NoDEA
Nonlinear differ. equ. appl. 9 (2002) 01–14
1021–9722/02/010001–14 \$ 1.50+0.20/0

Nonlinear Differential Equations
and Applications NoDEA

Positive solutions of $-\Delta_p u + u^{p-1} - q(x)u^\alpha = 0$ in $\mathbb{R}^N$

Giovanna CITTI, Francesco UGUZZONI
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40127 Bologna, Italy
e-mail: citti@dm.unibo.it; uguzzoni@dm.unibo.it

2000 *Mathematics Subject Classification*: 35J20, 35B40, 35J60.
Key words: p-Laplacian, Palais-Smale levels, uniqueness of ground states, asymptotic estimates.

1 Introduction

In this note we give an existence result for the following semilinear equation in $\mathbb{R}^N$:

$$\begin{cases} -\Delta_p u + u^{p-1} - q(x)u^\alpha = 0 & \text{in } \mathbb{R}^N, \\ u > 0, \qquad u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian on $\mathbb{R}^N$, $1 < p < 2 \leq N$, $1 < \alpha < p^* - 1 = \frac{pN}{N-p} - 1$ and $q \in L^\infty(\mathbb{R}^N)$ is a positive function with a positive limit q_∞ at infinity, satisfying (5) below.

Only recently it has been established in [DPR] a radial symmetry result for the solutions of the equation at infinity associated to (1) which, together with the previous results in [C2], implies the uniqueness of the ground state. Hence some sophisticated variational techniques known for the Laplace operator can now be applied also to the p-Laplacian. As a first example we prove Theorem 2 below.

Investigation supported by University of Bologna. Funds for selected research topics.

The classical results known for the p-Laplacian were based on some modifications of the mountain pass theorem and the concentration-compactness principle. The first existence results have been obtained when q is radially symmetric (see [BeL], [C1], [E]; see also the more general and recent results in [GST]). Other results have been established when q is not radial but $q_\infty = \inf_{\mathbb{R}^N} q$ or $q \in L^{p_0}$ for suitable values of p_0 (see [DN], [L1], [L2] for $p = 2$, [BC], [GS], [O], [Y] for general values of p). In all these works the hypothesis that α is smaller than the critical exponent is required; however we explicitly notice that existence results of Brezis-Nirenberg type with $\alpha = p^* - 1$ have been established also for the p-Laplacian operator on unbounded domains (see [GA], [NSJ], [SY]).

The uniqueness for the problem at infinity

$$-\Delta_p \omega + \omega^{p-1} - q_\infty \omega^\alpha = 0 \qquad \text{in } \mathbb{R}^N, \tag{2}$$

initially known only for $p = 2$, allows to perform a more delicate analysis of the problem. The Palais-Smale condition of the functional J naturally associated to problem (1) can be violated only at an increasing sequence of levels $\{S_n\}_{n\in\mathbb{N}}$ only dependent on the unique solution ω of (2) (see for example [BeC]). As a consequence very sophisticated existence results for (1) have been established in the case $p = 2$: see [BLn], [BeC] (on exterior domains) and [BL] (on $\mathbb{R}^N$). In particular Bahri and Lions introduced a deep topological argument (which contains several highly technical aspects) for studying the problem when the rate of convergence of q to q_∞ is comparable with the asymptotic behavior of ω. Afterwards Bahri and Li provided a simpler existence result, based on a min-max procedure, under a slightly stronger hypothesis on q. In [BL] the authors perform an estimate of the functional J on the set of functions

$$\{t\omega(\cdot - x_1) + (1-t)\omega(\cdot - x_2) | t \in [0,1]\}, \tag{3}$$

for suitable choices of $x_1, x_2 \in \mathbb{R}^N$, and prove the existence of an asymptotic critical level $c_0 \in]S_1, S_2[$, which is a true critical level by the (PS) property.

We also mention the papers [ABC], [DF], [W] where the perturbed equation

$$-\varepsilon\Delta u + u - q(x)u^\alpha = 0 \qquad \text{in } \mathbb{R}^N$$

is considered for $\varepsilon \to 0$ and only local conditions are required on q. However it does not seem possible to use these weak assumptions for problem (1) where the value of ε is fixed.

The new result of Damascelli, Pacella and Ramaswamy [DPR] and the uniqueness results contained in [C2], [PS] and [ST] allow to establish the uniqueness for ground states of the equation at infinity (2) for all values of $p < 2$.

Theorem 1 *For $p \in]1, 2[$, there exists a unique (up to translations) positive solution in $W^{1,p}(\mathbb{R}^N)$ of equation (2).*

Then also in this case it is possible to give a complete characterization of the compactness levels of the functional J (Theorem 3). In particular there exists a new sequence $\{S_n\}_{n\in\mathbb{N}}$ such that the (PS) condition is satisfied in $\mathbb{R}\backslash\{S_n\}_{n\in\mathbb{N}}$. Here we use this fact and the min-max procedure introduced by Bahri and Li in the case $p = 2$ to prove our existence theorem. First we establish a precise estimate of the asymptotic behavior at infinity of the ground state of (2):

$$\omega(x) \sim |x|^{-\frac{N-1}{p(p-1)}} \exp\left(-\frac{|x|}{(p-1)^{\frac{1}{p}}}\right) \sim \omega'(x). \tag{4}$$

In the case $p = 2$, (4) is well-known (see [GNN]) and is based on an estimate of the Green function of $-\Delta u + u$ in $\mathbb{R}^N$; here we obtain estimate (4) with a comparison method and o.d.e. techniques. Then, under the assumption that $q_\infty > 0$ and there exist $c > 0$ and $\mu > \frac{2}{(p-1)^{\frac{1}{p}}}$ such that

$$q(x) \geq q_\infty - c\exp(-\mu|x|), \tag{5}$$

we obtain an energy estimate on the set (3) analogous to the one of [BL]. This estimate is more delicate in our context, due to the nonlinearity of the second order term of J. Finally the min-max procedure yields the existence theorem.

Theorem 2 *Let $p \in]1,2[$, $\alpha \in]1, p^*-1[$ and let $q \in L^\infty(\mathbb{R}^N)$ be a positive function with a positive limit q_∞ at infinity satisfying (5). Then problem (1) has a solution $u \in W^{1,p}(\mathbb{R}^N) \cap C^{1+\beta}_{\mathrm{loc}}(\mathbb{R}^N)$, for a $\beta \in]0,1[$.*

2 Uniqueness of ground states

We first introduce some notation. We denote by $(X, \|\cdot\|)$ the Sobolev space $W^{1,p}(\mathbb{R}^N)$ with the norm $\|u\|^p = \|\nabla u\|_p^p + \|u\|_p^p$ and we introduce the functionals $J, J_\infty \in C^1(X\backslash\{0\}, \mathbb{R})$, $I, I_\infty \in C^1(X, \mathbb{R})$,

$$J(u) = \frac{\|u\|^p}{\left(\int q(x)|u|^{\alpha+1}\right)^{\frac{p}{\alpha+1}}}, \quad J_\infty(u) = \frac{\|u\|^p}{\left(q_\infty \int |u|^{\alpha+1}\right)^{\frac{p}{\alpha+1}}}, \tag{6}$$

$$I(u) = \frac{1}{p}\|u\|^p - \frac{1}{\alpha+1}\int q(x)|u|^{\alpha+1}, \quad I_\infty(u) = \frac{1}{p}\|u\|^p - \frac{q_\infty}{\alpha+1}\int |u|^{\alpha+1}. \tag{7}$$

Hereafter we always assume that p, α and q verify the hypotheses of Theorem 2. We also set

$$\Sigma = \{u \in X|\ \|u\| = 1\}, \quad \Sigma^+ = \{u \in \Sigma|\ u \geq 0\}$$

and

$$S_1 = \inf_{X\setminus\{0\}} J_\infty, \quad S_n = n^{1-\frac{p}{\alpha+1}} S_1 \qquad \forall n \in \mathbb{N}.$$

The following representation theorem of (PS) sequences is proved in [BC].

Theorem BC *Let u_m be a nonnegative sequence in X such that $I(u_m) \to \ell \in \mathbb{R}$ and $dI(u_m) \to 0$ as $m \to +\infty$. Then there exist a nonnegative function $u_0 \in X$, a nonnegative integer k, k nonnegative nontrivial functions $\omega_1, \dots, \omega_k \in X$ and k sequences $(y_{1,m}), \dots, (y_{k,m})$ in $\mathbb{R}^N$, such that $|y_{j,m}| \to +\infty$ as $m \to +\infty$ for every $j = 1, \dots, k$ and*

$$\begin{aligned} u_m &= u_0 + \sum_{j=1}^k \omega_j(\cdot - y_{j,m}) + o(1) \quad \text{in } X, \quad \text{as } m \to +\infty, \\ I(u_m) &= I(u_0) + \sum_{j=1}^k I_\infty(\omega_j) + o(1) \quad \text{as } m \to +\infty, \\ dI(u_0) &= 0, \qquad dI_\infty(\omega_j) = 0 \quad \forall j = 1, \dots, k. \end{aligned}$$

From the regularity results in [S] and [D] and from the strong maximum principle for Δ_p (see [V]) it follows that the functions ω_j in the above theorem are solutions of

$$\begin{cases} -\Delta_p \omega + \omega^{p-1} - q_\infty \omega^\alpha = 0 & \text{in } \mathbb{R}^N, \\ 0 < \omega \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), & \\ \omega(x) \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{8}$$

Since $1 < p < 2$, we can now deduce from [DPR, Theorem 1.1] that any ω_j is radially symmetric about some point in $\mathbb{R}^N$. Moreover the uniqueness results established in [C2] for the ordinary differential equation

$$\begin{cases} (|\omega'(r)|^{p-2}\omega'(r))' + \frac{N-1}{r}|\omega'(r)|^{p-2}\omega'(r) - \omega(r)^{p-1} + q_\infty \omega(r)^\alpha = 0 \\ \hfill 0 < r < +\infty, \\ \omega > 0, \quad \omega'(0) = 0, \quad \omega(+\infty) = 0, \end{cases} \tag{9}$$

ensure that the ω_j are, up to a translation, all equal to a positive radial function ω which is the unique radial solution of (8). We explicitly remark that such a solution ω exists and verifies

$$I_\infty(\omega) = \inf\{I_\infty(u) | u \in X\setminus\{0\}, dI_\infty(u)u = 0\} = \left(\frac{1}{p} - \frac{1}{\alpha+1}\right) S_1^{\frac{\alpha+1}{\alpha+1-p}},$$

$$J_\infty(\omega) = \inf_{X\setminus\{0\}} J_\infty = S_1;$$

moreover ω is radial decreasing (see for example [BC, Remark 1]). In Lemma 5 of the next section we shall study how ω behaves at infinity. If u_m is a (PS) sequence for $J\,|_\Sigma$ at a level $\ell \in \mathbb{R}$, then $J(u_m)^{\frac{\alpha+1}{p(\alpha+1-p)}} u_m$ is a (PS) sequence for I at the level $(\frac{1}{p} - \frac{1}{\alpha+1})\ell^{\frac{\alpha+1}{\alpha+1-p}}$; hence from Theorem BC and from the uniqueness of ground states of (8) we deduce the following compactness theorem for the (PS) sequences of J.

Theorem 3 *Let u_m be a sequence in Σ^+ such that $J(u_m) \to \ell \in \mathbb{R}$ and $dJ|_\Sigma(u_m) \to 0$ as $m \to +\infty$. If $\ell \notin \{S_n\}_{n\in\mathbb{N}}$, then there exists $u_0 \in X$ such that $u_0 \geq 0$, $u_0 \not\equiv 0$, $dI(u_0) = 0$. In other words u_0 is a weak solution of* (1).

Remark 4 If u_0 is a solution of (1), then by the results in [S] and [D], $u_0 \in C^{1+\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\beta > 0$, $u_0(x) \to 0$ as $|x| \to +\infty$ and u_0 is strictly positive, by the strong maximum principle proved in [V].

3 Proof of the existence theorem

We will prove our main result following an argument introduced for $p = 2$ by Bahri and Li [BL]. In order to do so we first study the asymptotic behavior of the solution ω of (8).

Lemma 5 *Let $\omega = \omega(|x|)$ be the unique radial solution of* (8). *Then there exist two positive constants c_1, c_2 such that, for large $r = |x|$,*

$$\begin{cases} c_1 r^{-\gamma^+}\exp(-\theta r) \leq \omega(r) \leq c_2 r^{-\gamma}\exp(-\theta r) \\ c_1 r^{-\gamma^+}\exp(-\theta r) \leq -\omega'(r) \leq c_2 r^{-\gamma}\exp(-\theta r) \end{cases} \qquad \text{if } N \geq 3;$$

$$\begin{cases} c_1 r^{-\gamma}\exp(-\theta r) \leq \omega(r) \leq c_2 r^{-\gamma^-}\exp(-\theta r) \\ c_1 r^{-\gamma}\exp(-\theta r) \leq -\omega'(r) \leq c_2 r^{-\gamma^-}\exp(-\theta r) \end{cases} \qquad \text{if } N = 2;$$

where we have set $\theta = (p-1)^{-\frac{1}{p}}$ and $\gamma = \frac{N-1}{p(p-1)}$; γ^+ (respectively γ^-) stands for any $\gamma + \varepsilon$ (respectively $\gamma - \varepsilon$) with $\varepsilon > 0$.

Proof. From the results in [DPR] we know that $\omega'(r) < 0$. Setting $k(r) = |\omega'(r)|^{p-2}\omega'(r) = -(-\omega'(r))^{p-1}$ and $f(\omega) = \omega^{p-1} - q_\infty\omega^\alpha$, the equation in (8) can be written as

$$f(\omega) = k' + \frac{N-1}{r}k = (p-1)(-\omega')^{p-2}\omega'' - \frac{N-1}{r}(-\omega')^{p-1} \qquad \forall r > 0. \quad (10)$$

We now set $F(\omega) = \int_0^\omega f(s)ds$ and $E = \frac{p-1}{p}|\omega'|^p - F(\omega)$. From (10) it follows

$$E' = -\frac{N-1}{r}|\omega'|^p < 0.$$

Hence E is decreasing and, since $F(\omega)$ vanishes at infinity, also E has a finite limit, and this is necessarily 0, since ω vanishes at infinity. This gives $E \geq 0$. On the other hand

$$0 \leq E = \frac{1}{p}((p-1)|\omega'|^p - \omega^p(1 + o(1))), \qquad \text{as } r \to +\infty,$$

which yields $|\omega'|^p \geq (\frac{1}{p-1})^- \omega^p$ (here and in the sequel we always assume that r is very large) where we have used the convention introduced in the statement of the lemma to denote γ^- a constant smaller than γ. Using the fact that $\omega' < 0$ and the definition of θ, we have

$$\omega' + \theta^- \omega \leq 0. \tag{11}$$

From (11) we immediately get

$$\omega(r) \leq M \exp(-\theta^- r). \tag{12}$$

In order to obtain more precise estimates we exploit the comparison results for Δ_p. For a β closed to γ we define $v = v_\beta$ as follows

$$v_\beta(r) = r^{-\beta} \exp(-\theta r).$$

Denoting $v(x) = v(|x|) = v(r)$ we compute

$$\begin{aligned} \Delta_p v &= (|v'|^{p-2}v')' + \frac{N-1}{r}|v'|^{p-2}v' \\ &= v^{p-1}\theta^{p-2}\left(1 + \frac{\beta}{r\theta}\right)^{p-2}\left((p-1)\theta^2 + \frac{1}{r}(2(p-1)\theta\beta - (N-1)\theta)\right. \\ &\quad \left. + \frac{1}{r^2}((p-1)\beta(1+\beta) - (N-1)\beta)\right). \end{aligned} \tag{13}$$

Taking the Taylor expansion $(1 + \frac{\beta}{r\theta})^{p-2} = 1 + \frac{(p-2)\beta}{r\theta} + O(\frac{1}{r^2})$, as $r \to +\infty$, and recalling that $\theta = (p-1)^{-\frac{1}{p}}$, $\gamma = \frac{N-1}{p(p-1)}$, we obtain

$$\frac{\Delta_p v}{v^{p-1}} = 1 + \theta^{-1} p(\beta - \gamma)\frac{1}{r} + O\left(\frac{1}{r^2}\right), \qquad \text{as } r \to +\infty. \tag{14}$$

We now set

$$\overline{v} = \Lambda v_{\gamma^-}, \qquad \underline{v} = \varepsilon v_{\gamma^+}.$$

We claim that we can choose $\Lambda > 0$, $\varepsilon > 0$ and $r_0 > 0$ so that

$$\begin{cases} \Delta_p \overline{v}(r) \leq f(\overline{v}(r)) & \forall r > r_0, \\ \overline{v}(r_0) \geq \omega(r_0), \end{cases} \tag{15}$$

$$\begin{cases} \Delta_p \underline{v}(r) \geq f(\underline{v}(r)) & \forall r > r_0, \\ \underline{v}(r_0) \leq \omega(r_0) \end{cases} \tag{16}$$

(we recall the definition of f: $f(s) = s^{p-1} - q_\infty s^\alpha$). For this purpose we exploit the estimate (12). We first fix $\lambda \in]\theta - \theta^-, \theta[$ and choose r_0 large enough to satisfy

$$M r_0^{\gamma^-} \exp(-(\lambda - (\theta - \theta^-))r_0) \leq 1, \tag{17}$$

$$\left(\frac{2q_\infty r}{\theta^{-1} p(\gamma - \gamma^-)} \right)^{\frac{1}{\alpha - p + 1}} \exp(-(\theta - \lambda)r) \leq 1 \qquad \forall r > r_0. \tag{18}$$

Then we set $\Lambda = \exp(\lambda r_0)$. From (17) and (12) it follows that $\overline{v}(r_0) \geq \omega(r_0)$. Moreover (14) and (18) yield

$$\frac{\Delta_p \overline{v}}{\overline{v}^{p-1}} = \frac{\Lambda^{p-1} \Delta_p v_{\gamma^-}}{\Lambda^{p-1} v_{\gamma^-}^{p-1}} \leq 1 - \frac{\theta^{-1} p(\gamma - \gamma^-)}{2r} \leq 1 - q_\infty \overline{v}^{\alpha - p + 1} = \frac{f(\overline{v})}{\overline{v}^{p-1}} \quad \forall r > r_0.$$

This proves (15). To prove (16) we only need to take $\varepsilon > 0$ small enough so that $\underline{v}(r_0) \leq \omega(r_0)$ and observe that

$$\begin{aligned} \frac{\Delta_p \underline{v}}{\underline{v}^{p-1}} &= \frac{\varepsilon^{p-1} \Delta_p v_{\gamma^+}}{\varepsilon^{p-1} v_{\gamma^+}^{p-1}} \geq 1 + \frac{\theta^{-1} p(\gamma^+ - \gamma)}{2r} \geq 1 \geq 1 - q_\infty \underline{v}^{\alpha - p + 1} \\ &= \frac{f(\underline{v})}{\underline{v}^{p-1}} \quad \forall r > r_0, \end{aligned}$$

by means of (14). We can now use the comparison principles in [DPR, Theorem 3.1]: from (8), (15) and (16) we deduce

$$\varepsilon r^{-\gamma^+} \exp(-\theta r) = \underline{v}(r) \leq \omega(r) \leq \overline{v}(r) = \Lambda r^{-\gamma^-} \exp(-\theta r) \qquad \forall r > r_0. \tag{19}$$

We now want to compare ω with $v = v_\gamma = r^{-\gamma} \exp(-\theta r)$. Instead of (14) we consider the second order expansion

$$\frac{\Delta_p v}{v^{p-1}} = 1 + \frac{1}{2} \theta^{p-2} \gamma (p-1)(3-N) \frac{1}{r^2} + O\left(\frac{1}{r^3}\right), \qquad \text{as } r \to +\infty,$$

which follows from (13) as well. This formula suggests to study separately the cases $N > 3$, $N = 3$ and $N < 3$. If $N > 3$ we set $\overline{v} = \Lambda' v_\gamma$ and, arguing as above, we can prove that

$$\omega(r) \leq \Lambda' r^{-\gamma} \exp(-\theta r) \tag{20}$$

(for large $r > 0$). Analogously, if $N = 2$ we set $\underline{v} = \varepsilon' v_\gamma$ and we obtain

$$\omega(r) \geq \varepsilon' r^{-\gamma} \exp(-\theta r). \tag{21}$$

Finally, if $N = 3$ we consider the third order expansion

$$\frac{\Delta_p v}{v^{p-1}} = 1 - \frac{4}{3} p^{-2}(p-1)^{\frac{3-2p}{p}}(2-p)\frac{1}{r^3} + O\left(\frac{1}{r^4}\right), \qquad \text{as } r \to +\infty,$$

and again we can prove (20).

We now look for the estimates of $-\omega'$. From (11) we immediately get the estimates from below. Moreover from (10) we infer $\omega'' > 0$ (for large $r > 0$); hence

$$-\omega'(r) \leq \int_{r-1}^{r} -\omega'(s)ds = \omega(r-1) - \omega(r) \leq \omega(r-1)$$

and we obtain the estimates from above. □

With Lemma 5 in hands, we can now establish an energy estimate for the functional J. This is the crucial estimate of the paper and we are able to prove it only if $\alpha > 1$.

Proposition 6 *There exists $R_0 > 0$ such that for every $R \geq R_0$, $|x_1| \geq R$, $|x_2| \geq R - \sqrt{R}$, $\sqrt{R} \leq |x_1 - x_2| \leq (2 + \frac{1}{\sqrt{R}-1}) \min\{|x_1|, |x_2|\}$, we have*

$$J(t\omega_1 + (1-t)\omega_2) < S_2 \qquad \forall t \in [0,1], \tag{22}$$

where $\omega_i = \omega(\cdot - x_i)$, $i = 1, 2$, being ω the unique radial solution of (8).

We shall need the following inequalities.

Lemma 7 1) *For every $\tau \in]0,1[$ and $x, y \in [0, +\infty[$ we have*

$$x^\tau + y^{1-\tau} \leq (1+x)^\tau (1+y)^{1-\tau} \tag{23}$$

and the equality holds iff $xy = 1$.

2) *For every $\tau \in]0,1[$ and $a_1, a_2, b_1, b_2 \in [0, +\infty[$ we have*

$$a_1^\tau a_2^{1-\tau} + b_1^\tau b_2^{1-\tau} \leq (a_1 + b_1)^\tau (a_2 + b_2)^{1-\tau} \tag{24}$$

and the equality holds if $a_1 b_2 = a_2 b_1$.

3) *For every $p \in]1,2[$ and $\xi, \eta \in \mathbb{R}^N$ we have*

$$|\xi + \eta|^p \leq (\langle |\xi|^{p-2}\xi + |\eta|^{p-2}\eta, \xi + \eta \rangle)^{\frac{p}{2}} (|\xi|^p + |\eta|^p)^{\frac{2-p}{2}}. \tag{25}$$

Proof. To prove (23) we study the function of one real variable $f_x(y) = (1+x)^\tau (1+y)^{1-\tau} - x^\tau - y^{1-\tau}$: since at the point $y = \frac{1}{x}$ it has a strong minimum point and it takes the value $f_x(\frac{1}{x}) = 0$, the assertion (23) immediately follows. From here also (24) is proved simply choosing (if $b_1 \neq 0 \neq a_2$) $x = \frac{a_1}{b_1}$, $y = \frac{b_2}{a_2}$ and dividing (24) by $b_1^\tau a_2^{1-\tau}$; if $b_1 = 0$ or $a_2 = 0$ (24) is trivial. Finally (25) can be found in [Y, p. 1041]. □

Proof of Proposition 6. We set for brevity

$$y = x_2 - x_1, \qquad A = \|\omega\|^p,$$
$$[u,v] = \int |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle + \int |u|^{p-2}uv \qquad \forall u, v \in X.$$

From the equation (8), we have

$$[\omega_1, \omega_2] = q_\infty \int \omega_1^\alpha \omega_2 = q_\infty \int \omega_2^\alpha \omega_1 = [\omega_2, \omega_1],$$
$$A = [\omega, \omega] = q_\infty \int \omega^{\alpha+1}.$$

We first consider the case $t = \frac{1}{2}$ and prove

$$J(\omega_1 + \omega_2) < S_2. \tag{26}$$

By means of inequality (25) we get

$$\begin{aligned}\|\omega_1 + \omega_2\|^p \quad &\le \int ((|\nabla \omega_1|^p + |\nabla \omega_2|^p + |\nabla \omega_1|^{p-2}\langle \nabla \omega_1, \nabla \omega_2\rangle \\ &\quad + |\nabla \omega_2|^{p-2}\langle \nabla \omega_2, \nabla \omega_1\rangle)^{\frac{p}{2}}(|\nabla \omega_1|^p + |\nabla \omega_2|^p)^{\frac{2-p}{2}} \\ &\quad + (\omega_1^p + \omega_2^p + \omega_1^{p-1}\omega_2 + \omega_2^{p-1}\omega_1)^{\frac{p}{2}}(\omega_1^p + \omega_2^p)^{\frac{2-p}{2}})\end{aligned}$$

(for (24))

$$\begin{aligned}&\le \int ((|\nabla \omega_1|^p + |\nabla \omega_2|^p + |\nabla \omega_1|^{p-2}\langle \nabla \omega_1, \nabla \omega_2\rangle + |\nabla \omega_2|^{p-2}\langle \nabla \omega_2, \nabla \omega_1\rangle \\ &\quad + \omega_1^p + \omega_2^p + \omega_1^{p-1}\omega_2 + \omega_2^{p-1}\omega_1)^{\frac{p}{2}}(|\nabla \omega_1|^p + |\nabla \omega_2|^p + \omega_1^p + \omega_2^p)^{\frac{2-p}{2}})\end{aligned}$$

(by Hölder inequality)

$$\begin{aligned}&\le (\|\omega_1\|^p + \|\omega_2\|^p + [\omega_1, \omega_2] + [\omega_2, \omega_1])^{\frac{p}{2}}(\|\omega_1\|^p + \|\omega_2\|^p)^{\frac{2-p}{2}} \\ &= 2A\left(1 + \frac{1}{A}[\omega_1, \omega_2]\right)^{\frac{p}{2}}.\end{aligned}$$

We now estimate $\int q(x)|\omega_1 + \omega_2|^{\alpha+1}$. Since $\alpha > 1$, there exists $c_\alpha > 0$ such that the following inequality holds for every nonnegative real numbers a, b:

$$(a+b)^{\alpha+1} \ge a^{\alpha+1} + b^{\alpha+1} + (\alpha+1)(a^\alpha b + b^\alpha a) - c_\alpha (ab)^{\frac{\alpha+1}{2}}$$

(see [BL]); hence we get

$$q_\infty \int (\omega_1 + \omega_2)^{\alpha+1} \ge 2A + 2(\alpha+1)[\omega_1, \omega_2] - c_\alpha q_\infty \int (\omega_1 \omega_2)^{\frac{\alpha+1}{2}}. \tag{27}$$

Using the estimates on ω established in Lemma 5, it is not difficult to see that

$$\int (\omega_1\omega_2)^{\frac{\alpha+1}{2}} = o([\omega_1,\omega_2]), \qquad \text{as } R\to+\infty. \tag{28}$$

Indeed $|y|\geq\sqrt{R}$ and for large $|y|$ we have

$$[\omega_1,\omega_2] = q_\infty\int\omega_1^\alpha\omega_2 = q_\infty\int\omega\omega(\cdot-y)^\alpha \geq c\int_{B(y,1)}\omega\geq c\exp(-\theta^+|y|) \tag{29}$$

and

$$\begin{aligned}\int(\omega_1\omega_2)^{\frac{\alpha+1}{2}} &= \int(\omega\omega(\cdot-y))^{\frac{\alpha+1}{2}} \leq c\int_{|z|\leq|y|}(\exp(-\theta|z|)\exp(-\theta|z-y|))^{\frac{\alpha+1}{2}}dz\\ &\quad +c\exp\left(-\theta|y|\frac{\alpha+1}{2}\right)\int_{|z|>|y|}\omega(z-y)^{\frac{\alpha+1}{2}}dz\\ &\leq c|y|^N\exp\left(-\theta|y|\frac{\alpha+1}{2}\right)+c\exp\left(-\theta|y|\frac{\alpha+1}{2}\right),\end{aligned}$$

with $\frac{\alpha+1}{2}>1$. Moreover from (5) it follows that

$$\begin{aligned}\int(q-q_\infty)(\omega_1+\omega_2)^{\alpha+1} &\geq -c\sum_{i=1}^2\int\exp(-\mu|x|)\omega(x-x_i)^{\alpha+1}dx\\ &\geq -c\sum_{i=1}^2\Bigg(\exp(-\theta(\alpha+1)|x_i|)\int_{|x-x_i|>|x_i|}\exp(-\mu|x|)dx\\ &\quad +\int_{|x-x_i|\leq|x_i|}\exp(-\mu|x|)\exp(-\theta(\alpha+1)|x-x_i|)dx\Bigg)\end{aligned}$$

(we can assume $\mu\in]2\theta,(\alpha+1)\theta]=]2(p-1)^{-\frac{1}{p}},(\alpha+1)(p-1)^{-\frac{1}{p}}]$)

$$\geq -c\sum_{i=1}^2(\exp(-\theta(\alpha+1)|x_i|)+|x_i|^N\exp(-\mu|x_i|))$$

(since $|y|=|x_1-x_2|\leq(2+\frac{1}{\sqrt{R}-1})|x_i|$ for $i=1,2$)

$$\geq -c\exp\left(-\frac{\mu^-|y|}{2+\frac{1}{\sqrt{R}-1}}\right)\geq -c\exp(-\theta^+|y|)$$

for large $R>0$. Recalling (29), where θ^+ is any number greater than θ, we get

$$\int(q-q_\infty)(\omega_1+\omega_2)^{\alpha+1}\geq -o([\omega_1,\omega_2]), \qquad \text{as } R\to+\infty. \tag{30}$$

Collecting (27), (28) and (30) we obtain

$$\int q(\omega_1+\omega_2)^{\alpha+1} \geq 2A + (2(\alpha+1) - o(1))[\omega_1,\omega_2], \qquad \text{as } R \to +\infty.$$

We can finally estimate

$$\begin{aligned} J(\omega_1+\omega_2) &= \frac{\|\omega_1+\omega_2\|^p}{(\int q|\omega_1+\omega_2|^{\alpha+1})^{\frac{p}{\alpha+1}}} \leq \frac{2A(1+\frac{1}{A}[\omega_1,\omega_2])^{\frac{p}{2}}}{(2A)^{\frac{p}{\alpha+1}}(1+(\frac{\alpha+1}{A}-o(1))[\omega_1,\omega_2])^{\frac{p}{\alpha+1}}} \\ &= (2A)^{1-\frac{p}{\alpha+1}} \frac{1+\frac{p}{2A}[\omega_1,\omega_2](1+o(1))}{1+\frac{p}{A}[\omega_1,\omega_2](1+o(1))}, \qquad \text{as } R \to +\infty. \end{aligned}$$

Since

$$\begin{aligned} S_2 &= 2^{1-\frac{p}{\alpha+1}} S_1 = 2^{1-\frac{p}{\alpha+1}} J_\infty(\omega) = 2^{1-\frac{p}{\alpha+1}} \frac{\|\omega\|^p}{(q_\infty \int \omega^{\alpha+1})^{\frac{p}{\alpha+1}}} \\ &= 2^{1-\frac{p}{\alpha+1}} \frac{A}{A^{\frac{p}{\alpha+1}}} = (2A)^{1-\frac{p}{\alpha+1}}, \end{aligned}$$

this proves (26).

We now turn to prove (22) for arbitrary values of $t \in [0,1]$. It is easy to verify that $J(\omega_2) \to J_\infty(\omega_2) = S_1 < S_2$, as $R \to +\infty$, and that $J(t\omega_1 + (1-t)\omega_2) \to J(\omega_2)$ as $t \to 0$, uniformly with respect to $R \geq R_0$. Hence there exists a small $\delta > 0$ not depending on R such that (22) holds for every $t \in [0,\delta]$. In the same way we see that (22) holds for $t \in [1-\delta, 1]$.

We now consider $t \in [\delta, 1-\delta]$ and we set $v_1 = t\omega_1$, $v_2 = (1-t)\omega_2$. Arguing as in the case $t = \frac{1}{2}$ we obtain

$$\begin{aligned} \|v_1+v_2\|^p &\leq (\|v_1\|^p + \|v_2\|^p + [v_1,v_2] + [v_2,v_1])^{\frac{p}{2}} (\|v_1\|^p + \|v_2\|^p)^{\frac{2-p}{2}} \\ &= (t^p + (1-t)^p) A \left(1 + \frac{1}{A}[\omega_1,\omega_2] \frac{t^{p-1}(1-t) + (1-t)^{p-1}t}{t^p + (1-t)^p}\right)^{\frac{p}{2}} \end{aligned}$$

and

$$\begin{aligned} \int q(v_1+v_2)^{\alpha+1} \;\geq\; & (t^{\alpha+1} + (1-t)^{\alpha+1}) A \\ & + (\alpha+1)(t^\alpha(1-t) + t(1-t)^\alpha)[\omega_1,\omega_2](1+o(1)), \end{aligned}$$

as $R \to +\infty$. Hence

$$J(v_1+v_2) \leq S_2 \varrho(t) \frac{1+\nu(t)\frac{p}{A}[\omega_1,\omega_2](1+o(1))}{1+\mu(t)\frac{p}{A}[\omega_1,\omega_2](1+o(1))}, \qquad \text{as } R \to +\infty, \tag{31}$$

where $\varrho(t) = \frac{t^p+(1-t)^p}{2}(\frac{t^{\alpha+1}+(1-t)^{\alpha+1}}{2})^{-\frac{p}{\alpha+1}}$, $\nu(t) = \frac{t^{p-1}(1-t)+(1-t)^{p-1}t}{2(t^p+(1-t)^p)}$, $\mu(t) = \frac{t^\alpha(1-t)+(1-t)^\alpha t}{t^{\alpha+1}+(1-t)^{\alpha+1}}$. We also recall that

$$[\omega_1,\omega_2] \to 0, \qquad \text{as } R \to +\infty. \tag{32}$$

Since $\frac{\nu(\frac{1}{2})}{\mu(\frac{1}{2})} = \frac{1}{2}$ and ν, μ are continuous functions, there exists $\sigma > 0$ such that $\max_{|t-\frac{1}{2}|\leq\sigma} \frac{\nu(t)}{\mu(t)} < 1$. Since $\varrho(t) \leq 1$ this implies (22) for $|t - \frac{1}{2}| \leq \sigma$. On the other hand, since

$$\varrho(t) = \frac{\varphi(t^{\alpha+1}) + \varphi((1-t)^{\alpha+1})}{2} \left(\varphi\left(\frac{t^{\alpha+1} + (1-t)^{\alpha+1}}{2} \right) \right)^{-1}$$

where $\varphi(s) = s^{\frac{p}{\alpha+1}}$ is a strictly concave function, we get $\max_{\sigma\leq|t-\frac{1}{2}|\leq\frac{1}{2}-\delta} \varrho(t) < 1$. Therefore (by (32) and (31)) (22) holds also for $\sigma \leq |t - \frac{1}{2}| \leq \frac{1}{2} - \delta$. □

We are now able to prove the existence result, using the min-max procedure of Bahri and Li. Indeed they define

$$h_0 : \overline{B(0,R)} \to \Sigma^+, \quad h_0(x_1) = \frac{\omega(\cdot - x_1)}{\|\omega(\cdot - x_1)\|},$$

$$\Gamma = \{h \in C(\overline{B(0,R)}, \Sigma^+) : h_{|\partial B(0,R)} = h_0\}$$

and

$$c_0 = \inf_{h\in\Gamma} \max_{y\in\overline{B(0,R)}} J(h(y)),$$

and they prove that $S_1 < c_0 < S_2$. The estimate $S_1 < c_0$ is performed using a baricenter-type function and a topological degree argument, while the energy estimates ensure that $c_0 < S_2$.

Proof of Theorem 2. The same procedure described above can be applied in our case using the energy estimates contained in Proposition 6. In this way we find a min-max value at a level $c_0 \in]S_1, S_2[$ which is a compactness level by means of Theorem 3. □

References

[ABC] A. AMBROSETTI, M. BADIALE, S. CINGOLANI, Semiclassical states of nonlinear Schrödinger equations, *Arch. Rational Mech. Anal.* **140** (1997), 285–300.

[BC] M. BADIALE, G. CITTI, Concentration compactness principle and quasilinear elliptic equations in $\mathbb{R}^n$, *Comm. Partial Differential Equations* **16** (1991), 1795–1818.

[BL] A. BAHRI, Y.Y. LI, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, *Rev. Mat. Iberoamericana* **6** (1990), 1–15.

[BLn] A. BAHRI, P.L. LIONS, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), 365–413.

[BeC] V. BENCI, G. CERAMI, Positive solutions of some nonlinear elliptic problems in exterior domains, *Arch. Rational Mech. Anal.* **99** (1987), 283–300.

[BeL] H. BERESTYCKI, P.L. LIONS, Nonlinear scalar field equations, I: Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983), 313–345.

[C1] G. CITTI, Positive solutions for a quasilinear degenerate elliptic equation in R^N, *Rend. Circ. Mat. Palermo* (2) **35** (1986), 364–375.

[C2] G. CITTI, A uniqueness theorem for radial ground states of the equation $\Delta_p u + f(u) = 0$, *Boll. Un. Mat. Ital.* (7) **7**-B (1993), 283–310.

[D] E. DI BENEDETTO, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), 827–850.

[DF] M. DEL PINO, P.L. FELMER, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* **4** (1996), 121–137.

[DN] W.Y. DING, W.M. NI, On the existence of positive entire solutions of a semilinear elliptic equation, *Arch. Rational Mech. Anal.* **91** (1986), 283–308.

[DPR] L. DAMASCELLI, F. PACELLA, M. RAMASWAMY, Symmetry of ground states of p-Laplace equations via the moving plane method, *Arch. Rational Mech. Anal.* **148** (1999), 291–308.

[E] H. EGNELL, Existence results for some quasilinear elliptic equations, Variational methods, *Proc. Conf., Paris/Fr. 1988, Prog. Nonlinear Differ. Equ. Appl.* **4** (1990), 61–76.

[GA] J.V. GONCALVES, C.O. ALVES, Existence of positive solutions for m-Laplacian equations in $\mathbb{R}^N$ involving critical Sobolev exponents, *Nonlinear Anal.* **32** (1998), 53–70.

[GNN] B. GIDAS, W.M. NI, L. NIRENBERG, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Adv. Math., Suppl. Stud.* **7A** (1981), 369–402.

[GS] L. GONGBAO, Y. SHUSEN, Eigenvalue problems for quasilinear elliptic equations on $\mathbb{R}^N$, *Comm. Partial Differential Equations* **14** (1989), 1291–1314.

[GST] F. GAZZOLA, J. SERRIN, M. TANG, Existence of ground states and free boundary problems for quasilinear elliptic operators, *Adv. Differential Equations* **5** (2000), 1–30.

[L1] P.L. LIONS, The concentration-compactness principle in the calculus of variations. The locally compact case, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 109–145, 223–283.

[L2] P.L. LIONS, On positive solutions of semilinear elliptic equations in unbounded domains, Nonlinear diffusion equations and their equilibrium states II, *Proc. Microprogram, Berkeley/Calif. 1986, Publ., Math. Sci. Res. Inst.* **13** (1988), 85–122.

[NSJ] E.S. NOUSSAIR, C.A. SWANSON, Y. JIANFU, Quasilinear elliptic problems with critical exponents, *Nonlinear Anal.* **20** (1993), 285–301.

[O] J.M.B. DO Ó, Solutions to perturbed eigenvalue problems of the p-Laplacian in $\mathbb{R}^N$, *Electron. J. Differential Equations* **1997** (1997), 1–15.

[PS] P. PUCCI, J. SERRIN, Uniqueness of ground states for quasilinear elliptic operators, *Indiana Univ. Math. J.* **47** (1998), 501–528.

[S] J. SERRIN, Local behavior of solutions of quasi-linear equations, *Acta Math.* **111** (1964), 247–302.

[ST] J. SERRIN, M. TANG, Uniqueness of ground states for quasilinear elliptic equations, preprint.

[SY] C.A. SWANSON, L.S. YU, Critical p-Laplacian problems in $\mathbb{R}^N$, *Ann. Mat. Pura Appl.* (4) **169** (1995), 233–250.

[V] J.L. VÁZQUEZ, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191–202.

[W] X. WANG, On concentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* **153** (1993), 229–244.

[Y] L.S. YU, Nonlinear p-Laplacian problems on unbounded domains, *Proc. Amer. Math. Soc.* **115** (1992), 1037–1045.

Received April 2000