

# A multiplicity result for a degenerate-elliptic equation with critical growth on noncontractible domains \*

Elisa Garagnani & Francesco Uguzzoni

Dipartimento di Matematica, Università di Bologna  
Piazza di Porta S. Donato 5, I-40126 Bologna, Italy  
e-mail: garagnan@dm.unibo.it, uguzzoni@dm.unibo.it

## Abstract

In this paper we consider the semilinear problem with critical growth in the Heisenberg group  $-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} + \lambda u$  in  $\Omega$ ,  $u > 0$  in  $\Omega$ ,  $u = 0$  in  $\partial\Omega$ , and we provide a multiplicity existence result involving Ljusternik-Schnirelmann category.

Keywords and phrases: multiplicity result, Heisenberg group, critical growth equation, Ljusternik-Schnirelmann category.

2000 Mathematics Subject Classification: 35H20, 43A80, 35B33, 35J70.

## 1 Introduction

We consider the critical boundary value problem

$$P_\lambda(\Omega) \quad \begin{cases} -\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where  $\Omega$  is a smooth bounded open subset of the Heisenberg group  $\mathbb{H}^n$ ,  $\Delta_{\mathbb{H}^n}$  is the subelliptic Laplacian (also called Kohn Laplacian) on  $\mathbb{H}^n$ ,  $Q = 2n + 2$  is the homogeneous dimension of  $\mathbb{H}^n$  and  $\lambda$  is a real parameter. In what follows we also denote by  $\lambda_1$  the first eigenvalue of  $-\Delta_{\mathbb{H}^n}$ .

When  $0 < \lambda < \lambda_1$ ,  $P_\lambda(\Omega)$  has at least a solution whatever the topology of  $\Omega$  is (see [8]). This is the Heisenberg-counterpart of a classical result by Brezis and Nirenberg [6] related to the Laplacian operator.

The aim of this paper is to show a first multiplicity result for  $P_\lambda(\Omega)$ . Indeed, we find at least  $m + 1$  solutions of  $P_\lambda(\Omega)$ , where  $m \equiv \text{cat}_\Omega(\Omega)$ , if  $\lambda$  is small enough and if  $\Omega$  is assumed to satisfy a suitable geometric condition of regularity, which, broadly speaking, means that  $\partial\Omega$  is “flat” near its characteristic points (see Definition 2 of  $\mathbb{H}$ -flat domains). Precisely, we prove the following result.

---

\*Investigation supported by University of Bologna. Funds for selected research topics.

**THEOREM 1** *Let  $\Omega$  be a  $\mathbb{H}$ -flat bounded domain of  $\mathbb{H}^n$  noncontractible in itself. Then there exists  $\bar{\lambda} \in (0, \lambda_1)$  such that for all  $\lambda \in (0, \bar{\lambda})$  problem  $P_\lambda(\Omega)$  has at least  $\text{cat}_\Omega(\Omega) + 1$  distinct solutions.*

Throughout the paper we denote by  $\text{cat}_X(Y)$  the Lyustenik-Schnirelman category of  $Y$  in  $X$ , i.e. the least nonnegative integer  $m$  such that  $Y$  can be covered by  $m$  closed and contractible subsets of  $X$ .

In order to prove the above theorem, we consider the functional  $f_\lambda(u) = \|\nabla_{\mathbb{H}^n} u\|_2^2 - \lambda \|u\|_2^2$  constrained to the manifold  $V = \{u \in S_0^1(\Omega) \mid \|u^+\|_{Q^*} = 1\}$ , whose critical points give solutions of  $P_\lambda(\Omega)$  (see below for the notations). In analogy with the Euclidean case in [21] and [16], we find  $m$  critical levels below the best Sobolev constant  $S$  (see (9)) corresponding at least to  $m$  distinct solutions (see Theorem 12). Moreover, we establish the existence of a further solution, as in [20] for the Euclidean setting. Indeed, in this paper, we find  $m - 1$  other critical levels greater than  $S$ , corresponding to different values of the category of a fixed set with respect to a varying ambient space (see Proposition 13). We expect that these critical levels give at least other  $m - 1$  distinct solutions (in addition to the one already found). However this is still an open problem, even in the setting of the classical Laplace operator.

The main tools in the proof of Theorem 1 are a representation theorem for Palais-Smale sequences (see Theorem 3) and some techniques introduced by Benci and Cerami in [1] and by Passaseo in [20]. We stress that the proof of the cited representation theorem is much more delicate than in the Euclidean setting and leads to the further  $\mathbb{H}$ -flat assumption for the domain  $\Omega$  (see also [9]).

For the reader convenience we now fix the main notation before recalling the background results related to the problem  $P_\lambda(\Omega)$ .

The Heisenberg group  $\mathbb{H}^n$  is the homogeneous Lie group whose underlying manifold is  $\mathbb{R}^{2n+1}$  with the group law defined by

$$(1) \quad \xi \cdot \xi' = (x + x', y + y', t + t' + 2(x'y - xy')),$$

for every  $\xi = (x, y, t)$ ,  $\xi' = (x', y', t') \in \mathbb{H}^n$ . The subelliptic Laplacian on  $\mathbb{H}^n$  is defined as

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^n (X_j^2 + Y_j^2), \quad \text{where } X_j = \partial_{x_j} + 2y_j \partial_t, \quad Y_j = \partial_{y_j} - 2x_j \partial_t.$$

Consider the left translations on  $\mathbb{H}^n$  and, for  $\lambda > 0$ , the natural  $\mathbb{H}$ -dilations so defined

$$(2) \quad \tau_\xi : \mathbb{H}^n \mapsto \mathbb{H}^n, \quad \tau_\xi(\xi') = \xi \cdot \xi'; \quad \delta_\lambda : \mathbb{H}^n \mapsto \mathbb{H}^n, \quad \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

Denoting by  $\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$  the subelliptic gradient on  $\mathbb{H}^n$ , then both  $\nabla_{\mathbb{H}^n}$  and  $\Delta_{\mathbb{H}^n}$  are invariant with respect to left translations and they are homogeneous (respectively of degree 1 and 2) with respect to the dilations. In other words we have  $\nabla_{\mathbb{H}^n}(u \circ \tau_\xi) = \nabla_{\mathbb{H}^n} u \circ \tau_\xi$ ,  $\nabla_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda \nabla_{\mathbb{H}^n} u \circ \delta_\lambda$ ,  $\Delta_{\mathbb{H}^n}(u \circ \tau_\xi) = \Delta_{\mathbb{H}^n} u \circ \tau_\xi$ ,  $\Delta_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{H}^n} u \circ \delta_\lambda$ . The Jacobian determinant of  $\delta_\lambda$  is  $\lambda^Q$ , where  $Q = 2n + 2$ . This number  $Q$  is called the homogeneous dimension of  $\mathbb{H}^n$  and it plays a role analogous to the topological dimension in the Euclidean case. The homogeneous norm of the space is

$$(3) \quad d_0(x, y, t) = ((|x|^2 + |y|^2)^2 + t^2)^{1/4}$$

and the natural distance is defined by  $d(\xi', \xi) = d_0(\xi^{-1} \cdot \xi')$ . We shall denote by  $B_d(\xi, r)$  the  $d$ -ball of center  $\xi$  and radius  $r$ . Notice that, by definition of  $d$ , we have  $\tau_\xi(B_d(0, r)) = B_d(\xi, r)$ ,  $\delta_r(B_d(0, 1)) = B_d(0, r)$ . A basic role is played by the following Sobolev-type inequality:

$$(4) \quad \|u\|_{Q^*}^2 \leq C \|\nabla_{\mathbb{H}^n} u\|_2^2, \quad \text{for all } u \in C_0^\infty(\mathbb{H}^n),$$

where  $Q^* = \frac{2Q}{Q-2}$  and  $C > 0$  only depends on the homogeneous dimension  $Q$ . This inequality ensures in particular that

$$(5) \quad \|u\| := \|\nabla_{\mathbb{H}^n} u\|_2$$

is a norm on  $C_0^\infty(\Omega)$ . We denote by  $S_0^1(\Omega)$  the closure of  $C_0^\infty(\Omega)$  with respect to this norm. Then with the inner product  $\langle u, v \rangle_{S_0^1(\Omega)} = \int_\Omega \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v \rangle$ ,  $S_0^1(\Omega)$  becomes a Hilbert space. Notice that the number  $Q^*$  in (4) is the critical Sobolev exponent for  $\Delta_{\mathbb{H}^n}$  since the embedding  $S_0^1(\Omega) \hookrightarrow L^{Q^*}(\Omega)$  is continuous but not compact, even if  $\Omega$  is bounded.

Following the arguments in [6] for the classical Laplacian case, it is easy to prove that if  $\lambda \geq \lambda_1$ , then  $P_\lambda(\Omega)$  has no solutions, where  $\lambda_1$  is the first eigenvalue of  $-\Delta_{\mathbb{H}^n}$  in  $S_0^1(\Omega)$ ,

$$(6) \quad \lambda_1 = \min_{\substack{u \in S_0^1(\Omega) \\ \|u\|_2=1}} \|\nabla_{\mathbb{H}^n} u\|_2^2.$$

The existence of a solution to  $P_\lambda(\Omega)$  is also strictly related to the topology and the geometry of  $\Omega$ . For instance, we refer to [13], where the notion of  $\delta$ -starshapedness is introduced. Let us define the vector field

$$X = \sum_{j=1}^n (x_j \partial_{x_j} + y_j \partial_{y_j}) + 2t \partial_t.$$

Then a piecewise  $C^1$  open set of  $\mathbb{H}^n$ ,  $\Omega \neq \mathbb{H}^n$ , is said to be  $\delta$ -starshaped with respect to a point  $\xi_0 \in \Omega$  if  $X \cdot N \geq 0$  at every point of  $\partial(\tau_{\xi_0^{-1}}(\Omega))$ , where  $N$  denote the outer unit normal to the boundary of  $\tau_{\xi_0^{-1}}(\Omega)$ . In [13], it is proved that if  $\Omega$  is  $\delta$ -starshaped and  $\lambda \leq 0$ , then  $P_\lambda(\Omega)$  has no solution. A remarkable fact is that in the same paper they give a first example of noncontractible domain, precisely  $\Omega = \{(z, t) \in \mathbb{H}^n \mid r_0 < |z| < r_1, |t| < T\}$  for fixed  $r_0, r_1, T > 0$ , in which  $P_\lambda(\Omega)$  has at least one solution for  $\lambda \leq 0$ .

The case  $\lambda = 0$ , i.e. the problem

$$P_0(\Omega) \quad \begin{cases} -\Delta_{\mathbb{H}^n} u = u^{Q^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

has been intensively studied. First of all in the case  $\Omega = \mathbb{H}^n$  Jerison and Lee found an explicit solution  $\omega$  of  $P_0(\mathbb{H}^n)$  (see [14], [15]) and proved that any other solution in  $S_0^1(\mathbb{H}^n)$  can be obtained from  $\omega$  by  $\mathbb{H}$ -dilations and left translations. On the other hand, if  $\Omega$  is a halfspace of  $\mathbb{H}^n$  then in [17, 23] it is shown that  $P_0(\Omega)$  has no solution in  $S_0^1(\Omega)$ . These uniqueness and non existence results allow the proof of a representation theorem based on the concentration compactness

principle. Since the exponent  $Q^*$  is the critical exponent, the Palais-Smale sequences of the functional naturally associated to the problem  $P_\lambda(\Omega)$  are in general not compact. In [9] the authors studied this loss of compactness for  $\lambda = 0$  and they proved that the nonnegative Palais-Smale sequences can be represented in terms of the solutions of the same problem on a different open subset  $\mathcal{D}$  of  $\mathbb{H}^n$ , called set at infinity. In the same paper, a complete description of such sets  $\mathcal{D}$  was given along with a definition of  $\mathbb{H}$ -flat domains for which the sets at infinity can only be the whole space  $\mathbb{H}^n$  or a halfspace. In conclusion, since in a halfspace there is no solution while in the whole space all the solutions are known, one can obtain a complete characterization of the compactness levels for a  $\mathbb{H}$ -flat domain (see [9, Theorem 3.5]). This allows the authors in [9] to prove a Bahri-Coron type existence result for the problem  $P_0(\Omega)$ . In [9] it is also proved that there exist contractible domains in which  $P_0(\Omega)$  has solution. As it is well known, the above mentioned characterization is crucial in the proof of the existence results for semilinear problems with critical growth. For this reason in Section 2 we sketch the proof of an analogous representation theorem for a functional related to  $P_\lambda(\Omega)$ . Then, in Section 3, properties of the functional  $f_\lambda$  constrained to  $V$  are studied and in Section 4 we prove our main results.

We would like to end this introduction by citing the recent papers [5, 4, 7, 18, 2, 3, 11, 12, 19] where related topics on the Heisenberg group are investigated.

## 2 Representation Theorem

Let us first recall the definition of  $\mathbb{H}$ -flat domain introduced in [9].

**DEFINITION 2** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{H}^n$ ,  $\xi_0 \in \partial\Omega$  and  $\varphi$  be a smooth function which describes  $\partial\Omega$  in a neighborhood of  $\xi_0$ , i.e.  $\varphi : B_d(\xi_0, R) \mapsto \mathbb{R}$  such that  $\varphi(\xi) = 0$  iff  $\xi \in \partial\Omega \cap B_d(\xi_0, R)$ ,  $\varphi(\xi) > 0$  iff  $\xi \in \Omega \cap B_d(\xi_0, R)$ ,  $\nabla\varphi(\xi) \neq 0$  for all  $\xi \in B_d(\xi_0, R)$  ( $\nabla$  denotes the Euclidean gradient). The point  $\xi_0$  is called characteristic if  $\nabla_{\mathbb{H}^n}\varphi(\xi_0) = 0$ . In this case,  $\Omega$  is called  $\mathbb{H}$ -flat at  $\xi_0$  if*

$$q_{\mathbb{H}}\varphi(\xi_0) = 0$$

where  $q_{\mathbb{H}}\varphi(\xi_0)$  is the quadratic form associated to the Hessian matrix  $\nabla_{\mathbb{H}^n}^2\varphi$  along the vector fields of the subelliptic gradient  $\nabla_{\mathbb{H}^n}$ , i.e.

$$(q_{\mathbb{H}}\varphi(\xi_0))(z) = \sum_{i,j=1}^{2n} ((\nabla_{\mathbb{H}^n})_i(\nabla_{\mathbb{H}^n})_j\varphi)(\xi_0)z_i z_j \quad \text{for all } z \in \mathbb{R}^{2n},$$

where  $(\nabla_{\mathbb{H}^n})_i$  denotes the  $i$ -th component of  $\nabla_{\mathbb{H}^n}$ . We say that  $\Omega$  is  $\mathbb{H}$ -flat if it is  $\mathbb{H}$ -flat at any characteristic point of its boundary.

If  $u \in S_0^1(\Omega)$ ,  $\mu > 0$ ,  $\xi \in \mathbb{H}^n$ , we will denote

$$(7) \quad u_{\mu\xi} = \mu^{\frac{Q-2}{2}} u \circ \delta_\mu \circ \tau_{\xi^{-1}}.$$

Then  $u_{\mu\xi} \in S_0^1(\tau_\xi(\delta_{\mu^{-1}}(\Omega)))$ ,  $\|u_{\mu\xi}\|_{Q^*} = \|u\|_{Q^*}$  and  $\|\nabla_{\mathbb{H}^n} u_{\mu\xi}\|_2 = \|\nabla_{\mathbb{H}^n} u\|_2$ .

For the case  $\lambda = 0$ , Jerison and Lee found an explicit solution of the problem

$$P_0(\Omega) \quad \begin{cases} -\Delta_{\mathbb{H}^n} u = u^{Q^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

when  $\Omega = \mathbb{H}^n$  (see [14], [15]). Precisely, let  $K : S_0^1(\mathbb{H}^n) \setminus \{0\} \mapsto \mathbb{R}$  be the functional

$$(8) \quad K(u) = \frac{\|\nabla_{\mathbb{H}^n} u\|_2^2}{\|u\|_{Q^*}^2}$$

and

$$(9) \quad S = \inf_{S_0^1(\mathbb{H}^n) \setminus \{0\}} K.$$

Then, by (4), we have  $S > 0$ .  $S$  is called the best Sobolev constant for the embedding of  $S_0^1(\Omega)$  in  $L^{Q^*}(\Omega)$ . Moreover  $S = \inf_{S_0^1(\Omega) \setminus \{0\}} K$  for all non empty open set  $\Omega \subseteq \mathbb{H}^n$  and the infimum is achieved when  $\Omega = \mathbb{H}^n$ , while this does not happen when  $\Omega \neq \mathbb{H}^n$ . Indeed Jerison and Lee proved that, up to a positive constant  $C$ , the function

$$(10) \quad \omega(\xi) = \frac{C}{((1 + |x|^2 + |y|^2)^2 + t^2)^{\frac{Q-2}{4}}}$$

is such that  $K(\omega) = K(\omega_{\mu\xi}) = S$ . Moreover  $\omega$  is the unique solution of the problem  $P_0(\Omega)$  in  $\Omega = \mathbb{H}^n$  (in the sense that all other solutions are of the form  $\omega_{\mu\xi}$ ).

We recall that in [9, Theorem 3.5] a representation theorem for Palais-Smale sequences of the functional  $F(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_{\Omega} |u|^{Q^*}$  associated to  $P_0(\Omega)$  is proved when  $\Omega$  is  $\mathbb{H}$ -flat. We shall now consider these functionals

$$(11) \quad F_{\lambda} : S_0^1(\Omega) \mapsto \mathbb{R}, \quad F_{\lambda}(u) = \frac{1}{2} \int_{\Omega} (|\nabla_{\mathbb{H}^n} u|^2 - \lambda u^2) - \frac{1}{Q^*} \int_{\Omega} |u^+|^{Q^*},$$

$$(12) \quad F_0 : S_0^1(\Omega) \mapsto \mathbb{R}, \quad F_0(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_{\Omega} |u^+|^{Q^*},$$

$$(13) \quad F_{\infty} : S_0^1(\mathbb{H}^n) \mapsto \mathbb{R}, \quad F_{\infty}(u) = \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_{\mathbb{H}^n} |u^+|^{Q^*}.$$

Our main goal of this section is to prove the following representation theorem for Palais-Smale sequences of  $F_{\lambda}$ .

**THEOREM 3** *Let  $\Omega$  be a  $\mathbb{H}$ -flat domain of  $\mathbb{H}^n$  and let  $(u_k)$  be a sequence in  $S_0^1(\Omega)$  such that*

$$F_{\lambda}(u_k) \rightarrow l \quad \text{and} \quad F'_{\lambda}(u_k) \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

*Then there exist a function  $u_0 \in S_0^1(\Omega)$ ,  $F'_{\lambda}(u_0) = 0$ , an integer  $m \geq 0$ ,  $m$  divergent sequences  $(\lambda_{1k}), \dots, (\lambda_{mk})$  in  $\mathbb{R}^+$  and  $m$  sequences  $(\xi_{1k}), \dots, (\xi_{mk})$  in  $\Omega$  such that (up to subsequences)*

$$(i) \quad u_k = u_0 + \sum_{i=1}^m \omega_{\lambda_{ik}\xi_{ik}} + o(1) \quad \text{in } S_0^1(\mathbb{H}^n) \quad \text{as } k \rightarrow +\infty,$$

$$(ii) \quad F_{\lambda}(u_k) = F_{\lambda}(u_0) + mF_{\infty}(\omega) + o(1), \quad \text{as } k \rightarrow +\infty,$$

*where  $\omega_{\lambda_{ik}\xi_{ik}}$  is according to (7) and  $\omega$  is defined in (10). We observe that  $F_{\infty}(\omega) = \frac{1}{Q} S^{\frac{Q}{2}}$ .*

**REMARK 4** *The hypothesis  $\Omega$   $\mathbb{H}$ -flat allows to characterize the problems at infinity and to obtain a complete description of the compactness levels of the Palais-Smale sequences. Without that hypothesis, one can still obtain existence results but only at low levels (as in [8]).*

The proof is based on standard techniques already adopted on proving analogous theorems present in literature. But, since Theorem 3 is not directly referable to the representation Theorems in [8], [9], neither to the results in the Euclidean case in [22], [1], it is opportune to give a sketch of the proof and underline the main differences. The first lemma allows us to consider Palais-Smale sequences of  $F_0$  weakly convergent to 0 instead of Palais-Smale sequences of  $F_\lambda$ .

LEMMA 5 *Let  $u_k$  be as in Theorem 3. Then there exist  $u_0 \in S_0^1(\Omega)$  such that (taking a subsequence if necessary)  $u_k \rightharpoonup u_0$  weakly in  $S_0^1(\Omega)$  and  $F'_\lambda(u_0) = 0$ . Moreover  $v_k \equiv u_k - u_0$  is a Palais-Smale sequence for  $F_0$ , precisely*

$$(i) \quad F_0(u_k - u_0) = F_\lambda(u_k) - F_\lambda(u_0) + o(1) ,$$

$$(ii) \quad F'_0(u_k - u_0) = o(1),$$

$$(iii) \quad \|\nabla_{\mathbb{H}^n}(u_k - u_0)\|_2^2 = \|\nabla_{\mathbb{H}^n} u_k\|_2^2 - \|\nabla_{\mathbb{H}^n} u_0\|_2^2 + o(1),$$

We refer to [1] or [22] for the proof of an analogous proposition.

If  $\xi \in \mathbb{H}^n$  and  $A$  is a subset of  $\mathbb{H}^n$ , we denote  $d(\xi, A) = \inf\{d(\xi, a) | a \in A\}$ .

LEMMA 6 *Let  $\Omega$  be a  $\mathbb{H}$ -flat domain of  $\mathbb{H}^n$  and let  $(v_k)$  be a sequence in  $S_0^1(\Omega)$  such that  $v_k \rightharpoonup 0$  weakly but  $v_k \not\rightarrow 0$  in  $S_0^1(\Omega)$  and  $F'_0(v_k) \rightarrow 0$ . Then there exist a sequence  $(\xi_k)$  in  $\Omega$ , a divergent sequence  $(\lambda_k)$  in  $\mathbb{R}^+$ ,  $\lambda_k d(\xi_k, \partial\Omega) \rightarrow \infty$ , and a function  $\varpi$  in  $S_0^1(\mathbb{H})$  such that  $F'_\infty(\varpi) = 0$  and*

$$(14) \quad \tilde{v}_k \equiv \lambda_k^{-\frac{Q-2}{2}} v_k \circ \tau_{\xi_k} \circ \delta_{\lambda_k}^{-1} \longrightarrow \varpi \quad \text{weakly in } S_0^1(\Omega).$$

*Proof.* The construction of the sequences  $(\lambda_k)$ ,  $(\xi_k)$  and of the function  $\varpi$  solution of the problem  $P_\infty^+(\mathcal{D}) : -\Delta_{\mathbb{H}^n} u = (u^+)^{Q^*-1}$  in  $\mathcal{D}$  (set at infinity related to  $\Omega$ , i.e.  $\mathcal{D}$  is obtained as limit of the subsequence of sets  $\Omega_k = \delta_{\lambda_k}(\tau_{\xi_k}^{-1}(\Omega))$ ), can be made as in [8, Lemma 2.3]. However in our case  $\Omega$  is  $\mathbb{H}$ -flat, so  $\mathcal{D} = \mathbb{H}^n$  or  $\mathcal{D}$  is a halfspace. The last case is not possible because we already pointed out that  $P_\infty^+(\mathcal{D})$  has no solution in an halfspace (see [17] and [23]). So we can conclude that  $\mathcal{D} = \mathbb{H}^n$  and  $F'_\infty(\varpi) = 0$ . We explicitly notice that in this case the hypothesis of  $v_k$  to be nonnegative is not necessary because  $\varpi > 0$  is consequence of our definition of  $F_\infty$ . In fact, because of  $F'_\infty(\varpi)h = \int_{\mathbb{H}^n} \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} h \rangle - \int_{\mathbb{H}^n} |u^+|^{Q^*-1} h$ , choosing as test function  $h = \varpi^-$  in  $F'_\infty(\varpi)h = 0$ , it is easy to conclude that  $\varpi^- \equiv 0$ .  $\square$

LEMMA 7 *Let  $v_k, \lambda_k, \xi_k, \varpi$  be as in Lemma 6. Then it is possible to consider a projection  $P_k : S_0^1(\mathbb{H}) \mapsto S_0^1(\Omega)$  (that can depend on  $k$ ) with the following properties*

$$(i) \quad \|\nabla_{\mathbb{H}^n}(P_k \varpi_{\lambda_k \xi_k})\|_2^2 = o(1),$$

$$(ii) \quad \|\nabla_{\mathbb{H}^n} v_k^{(2)}\|_2^2 = \|\nabla_{\mathbb{H}^n} v_k\|_2^2 - \|\nabla_{\mathbb{H}^n} \varpi\|_2^2 + o(1), \quad \text{where } v_k^{(2)} = v_k - P_k \varpi_{\lambda_k \xi_k},$$

$$(iii) \quad F_0(v_k^{(2)}) = F_0(v_k) - F_\infty(\varpi) + o(1),$$

$$(iv) \quad F'_0(v_k^{(2)}) = o(1)$$

*Proof.* Even if our metric is different from the Euclidean one, the idea introduced in [22] can be adopted also in this case. In particular we choose  $\varphi \in C_0^\infty(\mathbb{H}^n)$ ,  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  in  $B_d(0, 1)$ ,  $\varphi \equiv 0$  in  $\mathbb{H}^n - B_d(0, 2)$  and we denote  $\bar{\lambda}_k \equiv \lambda_k^{1/2}(d(\xi_k, \partial\Omega))^{1/2}$ . Then we define  $P_k \varpi_{\lambda_k \xi_k}(\xi) = \varpi_{\lambda_k \xi_k}(\xi) (\varphi \circ \delta_{\bar{\lambda}_k} \circ \tau_{\xi_k^{-1}})(\xi)$ . Using the invariance with respect to left translations of the subelliptic gradient, its homogeneous property with respect to dilations and (7), as in [22], all the properties can be verified.  $\square$

From these results, Theorem 3 easily follows.

*Proof of Theorem 3.* Using a standard iteration (see [8] or [22]) based on Lemma 6 and on the properties of Lemma 7, it is easy to find  $m$  solutions  $\omega^{(1)}, \dots, \omega^{(m)}$  of the problem at infinity  $F'_\infty(u) = 0$  such that  $u_k = u_0 + \sum_{i=1}^m \omega_{\lambda_{ik} \xi_{ik}} + o(1)$  in  $S_0^1(\mathbb{H}^n)$  as  $k \rightarrow +\infty$ . However we explicitly observe that if  $\omega^{(i)}$  is such that  $F'_\infty(\omega^{(i)}) = 0$ , then  $\omega^{(i)} > 0$  and so it is a solution of  $P_\infty(\mathbb{H}^n)$  and, in particular, it can be obtained from  $\omega$  by translations and dilations. In conclusion, noting that  $F_\infty(\omega_{\lambda_{ik} \xi_{ik}}^{(i)}) = F_\infty(\omega^{(i)}) = F_\infty(\omega)$ , we get the thesis.  $\square$

### 3 Preliminaries

Let  $\Omega$  be a bounded domain of  $\mathbb{H}^n$  and let  $f_\lambda : S_0^1(\Omega) \mapsto \mathbb{R}$  be the functional defined by

$$(15) \quad f_\lambda(u) = \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 - \lambda \int_\Omega u^2.$$

We consider the following  $C^2$ -manifold of  $S_0^1(\Omega)$

$$(16) \quad V = \{u \in S_0^1(\Omega) \mid \|u^+\|_{Q^*} = 1\}.$$

Then there is a correspondence between critical points of the restriction  $f_{\lambda|V}$  and solutions of the problem  $P_\lambda(\Omega)$ . More precisely

REMARK 8 *If  $\lambda \in (0, \lambda_1)$ , then  $u$  is a solution of  $P_\lambda(\Omega)$  iff  $\bar{u} = \frac{u}{\|u\|_{Q^*}}$  is a critical point of  $f_{\lambda|V}$  and  $u = (f_\lambda(\bar{u}))^{\frac{Q-2}{4}} \bar{u}$ .*

This fact becomes clear if we observe that every critical point  $\bar{u}$  of  $f_{\lambda|V}$  is nonnegative. In fact there exists  $\mu \in \mathbb{R}$  such that  $\Delta_{\mathbb{H}^n} \bar{u} + \lambda \bar{u} + \mu(\bar{u}^+)^{Q^*-1} = 0$ , but this implies (multiplying for  $\bar{u}^-$ , integrating and by definition of  $\lambda_1$ ) that  $(\lambda - \lambda_1) \|\bar{u}^-\|_2^2 \leq 0$ . So, since  $\lambda < \lambda_1$ ,  $\bar{u}^- \equiv 0$ . Then  $\bar{u} \geq 0$ .

PROPOSITION 9 *Denoting by  $S_\lambda = \inf_V f_\lambda$ , we have*

$$0 < S_\lambda < S \quad \text{for all } \lambda \in (0, \lambda_1).$$

*Proof.* First of all we notice that if  $u \in V$ , then  $1 = \|u^+\|_{Q^*}^2 \leq C \|\nabla_{\mathbb{H}^n} u^+\|_2^2 \leq C \|\nabla_{\mathbb{H}^n} u\|_2^2$  and so  $\|\nabla_{\mathbb{H}^n} u\|_2^2 \geq C^{-1} > 0$ . This implies

$$(17) \quad 0 < f_\lambda(u^+) \leq f_\lambda(u), \quad \text{for all } u \in S_0^1(\Omega) \text{ and } \lambda \in (0, \lambda_1).$$

In fact  $f_\lambda(u) = \|\nabla_{\mathbb{H}^n} u\|_2^2 \left(1 - \lambda \frac{\|u\|_2^2}{\|\nabla_{\mathbb{H}^n} u\|_2^2}\right) \geq C^{-1} \left(1 - \frac{\lambda}{\lambda_1}\right) > 0$ . In conclusion  $S_\lambda > 0$ . In order to prove that  $S_\lambda < S$ , the function  $\omega$  plays a basic role. In fact the following lemma shows that

if we multiply the functions  $\omega_{\mu 0}$  (see (7)) for a suitable cut-off function  $\varphi$ , then  $v_\mu = \varphi \omega_{\mu 0}$  is such that

$$(18) \quad f_\lambda \left( \frac{v_\mu}{\|v_\mu\|_{Q^*}} \right) < S, \quad \text{if } \mu \gg 1.$$

In particular,  $S_\lambda < S$ .  $\square$

LEMMA 10 *Let  $R > 0$  such that  $B_d(0, R) \subset \Omega$  (supposing  $0 \in \Omega$ ) and  $\varphi \in C_0^\infty(B_d(0, R))$ ,  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  in  $B_d(0, \frac{R}{2})$ . We consider the functions*

$$(19) \quad v_\mu = \varphi \omega_{\mu 0}$$

where  $\mu > 0$ ,  $\omega$  is defined in (10) and  $\omega_{\mu 0} = \mu^{\frac{Q-2}{2}} \omega \circ \delta_\mu$  in according with (7). Then

$$(20) \quad f_\lambda \left( \frac{v_\mu}{\|v_\mu\|_{Q^*}} \right) \leq \begin{cases} S - \frac{\lambda}{c\mu^2} (1 + o(1)) \text{ as } \mu \rightarrow +\infty, & \text{if } Q > 4 \\ S - \frac{\lambda}{c\mu^2} \log \mu (1 + o(1)) \text{ as } \mu \rightarrow +\infty, & \text{if } Q = 4 \end{cases}$$

for some positive constants  $c$ .

*Proof.* The main point of this proof is the strict relation between the functions  $\omega$  and the norm  $d_0$ . In fact these functions have the same behavior far from the origin. This fact allows us to use Heisenberg-polar coordinates. We recall in fact that for every  $0 \leq r_1 < r_2$  and for every measurable  $f : [r_1, r_2] \mapsto \mathbb{R}$ , we have

$$(21) \quad \int_{B_d(0, r_2) \setminus B_d(0, r_1)} f(d_0(\xi)) d\xi = Q |B_d(0, 1)| \int_{r_1}^{r_2} f(\rho) \rho^{Q-1} d\rho,$$

if at least one of the two integrals exists (we denote by  $|\cdot|$  the Lebesgue measure on  $\mathbb{R}^{2n+1}$ ).

Let us first consider  $\|v_\mu\|_2^2$

$$\begin{aligned} \|v_\mu\|_2^2 &\geq \int_{B_d(0, \frac{R}{2})} \omega_{\mu 0}^2 = \frac{1}{\mu^2} \int_{B_d(0, \frac{R}{2})} \omega^2 \geq \frac{1}{c\mu^2} \left( 1 + \int_{B_d(0, \frac{R}{2}) \setminus B_d(0, 1)} \frac{1}{(d_0(\xi))^{2Q-4}} d\xi \right) \\ &= \frac{1}{c\mu^2} \left( 1 + \int_1^{\mu \frac{R}{2}} \frac{1}{\rho^{Q-3}} d\rho \right). \end{aligned}$$

Then, if  $Q > 4$ ,  $\|v_\mu\|_2^2 = \frac{1}{c\mu^2} (1 + o(1))$  and, if  $Q = 4$ ,  $\|v_\mu\|_2^2 = \frac{1}{c\mu^2} \log \mu (1 + o(1))$ .

Concerning  $\|v_\mu\|_{Q^*}^{Q^*}$ , we have  $\|v_\mu\|_{Q^*}^{Q^*} = \int \omega_{\mu 0}^{Q^*} - \int (1 - \varphi^{Q^*}) \omega_{\mu 0}^{Q^*} = S^{\frac{Q}{2}} - \int (1 - \varphi^{Q^*}) \omega_{\mu 0}^{Q^*}$ .

Arguing as before we obtain

$$\begin{aligned} 0 \leq \int (1 - \varphi^{Q^*}) \omega_{\mu 0}^{Q^*} &\leq \int_{d_0(\xi) > \frac{\mu R}{2}} \omega_{\mu 0}^{Q^*} d\xi = \int_{d_0(\eta) > \frac{\mu R}{2}} \omega^{Q^*} d\eta \leq \frac{1}{c} \int_{d_0(\eta) > \frac{\mu R}{2}} \frac{1}{d_0(\eta)^{2Q}} d\eta \\ &= \frac{1}{c} \int_{\frac{\mu R}{2}}^\infty \frac{1}{\rho^{Q+1}} d\rho = O(\mu^{-Q}) \text{ as } \mu \rightarrow +\infty. \end{aligned}$$

This proves that  $\|v_\mu\|_{Q^*}^{Q^*} = S^{\frac{Q}{2}} + O(\mu^{-Q})$ .

Let us now estimate  $\|\nabla_{\mathbb{H}^n} v_\mu\|_2^2$ , using the fact that  $\omega_{\mu 0}$  is a solution of  $P_\infty(\mathbb{H}^n)$ ,

$$\int |\nabla_{\mathbb{H}^n} v_\mu|^2 = \int \langle \nabla_{\mathbb{H}^n} \omega_{\mu 0}, \nabla_{\mathbb{H}^n} (\omega_{\mu 0} \varphi^2) \rangle + \int |\nabla_{\mathbb{H}^n} \varphi|^2 \omega_{\mu 0}^2 = \int \omega_{\mu 0}^{Q^*} \varphi + \int |\nabla_{\mathbb{H}^n} \varphi|^2 \omega_{\mu 0}.$$

Finally, following the lines of the other estimates, we obtain

$$\|\nabla_{\mathbb{H}^n} v_\mu\|_2^2 = S^{\frac{Q}{2}} + O(\mu^{-Q}) + O(\mu^{-Q+2}) = S^{\frac{Q}{2}} + O(\mu^{-Q+2})$$

Consequently, if  $Q > 4$ ,

$$\begin{aligned} f_\lambda \left( \frac{v_\mu}{\|v_\mu\|_{Q^*}} \right) &= \frac{\|\nabla_{\mathbb{H}^n} v_\mu\|_2^2 - \lambda \|v_\mu\|_2^2}{\|v_\mu\|_{Q^*}^2} \leq \frac{S^{\frac{Q}{2}} - \frac{\lambda}{c\mu^2}(1 + o(1))}{S^{\frac{Q-2}{2}}(1 + O(\mu^{-Q}))^{\frac{Q-2}{Q}}} \\ (22) \quad &= \left( S - \frac{\lambda}{c\mu^2}(1 + o(1)) \right) (1 + O(\mu^{-Q}))^{\frac{Q-2}{Q}} \\ &= \left( S - \frac{\lambda}{c\mu^2}(1 + o(1)) \right) \left( 1 + \frac{2-Q}{Q} O(\mu^{-Q}) \right) (1 + o(1)) \\ &= S - \frac{\lambda}{c\mu^2}(1 + o(1)). \end{aligned}$$

Arguing in the same way we also obtain the estimate for the case  $Q = 4$ .  $\square$

Our last goal for this section is to exhibit some levels where  $f_{\lambda|V}$  satisfies the Palais-Smale condition. Precisely,

**THEOREM 11** *For all  $\lambda \in (0, \lambda_1)$  and for all  $c \in \mathbb{R}$  such that  $0 < c < (S_\lambda^{\frac{Q}{2}} + S^{\frac{Q}{2}})^{\frac{2}{Q}}$  and  $c \neq S$ , if  $(u_k)$  is a sequence in  $V$  satisfying*

$$f_\lambda(u_k) \rightarrow l > 0 \quad \text{and} \quad f'_{\lambda|V}(u_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

*then there exists a subsequence of  $(u_k)$  that converges in  $S_0^1(\Omega)$ .*

The proof of the analogous result in the Euclidean case, given in [20], is also valid in our contest. But in order to apply those technique we have to use Theorem 3.

## 4 Multiplicity results

The aim of this section is to show that, if  $\lambda$  is small enough, then for noncontractible  $\mathbb{H}$ -flat domains it is possible to find more than one solution of  $P_\lambda(\Omega)$ . More precisely

**THEOREM 12** *Let  $\Omega$  be a bounded  $\mathbb{H}$ -flat domain of  $\mathbb{H}^n$ . If  $\Omega$  is noncontractible, i.e.  $m = \text{cat}_\Omega(\Omega) > 1$ , then there exists  $\bar{\lambda} \in (0, \lambda_1)$  such that, for all  $\lambda \in (0, \bar{\lambda})$ , the problem  $P_\lambda(\Omega)$  has a least  $m$  solutions  $u_{1,\lambda}, \dots, u_{m,\lambda} \in S_0^1(\Omega)$  satisfying  $f_\lambda(\bar{u}_{i,\lambda}) \in [S_\lambda, S[$ , where  $\bar{u}_{i,\lambda} = \frac{u_{i,\lambda}}{\|u_{i,\lambda}\|_{Q^*}}$ , for  $i = 1, \dots, m$ .*

Moreover we will find another solution in the following way. First of all, fixed  $r > 0$  small enough, consider these two sets homotopically equivalent to  $\Omega$

$$(23) \quad \Omega_r^+ = \{\xi \in \mathbb{H}^n / d(\xi, \Omega) \leq r\}, \quad \Omega_r^- = \{\xi \in \Omega / d(\xi, \partial\Omega) \geq r\}$$

and the map  $\psi_\mu : \Omega_r^- \mapsto V$  defined by

$$(24) \quad \psi_\mu(\xi) = \frac{v_\mu \circ \tau_{\xi^{-1}}}{\|v_\mu \circ \tau_{\xi^{-1}}\|_{Q^*}},$$

where  $v_\mu = \varphi \omega_{\mu 0}$  (see (19)). From now on, we will denote by  $f_\lambda^c$  the sublevels

$$f_\lambda^c = \{u \in V \mid f_\lambda(u) \leq c\},$$

for  $c \in \mathbb{R}$ , where  $V$  is the manifold defined in (16). We define the levels

$$(25) \quad \widehat{c}_{\lambda, \mu, k} = \inf \left\{ c \in \mathbb{R} \mid \text{cat}_{f_\lambda^c}(\psi_\mu(\Omega_r^-)) \leq k \right\}$$

for  $k \in \mathbb{N}$ ,  $1 \leq k \leq m - 1$ . We will prove next proposition that is closely inspired by Passaseo method in [20].

**PROPOSITION 13** *Let  $\Omega$  be a  $\mathbb{H}$ -flat domain of  $\mathbb{H}^n$ . If  $m = \text{cat}_\Omega(\Omega) > 1$ , then for all  $\lambda \in (0, \bar{\lambda})$  ( $\bar{\lambda}$  as in Theorem 12) there exists  $\widehat{\mu} > 0$  such that for all  $\mu > \widehat{\mu}$  we have*

$$(26) \quad \widehat{c}_{\lambda, \mu, k} \in \left[ S, \left( S_\lambda^{\frac{Q}{2}} + S^{\frac{Q}{2}} \right)^{\frac{2}{Q}} \right] \quad \text{for } k = 1, \dots, m - 1.$$

Moreover  $\widehat{c}_{\lambda, \mu, k}$  are  $m - 1$  critical levels of  $f_{\lambda|V}$ .

Probably these critical levels correspond to distinct solutions but this is still an open problem. However we can certainly conclude as follows

**COROLLARY 14** *In the same hypothesis of Theorem 12, there exists also at least one solution  $\widehat{u}_\lambda$  of  $P_\lambda(\Omega)$  such that  $f_\lambda \left( \frac{\widehat{u}_\lambda}{\|\widehat{u}_\lambda\|_{Q^*}} \right) \in \left[ S, \left( S_\lambda^{\frac{Q}{2}} + S^{\frac{Q}{2}} \right)^{\frac{2}{Q}} \right]$ .*

Let now start with the proof of Theorem 12. We define the function  $\beta : V \mapsto \mathbb{H}^n$  by

$$(27) \quad \beta(u) = \int_\Omega (u^+(\xi))^{Q^*} \xi \, d\xi.$$

**REMARK 15**  $\beta(\psi_\mu(\xi)) = \xi$  for all  $\xi \in \Omega_r^-$ .

*Proof.* Applying the group law (1), we get

$$\begin{aligned} \beta(\psi_\mu(\xi)) &= \|v_\mu\|_{Q^*}^{-Q^*} \int_{B_d(\xi, r)} (v_\mu(\xi^{-1} \circ \eta))^{Q^*} \eta \, d\eta = \|v_\mu\|_{Q^*}^{-Q^*} \int_{B_d(0, r)} (\xi \circ \eta') (v_\mu(\eta'))^{Q^*} \, d\eta' = \\ &= \|v_\mu\|_{Q^*}^{-Q^*} \left( \int_{B_d(0, r)} (z + z') v_\mu^{Q^*}(\eta') \, d\eta', \int_{B_d(0, r)} (t + t') v_\mu^{Q^*}(\eta') \, d\eta' + 2 \int_{B_d(0, r)} (x'y - xy') v_\mu^{Q^*}(\eta') \, d\eta' \right), \end{aligned}$$

where we denote  $\xi = (z, t) = (x, y, t)$  and  $\eta' = (z', t') = (x', y', t')$ . Now  $v_\mu^{-Q^*}$  is an even function, then we obtain  $\beta(\psi_\mu(\xi)) = \|v_\mu\|_{Q^*}^{-Q^*} \left( z \int_{B_d(0, r)} v_\mu^{Q^*}, t \int_{B_d(0, r)} v_\mu^{Q^*} \right) = (z, t) = \xi \quad \square$

The main step in the proof of Theorem 12 is the following result.

**LEMMA 16** *We have*

$$\lim_{\lambda \rightarrow 0^+} I_\lambda > S, \quad \text{where } I_\lambda \equiv \inf_{\substack{u \in V \\ \beta(u) \notin \Omega_r^+}} f_\lambda(u).$$

*Proof.* We give a direct proof of this lemma making use of the representation theorem cited in [9]. First of all we notice that this limit exists, in particular  $\lim_{\lambda \rightarrow 0^+} I_\lambda = \sup_{\lambda > 0} I_\lambda$ . By contradiction, consider a sequence  $(\mu_i)$  in  $\mathbb{R}^+$ ,  $\lim_{i \rightarrow +\infty} \mu_i = 0$ , and a sequence  $(u_i)$  in  $V$ ,

$\beta(u_i) \notin \Omega_r^+$  such that  $\limsup_{i \rightarrow +\infty} f_{\mu_i}(u_i) \leq S$ . Then, by (17),  $\limsup_{i \rightarrow +\infty} f_{\mu_i}(u_i^+) \leq S$ . Moreover  $\|\nabla_{\mathbb{H}^n} u_i^+\|_2^2 \geq S$  and (using Hölder inequality)  $\|u_i^+\|_2^2 \leq |\Omega|^{2/Q}$ . In conclusion we get

$$\lim_{i \rightarrow +\infty} \|\nabla_{\mathbb{H}^n} u_i^+\|_2^2 = S.$$

We prove that this implies  $\beta(u_i) \in \Omega_r^+$  for  $i \gg 1$ .

In what follows, for brevity, we will denote by  $\|\cdot\|$  the norm in  $S_0^1(\Omega)$  (see (5)) and by  $F$  the functional  $F_0$  defined in (12).

We fix  $\delta_1 > 0$  such that if  $u \in V, v \in S_0^1(\Omega), \|u - v\| \leq \delta_1$  then  $\beta(v) \in \Omega_{\frac{r}{2}}^+$  implies  $\beta(u) \in \Omega_r^+$ . This is possible; in fact, for example, fix  $R$  such that  $\Omega \subset B_d(0, R)$  and  $0 < \delta_1 \leq \frac{r}{C2^{Q^*}Q^*R}$  where  $C$  is the constant in (4) and suppose  $\|u - v\| \leq \delta_1$ . We have  $\beta'(u)h = Q^* \int_{\Omega} \xi(u^+)^{Q^*-1} h$ , for all  $h \in S_0^1(\Omega)$ . Then

$$|\beta'(u)h| \leq Q^* R \int_{\Omega} (u^+)^{Q^*-1} |h| \leq Q^* R \|u^+\|_{Q^*}^{Q^*-1} \|h\|_{Q^*} \leq CQ^* R \|u\|_{Q^*}^{Q^*-1} \|h\|.$$

In conclusion  $\|\beta'(u)\| \leq CQ^* R \|u\|_{Q^*}^{Q^*-1}$ . Moreover there exists  $t \in [0, 1]$  such that  $h = (1-t)v - tu$  satisfies  $d_0(\beta(v) - \beta(u)) \leq \|\beta'(h)\| \|v - u\| \leq CQ^* R \delta_1 \|h\|_{Q^*}^{Q^*-1}$ . But

$$\|h\|_{Q^*} \leq \|h - u\|_{Q^*} + \|u\|_{Q^*} = \|(1-t)(v - u)\|_{Q^*} + 1 \leq C\|v - u\| + 1 \leq C\delta_1 + 1$$

and so, since  $\delta_1 \leq \frac{r}{C2^{Q^*}Q^*R} \leq \frac{1}{C}$  because  $r$  is small,

$$|\beta(v) - \beta(u)| \leq CQ^* R \delta_1 (C\delta_1 + 1)^{Q^*-1} \leq CQ^* R \delta_1 2^{Q^*-1} \leq \frac{r}{2}.$$

Then, if  $\beta(v) \in \Omega_{\frac{r}{2}}^+$ ,  $|\beta(u)| \leq |\beta(v) - \beta(u)| + |\beta(v)| \leq r$ .

Consider now the functional  $K$  defined in (8). We extend  $K$  on  $S_0^1(\Omega)$  defining  $K(0) = S$ . Then  $K : S_0^1(\Omega) \mapsto \mathbb{R}$  is lower semi-continuous. Moreover, since  $u_i \in V$ , there exists a sequence  $(\varepsilon_i)$  in  $\mathbb{R}^+$ ,  $\lim_{i \rightarrow \infty} \varepsilon_i = 0$  such that  $K(u_i^+) = \|u_i^+\|^2 = S + \varepsilon_i$ . Applying Ekeland Theorem (see Theorem 1.1 in [10]) we can conclude that for all  $\delta > 0$  and for all  $i \in \mathbb{N}$  there exists  $v_i \in S_0^1(\Omega)$  such that

- (i)  $K(v_i) = K(u) + \frac{\varepsilon_i}{\delta} \|v_i - u\|$  for all  $u \in S_0^1(\Omega)$ ,
- (ii)  $K(v_i) \leq K(u_i^+)$ ,
- (iii)  $\|v_i - u_i^+\| \leq \delta$

From (i) and (ii), we have  $K(v_i) \rightarrow S$  and  $K'(v_i) \rightarrow 0$  for  $i \rightarrow +\infty$ . Define now

$$\bar{v}_i = \rho_i |v_i| \quad \text{where } \rho_i \equiv \|v_i\|^{\frac{Q-2}{2}} \|v_i\|_{Q^*}^{-\frac{Q}{2}}.$$

Then  $F(\bar{v}_i) = \frac{1}{Q} K(v_i)^{\frac{Q}{2}}$  and so  $\lim_{i \rightarrow \infty} F(\bar{v}_i) = \frac{1}{Q} S^{\frac{Q}{2}}$ . It is also easy to verify that  $\lim_{i \rightarrow \infty} F'(\bar{v}_i) = 0$ . Then  $\bar{v}_i$  is a Palais-Smale sequence of  $F$  at the level  $F_{\infty}(\omega) = \frac{1}{Q} S^{\frac{Q}{2}}$ . Then from representation Theorem 3.5 in [9], there exist a divergent sequence  $(\mu_i), \mu_i > 0$  and a sequence  $(\xi_i)$  in  $\Omega$  such that (taking a subsequence of  $\bar{v}_i$  if necessary)

$$\|\bar{v}_i - \omega_{\mu_i \xi_i}\| \longrightarrow 0 \quad \text{for } i \rightarrow +\infty.$$

Since  $\Omega$  is bounded, we can suppose  $\xi \rightarrow \xi_o$ . By use of (7) and Lebesgue Theorem, we obtain

$$\lim_{i \rightarrow +\infty} \int_{B_d(0,R)} \xi(\omega_{\mu_i \xi_i}(\xi))^{Q^*} d\xi = \xi_o S^{\frac{Q}{2}}.$$

Denoting by  $\tilde{h}_i = \bar{v}_i^+ - \omega_{\mu_i \xi_i}$  we have

$$\begin{aligned} \beta(\bar{v}_i) &= \int_{B_d(0,R)} \xi(\tilde{h}_i + \omega_{\mu_i \xi_i})^{Q^*} = \int_{B_d(0,R)} \xi \omega_{\mu_i \xi_i}^{Q^*} + \sum_{k=0}^{Q^*-1} c_k \int_{B_d(0,R)} \xi(\tilde{h}_i)^{Q^*-k} \omega_{\mu_i \xi_i}^k, \\ \left| \int_{B_d(0,R)} \xi(\tilde{h}_i)^{Q^*-k} \omega_{\mu_i \xi_i}^k \right| &\leq R \int_{B_d(0,R)} |\bar{v}_i - \omega_{\mu_i \xi_i}|^{Q^*-k} \omega_{\mu_i \xi_i}^k \\ &\leq R \|\bar{v}_i - \omega_{\mu_i \xi_i}\|_{Q^*}^{Q^*-k} \|\omega_{\mu_i \xi_i}\|_{Q^*}^k \\ &= CR \left( S^{\frac{Q-2}{4}} \right)^k \|\bar{v}_i - \omega_{\mu_i \xi_i}\|_{Q^*}^{Q^*-k} \rightarrow 0 \text{ for all } k = 0, \dots, Q^* - 1 \end{aligned}$$

and so  $\lim_{i \rightarrow +\infty} \beta(\bar{v}_i) = \xi_o S^{\frac{Q}{2}}$ . In particular

$$\lim_{i \rightarrow \infty} \frac{\beta(\bar{v}_i)}{K(v_i)^{\frac{Q}{2}}} = \xi_o.$$

We infer that this implies, if  $\delta$  is small enough, the existence of  $i_o \in \mathbb{N}$  such that  $|\beta(v_{i_o}) - \xi_o| < \frac{r}{2}$ . In fact, from (iii), follows  $\|v_i\|_{Q^*} \leq C\delta + 1$ , since

$$\| \|v_i\|_{Q^*} - 1 \| = \left| \|v_i\|_{Q^*} - \|u_i^+\|_{Q^*} \right| \leq \|v_i - u_i^+\|_{Q^*} \leq C \|v_i - u_i^+\| \leq C\delta.$$

Then

$$\begin{aligned} |\beta(v_i) - \xi_o| &= \left| \|v_i\|_{Q^*}^{\frac{Q}{Q-2}} \frac{\beta(\bar{v}_i)}{K(v_i)^{\frac{Q}{2}}} - \xi_o \right| = \left| \|v_i\|_{Q^*}^{\frac{Q}{Q-2}} \left( \frac{\beta(\bar{v}_i)}{K(v_i)^{\frac{Q}{2}}} - \xi_o \right) + \left( \|v_i\|_{Q^*}^{\frac{Q}{Q-2}} - 1 \right) \xi_o \right| \\ &= o(1) + R \left( (C\delta + 1)^{\frac{Q}{2}} - 1 \right) \text{ for } i \rightarrow +\infty. \end{aligned}$$

Taking now  $\delta < \delta_1$ ,  $\delta$  small enough and  $i_0$  large we obtain  $|\beta(v_{i_0}) - \xi_o| < \frac{r}{2}$ . This implies, up to a new choice of  $r$ , that also  $d(v_{i_0}, \xi_o) < \frac{r}{2}$ . By the choice of  $\delta_1$ , we get  $\beta(u_{i_0}) \in \Omega_r^+$ , in contradiction with the hypothesis.  $\square$

We are now able to prove Theorem 12.

*Proof of Theorem 12.* We set  $c_{\lambda, \mu} \equiv f_\lambda \left( \frac{v_\mu}{\|v_\mu\|_{Q^*}} \right)$ . Then, by Lemma 10,  $c_{\lambda, \mu} < S$  for  $\mu \gg 1$ .

Since Theorem 11, it is verified (P-S) condition for  $f_\lambda$  in  $f_\lambda^{c_{\lambda, \mu}}$ . Moreover, as in [6], using Lemma 16 and Remark 15, we can prove that

$$(28) \quad \text{cat}_{f_\lambda^c}(\psi(\Omega_r^-)) \geq \text{cat}_{\Omega_r^+}(\Omega_r^-) = \text{cat}_\Omega(\Omega) = m \quad \text{for all } c \in [c_{\lambda, \mu}, I_\lambda],$$

and so there exist almost  $m$  critical points of  $f_{\lambda|V}$  in  $f_\lambda^{c_{\lambda, \mu}}$ .  $\square$

We conclude with the

*Proof of Proposition 13.* For simplicity we write  $\widehat{c}_k$  instead of  $\widehat{c}_{\lambda,\mu,k}$ . Obviously we have  $\widehat{c}_k \leq \widehat{c}_{k-1}$ , for all  $k = 1, \dots, m-1$ . Moreover  $I_\lambda \leq \widehat{c}_{m-1}$ , otherwise there exist  $c \in [c_{\lambda,\mu}, I_\lambda[$  such that  $\widehat{c}_{m-1} \leq c$ . Then, by definition of  $\widehat{c}_{m-1}$  and by (28), we get  $cat_{f_\lambda^c}(\psi_\mu(\Omega_r^-)) \leq cat_{f_\lambda^{\widehat{c}_{m-1}}}(\psi_\mu(\Omega_r^-)) < cat_\Omega(\Omega) \leq cat_{f_\lambda^c}(\psi_\mu(\Omega_r^-))$ , that is a contradiction. Let now prove that  $\widehat{c}_k$  is an asymptotic critical level, for all  $k = 1, \dots, m-1$ . In fact, if  $\widehat{c}_k$  is not a critical level, we can consider  $\varepsilon \in (0, \widehat{c}_k - S)$  and a homeomorphism  $\eta : V \mapsto V$  such that

- (i)  $\eta(u) = u$  if  $|f_\lambda(u) - \widehat{c}_k| \geq \widehat{c}_k - S$ ,
- (ii)  $\eta(f_\lambda^{\widehat{c}_k + \varepsilon}) \subset f_\lambda^{\widehat{c}_k - \varepsilon}$ .

But  $\psi_\mu(\Omega_r^-) \subset f_\lambda^{\widehat{c}_k - \varepsilon}$ . Moreover (i) leads to the identity  $\eta(\psi_\mu(\Omega_r^-)) = (\psi_\mu(\Omega_r^-))$ . Consequently, using (ii),  $k \geq cat_{f_\lambda^{\widehat{c}_k + \varepsilon}}(\psi_\mu(\Omega_r^-)) = cat_{\eta(f_\lambda^{\widehat{c}_k + \varepsilon})}(\psi_\mu(\Omega_r^-)) \geq cat_{f_\lambda^{\widehat{c}_k - \varepsilon}}(\psi_\mu(\Omega_r^-))$ , in contradiction with the definition of  $\widehat{c}_k$ .

Finally, in order to prove that, when  $\mu \gg 1$ ,  $\widehat{c}_k \in \left] S, (S_\lambda^{\frac{Q}{2}} + S^{\frac{Q}{2}})^{\frac{2}{Q}} \right[$ , we follow the lines of [20, Lemma 3.6 and Remark 3.7] with minor modifications and using the estimates (20). We omit the computation for the sake of brevity. So, since Theorem 11,  $\widehat{c}_k$  are  $m-1$  critical levels of  $f_{\lambda|V}$ .  $\square$

## References

- [1] V. BENCI, G. CERAMI, *Existence of positive solutions of the equation  $-\Delta u + a(x)u = u^{(n+2)/(n-2)}$  in  $\mathbb{R}^N$* , J. Funct. Anal. **88** (1990), 90–117.
- [2] I. BIRINDELLI, I. CAPUZZO DOLCETTA, A. CUTRÌ, *Liouville theorems for semilinear equations on the Heisenberg group*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 295–308.
- [3] I. BIRINDELLI, I. CAPUZZO DOLCETTA, A. CUTRÌ, *Indefinite semi-linear equations on the Heisenberg group: a priori bounds and existence*, Comm. Partial Differential Equations **23** (1998), 1123–1157.
- [4] I. BIRINDELLI, A. CUTRÌ, *A semi-linear problem for the Heisenberg Laplacian*, Rend. Sem. Mat. Univ. Padova **94** (1995), 137–153.
- [5] S. BIAGINI, *Positive solutions for a semilinear equation on the Heisenberg group*, Boll. Un. Mat. Ital. (7) **9-B** (1995), 883–900.
- [6] H. BREZIS, L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.
- [7] L. BRANDOLINI, M. RIGOLI, A.G. SETTI, *On the existence of positive solutions of Yamabe-type equations on the Heisenberg group*, Duke Math. J. **2**, n.3 (1996), 101–107.
- [8] G. CITTI, *Semilinear Dirichlet problem involving critical exponent for the Kohn Laplacian*, Ann. Mat. Pura Appl. **169** (1995), 375–392.

- [9] G. CITTI, F. UGUZZONI, *Critical semilinear equations on the Heisenberg group: the effect of the topology of the domain*, *Nonlinear Anal.* **46** (2001), 399–417.
- [10] I. EKELAND, *On the variational principle*, *J. Math. Anal. Appl.* **47** (1974), 324–353.
- [11] N. GAMARA, *The CR Yamabe conjecture: the case  $n = 1$* , *J. Eur. Math. Soc.* **3** (2001), no. 2, 105–137.
- [12] N. GAMARA, R. YACOB, *CR Yamabe conjecture—the conformally flat case*, *Pacific J. Math.* **201** (2001), no. 1, 121–175.
- [13] N. GAROFALO, E. LANCONELLI, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, *Indiana Univ. Math. J.* **41** (1992), 71–98.
- [14] D.S. JERISON, J.M. LEE, *Intrinsic CR normal coordinates and the CR Yamabe problem*, *J. Differential Geom.* **29** (1989), 303–343.
- [15] D.S. JERISON, J.M. LEE, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, *J. Amer. Math. Soc.* **1** (1988), 1–13.
- [16] M. LAZZO, *Multiple positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, *C. R. Acad. Sci. Paris Sr. I Math. I* (1992), 61–64.
- [17] E. LANCONELLI, F. UGUZZONI, *Asymptotic behaviour and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, *Boll. Unione Mat. Ital.* (8) 1-B (1998), 139–168.
- [18] G. LU, J. WEI, *On positive entire solutions to the Yamabe-type problem on the Heisenberg and stratified groups*, *Electron. Res. Announc. Amer. Math. Soc.* **3** (1997), 83–89.
- [19] A. MALCHIODI, F. UGUZZONI, *A perturbation result for the Webster scalar curvature problem on the CR sphere*, *J. Math. Pures Appl* **81** (2002), 983–997.
- [20] D. PASSASEO, *Multiplicity of positive solutions for the equation  $\Delta u + \lambda u + u^{2^*-1} = 0$  in noncontractible domains*, *Topol. Methods Nonlinear Anal.* **2** (1993), 343–366.
- [21] O. REY, *A multiplicity result for a variational problem with lack of compactness*, *Nonlinear Anal.* **13** (1989), 1241–1249.
- [22] M. STRUWE, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, *Math. Z.* **187** (1984), 511–517.
- [23] F. UGUZZONI, *A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on halfspaces of the Heisenberg group*, *NoDEA Nonlinear Differential Equations Appl.* **6** (1999), 191–206.