{q/2}}\right),$$

where $C_2 = \int_{\mathbb{H}^n} \omega_0^{(q+2)/(q-2)}$.

Now, using (5.2), we find

$$\int_\Omega \omega_i^{(q+2)/(q-2)} h_i = \frac{C_0 C_2}{c_q} \frac{H(\xi_i, \xi_i)}{\lambda^{q-2}} + O\left(\frac{1}{\lambda^{q-1}}\right).$$

From this and (5.1), with some easy computations, we get

$$\int D_H P\omega_i \cdot D_H P\omega_i = q I_\infty(\omega_0) - \frac{C_0 C_2}{c_q} \frac{H(\xi_i, \xi_i)}{\lambda^{q-2}} + O\left(\frac{1}{\lambda^{q-1}}\right).$$

The case $i \neq j$ is more technical but does not present new difficulties with respect to the proof in [2] (as well as the estimate in norm $\|\cdot\|_{q^*}$).

Setting for brevity $\varrho = \varrho(\alpha, \xi, \lambda)$, we finally get

$$\begin{aligned}
\left(\int |D_H \varrho|^2\right)^{q/2} &= (q I_\infty(\omega_0) \|\alpha\|^2)^{q/2} \left(1 + \frac{C_0 C_2}{2 c_q I_\infty(\omega_0) \lambda^{q-2}} \left(\sum_{i \neq j} \frac{\alpha_i \alpha_j}{\|\alpha\|^2} G(\xi_i, \xi_j)\right.\right. \\
&\qquad \left.\left. - \sum_{i=1}^m \frac{\alpha_i^2}{\|\alpha\|^2} H(\xi_i, \xi_i)\right)\right)^{\frac{q}{2}} + O\left(\frac{1}{(\lambda\delta)^{q-1}}\right).
\end{aligned}$$

Analogously we can prove that

$$\begin{aligned}
&\left(\int_\Omega \varrho^{2q/(q-2)}\right)^{-(q-2)/2} \\
&\quad = (q I_\infty(\omega_0))^{-(q-2)/2} |\alpha|^{-q} \left(1 + \frac{C_0 C_2}{c_q I_\infty(\omega_0) \lambda^{q-2}} \left(\sum_{i=1}^m \left(\frac{\alpha_i}{|\alpha|}\right)^{2q/(q-2)} H(\xi_i, \xi_i)\right.\right. \\
&\qquad \left.\left. - \sum_{i \neq j} \frac{\alpha_i^{(q+2)/(q-2)} \alpha_j}{|\alpha|^{2q/(q-2)}} G(\xi_i, \xi_j)\right)\right) + O\left(\frac{1}{(\lambda\delta)^{q-1}}\right)
\end{aligned}$$

and we can conclude

$$I(\theta(\varrho)\varrho) = \frac{1}{q}\left(\frac{\|\varrho\|}{\|\varrho\|_{q^*}}\right)^q$$

$$= I_\infty(\omega_0)\left(\frac{\|\alpha\|}{|\alpha|}\right)^q \left(1 + \frac{C_0C_2}{2c_qI_\infty(\omega_0)\lambda^{q-2}}\sum_{i=1}^m H(\xi_i,\xi_i)\right.$$

$$\left(2\left(\frac{\alpha_i}{|\alpha|}\right)^{2q/(q-2)} - \frac{\alpha_i^2}{\|\alpha\|^2}\right) - \frac{C_0C_2}{2c_qI_\infty(\omega_0)\lambda^{q-2}}$$

$$\left.\sum_{i\neq j}\left(2\frac{\alpha_i^{(q+2)/(q-2)}\alpha_j}{|\alpha|^{2q/(q-2)}} - \frac{\alpha_i\alpha_j}{\|\alpha\|^2}\right)G(\xi_i,\xi_j)\right) + O\left(\frac{1}{(\lambda\delta)^{q-1}}\right). \qquad \square$$

From the previous proposition we can deduce

Lemma 5.2. *Let K and Δ_{m-1} be as in the previous proposition.*

(a) *For every $m \in \mathbb{N}$, $m \geq 2$, there exist $\varepsilon > 0$ and $\lambda > 0$ such that*

$$I(\theta(\varrho(\alpha,\xi,\lambda))\varrho(\alpha,\xi,\lambda)) \leq mI_\infty(\omega_0), \tag{5.3}$$

for every $(\alpha,\xi) \in \Delta_{m-1} \times K^m$ with $\alpha_i \leq \varepsilon$ for some i.

(b) *There exists $m_0 \in \mathbb{N}$ such that for every $\delta > 0$ there exists $\bar{\lambda} > 0$ such that* (5.3) *holds for every $m \geq m_0$, $\lambda \geq \bar{\lambda}$, $\alpha \in \Delta_{m-1}$ and $\xi \in K^m$ with* $\min_{i\neq j} d(\xi_i,\xi_j) \geq \delta$.

(c) *For every $m \in \mathbb{N}$, $m \geq 2$, and for every $\varepsilon > 0$ there exist $\delta_0 > 0$ and $\bar{\lambda} > 0$ such that* (5.3) *holds for every $\alpha \in \Delta_{m-1} \cap [\varepsilon,1]^m$, $\lambda \geq \bar{\lambda}$ and $\xi \in K^m$ with* $\min_{i\neq j} d(\xi_i,\xi_j) \leq \delta_0$.

(d) *There exist $m_0 \in \mathbb{N}$ and $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$* (5.3) *holds for every $\alpha \in \Delta_{m_0-1}$ and $\xi \in K^{m_0}$.*

We omit the proof since in Proposition 5.1 we provide an expression of the functional I which is formally the same as the expression known for the analogous problem associated to the classical Laplacian. Then Lemma 5.2 follows as in [2].

6. Proof of the existence result

Theorem 1.1 will be proved as an application of the well-known theory introduced by Bahri and Coron in the Laplacian case and extended by Bahri and Brezis to the Riemannian case. In their paper Bahri and Brezis pointed out that this technique can be applied to look for critical points of a functional any time the structure of its (PS) sequences is known. Here we look for critical points of the functional I constrained on the manifold M and we state their theorem in our particular situation.

Theorem 6.1. *Let Ω be a smooth connected bounded domain of $\mathbb{H}^n$. If Theorem* 3.5 *and Lemmas* 4.1, 5.2 *are satisfied and there exists a positive integer d such that the homology group $H_d(\Omega)$ is nontrivial, then problem* (1.2) *has a solution.*

This theorem is proved in [1, Section 8] and we do not repeat the proof here but we only want to give an idea of it for the reader's convenience. It is based on the topological properties of the sublevels of the functional $I_{|_M}$

$$W_m = \{u \in M^+ | I(u) \leq (m+1) I_\infty(\omega_0)\}.$$

If by contradiction (1.2) has no solutions, then the representation Theorem 3.5 implies that Ω is homeomorphic to a retract of W_1. Indeed the set $V(1,\varepsilon)$ defined in (4.3) is a deformation retract of W_1 and, roughly speaking, the function

$$\varrho_1(e,.,\lambda) : \Omega \to V(1,\varepsilon)$$

induces isomorphisms on homology groups with inverses induced by the function

$$\Xi : V(1,\varepsilon) \to \Omega$$

defined in Lemma 4.1. The estimates of the functional I on the image of ϱ provided in Lemma 5.2, ensure that there exists m_0 such that ϱ_{m_0} is homologically trivial. The main point of the proof of [1] is a topological argument which leads to a commutative diagram which shows that ϱ_{m_0} is trivial if and only if ϱ_1 is. This gives rise to a contradiction.

Proof of Theorem 1.1. Theorem 3.5, Lemmas 4.1 and 5.2 have already been proved in Sections 3, 4 and 5, respectively. Hence Theorem 1.1 directly follows from Theorem 6.1. □

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