Parameter Identification by Iterative Constrained Regularization

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Problem

Identification of the parameter q of a model (state equation)

$$c(u, q) = 0 \quad \text{ODE-PDE model}$$

Discrete reduced problem:

$$\min_{\mathbf{q}} \|F(\mathbf{q}) - \mathbf{y}^\delta\|_2^2, \quad \mathbf{q} \in \mathbb{R}^K$$

- $q(x) = \sum_{j=1}^K q_i \phi_i(x)$ (ϕ shape functions)
- Noisy data $\mathbf{y}^\delta = \mathbf{y} + \delta \boldsymbol{\eta}$, $\mathbf{y}^\delta \in \mathbb{R}^N$
- $\mathbf{y} = F(\mathbf{q}) \equiv \mathcal{C}(\mathbf{u}(\mathbf{q}))$, F reduced measurement function.
- $\delta \boldsymbol{\eta}$ random noise $\|\boldsymbol{\eta}\| = 1$, $\delta > 0$

Ill-posed problem $\Rightarrow$ Regularization

$$min_{\mathbf{q}} \|F(\mathbf{q}) - \mathbf{y}^\delta\|_2^2, \quad \text{s.t.} \quad \|L_2 \mathbf{q}\|_2^2 \le \gamma \qquad (1)$$

- L_2 second order difference matrix.
- $\gamma > 0$ smoothing parameter.

Aims:

- Compute the parameter $\mathbf{q}$ by solving (1): `CLSId` algorithm.
- Define suitable γ: adapting procedure.

Adaptive Algorithm

compute $\mathbf{q}_{cg}$, $\ell = 0$, $\gamma_0 = ||L_2 \mathbf{q}_{cg}||$
compute θ as in (3)
`repeat`
  $\gamma_{\ell+1} = (1+\theta)\gamma_\ell$; $\ell = \ell + 1$;
  $\mathbf{q}_\ell = CLSId(\gamma_\ell)$;
`until` $(J_\ell - J_{\ell-1}) < 0.1(\gamma_\ell - \gamma_{\ell-1})$

Test Problems

- **T1** Identification of $q(x)$ in:

$$u'' - q(x)u(x) = f, \quad u(0) = u(1) = 0$$

$u(x) = sin(\pi x)$, $q(x) = 10(x^4 - sin(\pi x))$ $q_0(x) = 10(2x^3 - (1+\pi)x^2 + \pi x)$ [5].

- **T2** Identification of $c(x)$ in:

$$(c(x)u')' = f(x), \quad u(0) = u(1) = 0$$

$u(x) = \sin(\pi(x(1-x))$, $c_0(x) = 1$, $c(x) = 1 + e^{-(10(x-0.3))^2)} + e^{-(10(x-0.7))^2)}$

- **T3** Identification of $r(x)$ in:

$$u'' + u' - \frac{u(x)}{r(x) + u(x)} = f$$

$u(0) = u(1) = 0$, $u(x) = x^2 + \sin(\pi x)$, $r(x) = x(1-x)$, $r_0(x) = \sin(\pi x)$

Constrained Least Squares Identification

Computation of parameter q Solve (1) in Lagrangian dual form:

$$\max_\lambda \left\{ \min_{\mathbf{q}} J(\mathbf{q}) + \lambda (\mathcal{R}(\mathbf{q}) - \gamma) \right\}, \quad J(\mathbf{q}) = \|F(\mathbf{q}) - \mathbf{y}^\delta\|_2^2, \quad \mathcal{R}(\mathbf{q}) = \|L_2 \mathbf{q}\|_2^2 \qquad (2)$$

$$\text{find} \quad \hat{\lambda} \quad \text{s.t.} \quad \mathcal{R}(\hat{\mathbf{q}}) - \gamma = 0, \quad \text{where } \hat{\mathbf{q}} = argmin_{\mathbf{q}} \left\{ J(\mathbf{q}) + \hat{\lambda}(\mathcal{R}(\mathbf{q}) - \gamma) \right\}$$

Algorithm `CLSId`(γ)

$k = 0$; $\lambda_0 = 1$,
`repeat`
  $\mathbf{q}(\lambda_k) = argmin_{\mathbf{q}} \{J(\mathbf{q}) + \lambda_k(\mathcal{R}(\mathbf{q}) - \gamma)\}$ `(Gauss Newton + Armijo)`
  $G_k = \mathcal{R}(\mathbf{q}(\lambda_k)) - \gamma$
  $\lambda_{k+1} = \lambda_k +$ `hybrid bisection-secant step`
  $k = k + 1$
`until` $|G_k - \gamma| < \tau_r G_0 + \tau_a$ or $|\lambda_k - \lambda_{k-1}| < \tau_a$ or $k > maxit$

- τ_a, τ_r are relative and absolute tolerance parameters:($\tau_a = 1.e-5$, $\tau_r = 1.e-5$)
- From Proposition 1 in [2], λ_k converge to $\hat{\lambda}$, solution of (2), provided $\mathcal{R}(\mathbf{q}(\lambda))$ is strictly decreasing in $0 \le \lambda_- < \hat{\lambda} < \lambda_+$.
- Using Lemma 2.1 in [4], we prove that $\|L_2 \mathbf{q}(\lambda_k)\|_2^2$ is strictly decreasing $\forall \lambda_k$.

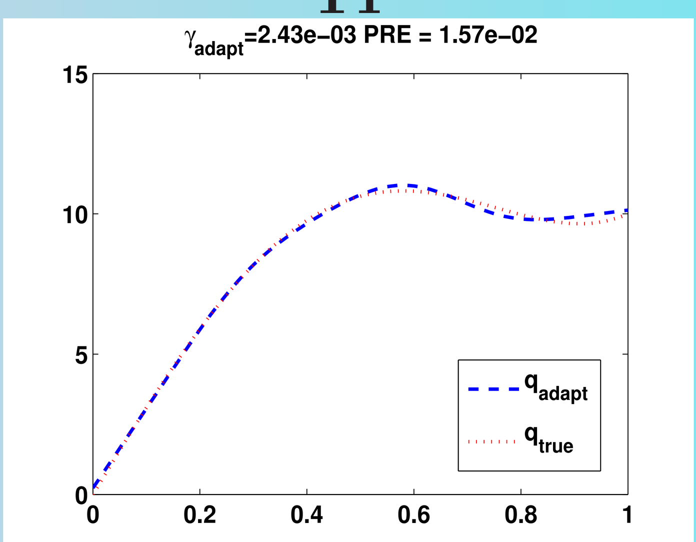
Adaptive update of γ:

$$\boxed{\gamma_{\ell+1} = (1+\theta)\gamma_\ell, \quad \texttt{if} \quad (J_\ell - J_{\ell-1}) > 0.1(\gamma_\ell - \gamma_{\ell-1}), \quad \gamma_0 = ||L_2 \mathbf{q}_{cg}||, \quad \theta = 30\% \text{ or } 130\%}$$

- $\mathbf{q}_{cg}$ is computed by few iterations of Nonlinear Conjugate Gradient applied to $min_{\mathbf{q}} J(\mathbf{q})$.
- Value θ: $$\theta = \begin{cases} 0.3 & |J(\mathbf{q}_{cg}) - J(0.8\mathbf{q}_{cg})| > 1.e-5 \\ 1.3 & \text{otherwise} \end{cases} \qquad (3)$$
- $J_\ell \equiv J(\mathbf{q}_\ell)$ where $\mathbf{q}_\ell$ is the solution of (2) with $\gamma = \gamma_\ell$.
- Assumption: $J_\ell < J_{\ell-1}$, $\gamma_{opt} > \gamma_\ell > \gamma_{\ell-1}$ where $\gamma_{opt} = min_\gamma ||\mathbf{q}_\gamma - \mathbf{q}^*||$, $F(\mathbf{q}^*) = \mathbf{y}$.

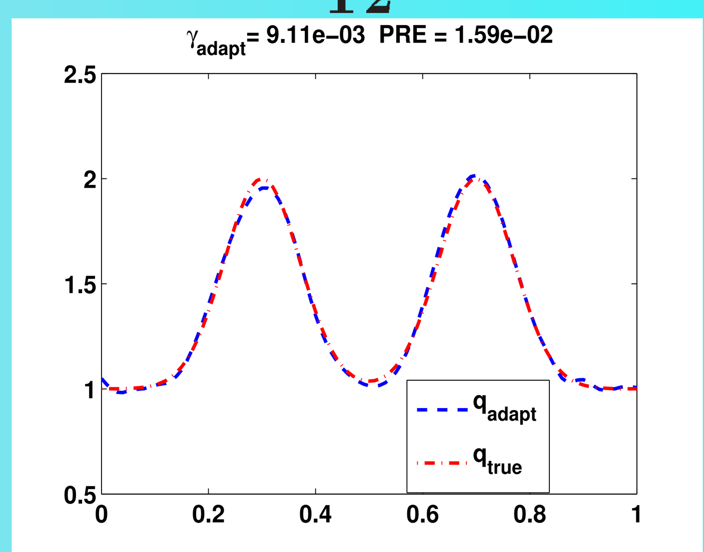
Numerical Experiments

T1



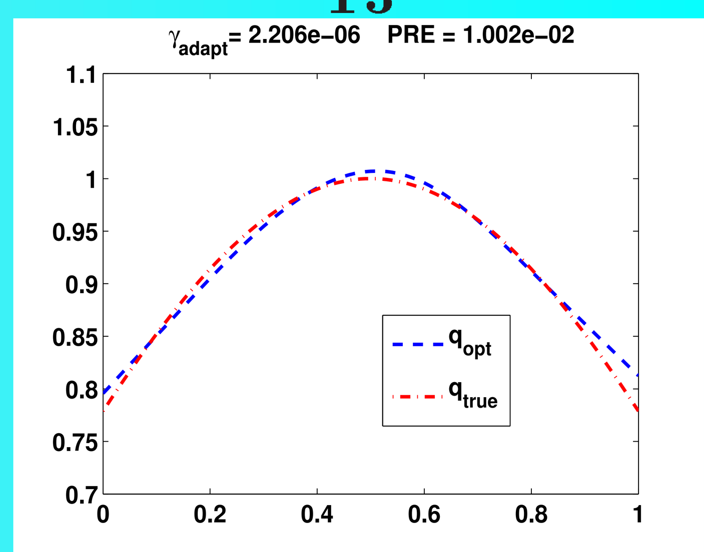
$\delta = 1.e-2$
$\gamma_{opt} = 2.916e-3$
$PRE_{opt} = 1.357e-2$
$\gamma_{adapt} = 2.429e-3$
$PRE_{adapt} = 1.571e-2$
$iter\ adapt\ \ell = 1$,
$iter\ Gauss\ Newton = 17$

T2

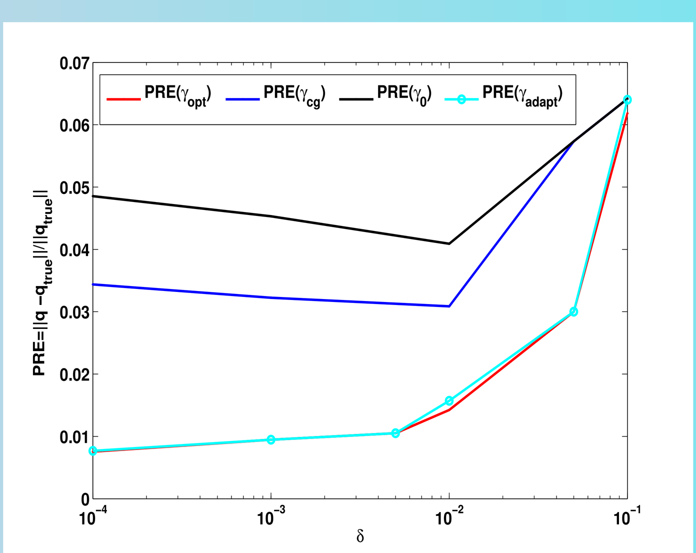


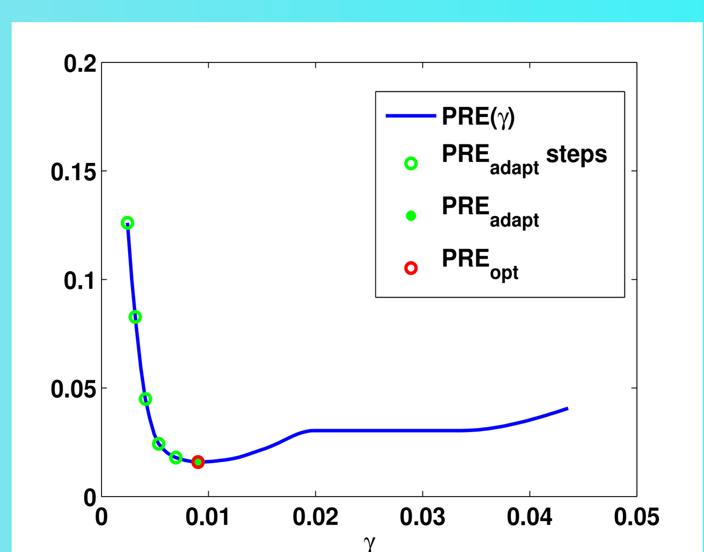
$\delta = 5.e-3$
$\gamma_{opt} = 9.37e-3$,
$PRE_{opt} = 1.58e-2$
$\gamma_{adapt} = 9.11e-3$,
$PRE_{adapt} = 1.59e-2$
$iter\ adapt\ \ell = 6$,
$iter\ Gauss\ Newton = 58$

T3



$\delta = 5.e-3$
$\gamma_{opt} = 3.960e-6$,
$PRE_{opt} = 9.789e-3$
$\gamma_{adapt} = 2.205e-6$,
$PRE_{adapt} = 1.002e-2$
$iter\ adapt\ \ell = 3$,
$iter\ Gauss\ Newton = 38$





PRE: Parameter Relative Error

$$PRE_v = \frac{\|\mathbf{q}_v - \mathbf{q}_{true}\|_2}{\|\mathbf{q}_{true}\|_2}$$

- $v = opt$, $\mathbf{q}_{opt} = \mathbf{q}(\gamma_{opt})$
- $v = adapt$, $\mathbf{q}_{adapt} = \mathbf{q}(\gamma_{adapt})$

References

[1] Zama F, Ciavarelli R, Frascari D and Pinelli D 2013 (*IFIP Advances in Information and Communication Technology* vol 391)
[2] Piccolomini E L and Zama F 2011 *Applied Mathematics and Computation* **217**
[3] Chen K, Piccolomini E and Zama F 2013 *Numerical Algorithms*
[4] Cartis C, Gould N and Toint P 2009 *BIT*
[5] Jin B and Maass P 2012 *Inverse Problems* **28** 123001