Scaling Limits for Multispecies Statistical Mechanics Mean-Field Models

Micaela Fedele and Pierluigi Contucci

Dipartimento di Matematica

Università di Bologna, 40127 Bologna, Italy

e-mail: *fedele@dm.unibo.it*, *contucci@dm.unibo.it*

Abstract

We study the limiting thermodynamic behavior of the normalized sums of spins in multi-species Curie-Weiss models. We find sufficient conditions for the limiting random vector to be Gaussian (or to have an exponential distribution of higher order) and compute the covariance matrix in terms of model parameters.

Introduction

The study of the normalized sum of random variables and its asymptotic behavior has been and continues to be a central chapter in probability and statistical mechanics. When those variables are independent the central limit theorem ensures that the sum with square-root normalization converges toward a Gaussian distribution. The generalization of that result to non-independent variables is particularly interesting in statistical mechanics where the random variables have an Hamiltonian interaction.

Ellis and Newman [EN78a,EN78b,ENR80] have studied the distribution of the normalized sums of spins whose interaction is described by a wide class of mean field Hamiltonian a la Curie-Weiss. They have found the conditions, in terms of the interaction, that lead in the thermodynamic limit to a Gaussian behavior and those who lead to a higher order exponential probability distribution.

In recent times a multispecies extension of the Curie-Weiss model has been proposed in the attempt to describe the large scale behavior of some socio-economic systems [CG07]. Multi-populated non-interacting spin models are at the basis of the so called Mc Fadden discrete choice [McF01] theory. The extension of the discrete choice theory to the interacting, and more realistic, case is an important problem toward the understanding of the collective behavior of social and economical systems. The investigation of the model introduced in [CG07] has been pursued at a mathematical level [GC08] where they have been proved properties like the existence of the thermodynamic limit by monotonicity, the computation of the free energy and of the intensive quantities like local magnetizations. The phenomenological test of the model has been started in [GBC08] and it is a topic of current investigations.

In this paper we deal with the study of the normalized sum behavior for a multi-populated model with mean field interaction. We prove that, under the assumption that the mean field Hamiltonian interaction has a convexity property, when the system reaches its thermodynamic limit, the random vector whose components are the sums of spins on each population, converges to a nontrivial random variable S. The behavior of S depends crucially upon the nature of the minima points of a function G (the pressure functional) which we associate to the model interaction type. In particular it is the value of the determinant of the Hessian matrix of G computed on the minima points that establish the Gaussian or non Gaussian behavior of the random vector. In the case of a unique minimum point if the determinant is different from zero then S is a multivariate Gaussian of covariance that can be computed from the mean field equations. Otherwise S has a distribution whose density is proportional to a higher order exponential.

This work is organized as follows. Chapter one introduces the language and the notations and states the main results in theorems 1 and 2. Chapter 2 contains the proofs. Chapter 3 describes specific cases in which the distribution is Gaussian and others in which is not. The appendix contains the proof of two lemmas that make the paper self contained.

1 Definitions and statements

We consider a system of N particles that can be divided into n subsets $P_1, \dots, P_n$ with $P_l \cap P_s = \emptyset$, for $l \neq s$ and sizes $|P_l| = N_l$, where $\sum_{l=1}^{n} N_l = N$. Particles interact with each other and with an extern field according to the mean field Hamiltonian:

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} h_i\sigma_i \ . \tag{1}$$

The σ_i represents the spin of the particle i, $\sigma_i = \pm 1$ while J_{ij} is the parameter that tunes the mutual interaction between the particle i and the particle j and takes values according to the following simmettrix matrix:

$$\begin{array}{cc}
 & \begin{array}{cccc} \overbrace{\qquad}^{N_1} & \overbrace{\qquad}^{N_2} & \qquad\quad & \overbrace{\qquad\qquad}^{N_n} \end{array} \\
\begin{array}{r} N_1 \{ \\ N_2 \{ \\ \\ \\ N_n \{ \end{array} &
\left(\begin{array}{c|c|c|c}
\mathbf{J}_{11} & \mathbf{J}_{12} & \dots & \mathbf{J}_{1n} \\ \hline
\mathbf{J}_{12} & \mathbf{J}_{22} & & \\ \hline
\vdots & & & \\ \hline
\mathbf{J}_{1n} & \mathbf{J}_{2n} & \dots & \mathbf{J}_{nn}
\end{array}\right)
\end{array}$$

where each block $\mathbf{J}_{ls}$ has constant elements J_{ls}. For $l = s$, $\mathbf{J}_{ll}$ is a square matrix, whereas the matrix $\mathbf{J}_{ls}$ is rectangular. We assume $J_{11}, J_{22}, \dots, J_{nn}$ be positive, whereas J_{ls} with $l \neq s$ can be either positive or negative allowing both ferromagnetic and antiferromagnetic interactions. The vector field takes also different values depending on the subset the particles belong to:

$$
\begin{array}{r}
N_1 \left\{ \vphantom{\mathbf{h}_1} \right. \\
N_2 \left\{ \vphantom{\mathbf{h}_2} \right. \\
\\
N_n \left\{ \vphantom{\begin{array}{c}\\ \\ \mathbf{h}_n \\ \\ \\ \end{array}} \right.
\end{array}
\left(
\begin{array}{c}
\mathbf{h}_1 \\ \hline
\mathbf{h}_2 \\ \hline
\vdots \\ \hline
\\ \\ \mathbf{h}_n \\ \\ \\
\end{array}
\right)
$$

where each $\mathbf{h}_l$ is a vector of costant elements h_l.

The distribution of a spin configuration σ is given by:

$$
\frac{e^{-H_N(\sigma)} \prod_{i=1}^{N} d\rho(\sigma_i)}{\int_{\mathbb{R}^N} e^{-H_N(\sigma)} \prod_{i=1}^{N} d\rho(\sigma_i)} \tag{2}
$$

whit:

$$
\rho(x) = \frac{1}{2}\Big[\delta(x-1) + \delta(x+1)\Big] \tag{3}
$$

where $\delta(x - x_0)$ $x_0 \in \mathbb{R}$ denotes the unit point mass with support at x_0.

By introducing the magnetization of a set A as:

$$
m_A(\sigma) = \frac{1}{|A|} \sum_{i \in A} \sigma_i \tag{4}
$$

and indicating by $m_l(\sigma)$ the magnetization of the set P_l, and by $\alpha_l = N_l/N$ the relative size of the set P_l, we may easily express the Hamiltonian (1) as:

$$
H_N(\sigma) = -Ng(m_1(\sigma), \dots, m_n(\sigma)) \tag{5}
$$

where:

$$
g(m_1(\sigma), \dots, m_n(\sigma)) = \frac{1}{2}\Bigg(\sum_{l=1}^{n} \alpha_l^2 J_{ll} m_l^2(\sigma) + \sum_{l \neq s} \alpha_l \alpha_s J_{ls} m_l(\sigma) m_s(\sigma)\Bigg) + \sum_{l=1}^{n} \alpha_l h_l m_l(\sigma) \tag{6}
$$

In [GC08] it is shown that the thermodynamic limit of the pressure function

$$
p_N = \frac{1}{N} \ln \sum_{\sigma} e^{-H_N(\sigma)} \tag{7}
$$

exists and is reached monotonically if the the function $g(m_1(\sigma),\dots,m_n(\sigma))$ is convex and bounded (see also [BCG03]). In the first paper the limiting value is computed:

$$\lim_{N\to\infty} p_N = \sup_{\mathbf{x}\in[-1,1]^n} \overline{p}(\mathbf{x}) \tag{8}$$

where the functional $\overline{p}(\mathbf{x})$ is:

$$\begin{aligned}\overline{p}(\mathbf{x}) = \overline{p}(x_1,\dots,x_n) = \ln 2 - \frac{1}{2}\Big(\sum_{l=1}^{n} \alpha_l^2\ J_{ll}\ x_l^2 + \sum_{l\neq s} \alpha_l\alpha_s\ J_{ls}\ x_l x_s\Big) \\ + \sum_{l=1}^{n} \alpha_l \ln\Big[\cosh\Big(\sum_{s=1}^{n} \alpha_s J_{ls}\ x_s + h_l\Big)\Big]\end{aligned} \tag{9}$$

The extremality conditions of $\overline{p}(x_1,\dots,x_n)$ give the Mean Field Equations of the model.

$$\begin{cases} \mu_1 &= \tanh\Big(\sum\limits_{l=1}^{n}\ \alpha_l J_{1l}\ \mu_l + h_1\Big) \\ \mu_2 &= \tanh\Big(\sum\limits_{l=1}^{n}\ \alpha_l J_{2l}\ \mu_l + h_2\Big) \\ \vdots \\ \mu_n &= \tanh\Big(\sum\limits_{l=1}^{n}\ \alpha_l J_{ln}\ \mu_l + h_n\Big)\ . \end{cases} \tag{10}$$

In the themodynamic limit the random vector $(m_1(\sigma),\dots,m_n(\sigma))$ converges, with respect to the Boltzamann-Gibbs measure, to the deterministic vector $(\mu_1,\dots,\mu_n)$ solution of the Mean Field Equations. This means that the variances of the magnetizations vanish for large N (see [GC08] for the precise statement).

In this paper, defined the sum of the spins of a set A as:

$$S_A(\sigma) = \sum_{i\in A} \sigma_i \tag{11}$$

and indicating by $S_l(\sigma)$ the sum of the spins of the set P_l we want to determinate a suitable normalization for $S_1(\sigma),\dots,S_n(\sigma)$ so that in the thermodinamic limit they converges to well defined random variables with finite (non zero) variance. The problem in $n=1$ has been solved in [EN78a] and [ENR80].

We consider the function $G(\mathbf{x}) = -\overline{p}(\mathbf{x})$

$$
\begin{aligned}
G(x_1,\dots,x_n) = &-\ln 2 + \frac{1}{2}\Big(\sum_{l=1}^{n} \alpha_l^2\; J_{ll}\; x_l^2 + \sum_{l\neq s} \alpha_l\alpha_s\; J_{ls}\; x_l x_s\Big) \\
&- \sum_{l=1}^{n} \alpha_l \ln\Big[\cosh\Big(\sum_{s=1}^{n} \alpha_s J_{ls}\; x_s + h_l\Big)\Big].
\end{aligned} \tag{12}
$$

It is a real analytic function. Since

$$
\begin{aligned}
G(x_1,\dots,x_n) \geq &-\ln 2 + \frac{1}{2}\Big(\sum_{l=1}^{n} \alpha_l^2\; J_{ll}\; x_l^2 + \sum_{l\neq s} \alpha_l\alpha_s\; J_{ls}\; x_l x_s\Big) \\
&- \sum_{l=1}^{n} \alpha_l \Big|\sum_{s=1}^{n} \alpha_s J_{ls}\; x_s + h_l\Big|
\end{aligned} \tag{13}
$$

If the r.h.s. function is convex, G must have a finite number of global minimum points. In this case also it is true that:

$$
\int_{\mathbb{R}^n} \exp[-NG(x_1,\dots,x_n)]dx_1\dots dx_n < \infty \qquad \text{for any } N \in \{1,2,\dots\} \tag{14}
$$

Proof. We prove the statement by induction. For $N=1$ we have:

$$
\begin{aligned}
\int_{\mathbb{R}^n} \exp[-G(x_1,\dots,x_n)]dx_1\dots dx_n \leq &\int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\Big(\sum_{l=1}^{n} \alpha_l^2\; J_{ll}\; x_l^2 + \sum_{l\neq s} \alpha_l\alpha_s\; J_{ls}\; x_l x_s\Big)\Big] \\
&\times \prod_{l=1}^{n}\Big(\exp\Big[\sum_{s=1}^{n} \alpha_s J_{ls}\; x_s + h_l\Big] + \exp\Big[-\sum_{s=1}^{n} \alpha_s J_{ls}\; x_s - h_l\Big]\Big)dx_1\dots dx_n \\
&\leq 2^n \exp\Big[n \max_i\{h_i\} + \max_{\mathbf{b}_\sigma}\{\frac{1}{2}\langle \mathbf{Q}^{-1}\mathbf{b}_\sigma, \mathbf{b}_\sigma\rangle\}\Big]
\end{aligned}
$$

where $\mathbf{Q} = \mathbf{D}_{\boldsymbol{\alpha}}\mathbf{D}_{\boldsymbol{\alpha}}\mathbf{J}\mathbf{D}_{\boldsymbol{\alpha}}\mathbf{D}_{\boldsymbol{\alpha}}$ with

$$
\mathbf{D}_{\boldsymbol{\alpha}} = \begin{pmatrix} \sqrt{\alpha_1} & 0 & \dots & 0 \\ 0 & \sqrt{\alpha_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sqrt{\alpha_n} \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1n} \\ J_{12} & J_{22} & \dots & J_{2n} \\ \vdots & \vdots & & \vdots \\ J_{1n} & J_{2n} & \dots & J_{nn} \end{pmatrix} \tag{15}
$$

and $\mathbf{b}_\sigma$ is the vector:

$$\mathbf{b}_\sigma = \begin{pmatrix} \alpha_1 \sum_{i=1}^n \sigma_i J_{1i} \\ \\ \alpha_2 \sum_{i=1}^n \sigma_i J_{2i} \\ \\ \vdots \\ \\ \alpha_n \sum_{i=1}^n \sigma_i J_{in} \end{pmatrix} \tag{16}$$

The matrix $\mathbf{J}$ is called the redux interaction matrix. Defined $f = \min\{G(\mathbf{x})|\mathbf{x} \in \mathbb{R}^n\}$ and supposed true:

$$\int_{\mathbb{R}^n} e^{-(N-1)G(\mathbf{x})} d\mathbf{x} < \infty \tag{17}$$

we have:

$$\int_{\mathbb{R}^n} e^{-NG(\mathbf{x})} d\mathbf{x} < \int_{\mathbb{R}^n} e^{-(N-1)G(\mathbf{x})} e^{-G(\mathbf{x})} d\mathbf{x} < e^f \int_{\mathbb{R}^n} e^{-(N-1)G(\mathbf{x})} d\mathbf{x} < \infty \tag{18}$$

$\square$

Consider a (global or local) minimum point $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ of the function $G(\mathbf{x})$, we can write the Taylor expansion around $\boldsymbol{\mu}$:

$$G(\mathbf{x}) = G(\boldsymbol{\mu}) + \sum_{2 \le |\boldsymbol{\eta}| \le 2k(\boldsymbol{\mu})} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} (\mathbf{x} - \boldsymbol{\mu})^{\boldsymbol{\eta}} + o\left(\left\| \sum_{|\boldsymbol{\eta}| = 2k(\boldsymbol{\mu})} (\mathbf{x} - \boldsymbol{\mu})^{\boldsymbol{\eta}} \right\| \right) \tag{19}$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ is a multi-index. We recall that:

- $|\boldsymbol{\eta}| = \eta_1 + \cdots + \eta_n$
- $\boldsymbol{\eta}! = \eta_1! \eta_2! \ldots \eta_n!$
- $\partial^{\boldsymbol{\eta}} G = \frac{\partial^{\eta_1}}{\partial x_1^{\eta_1}} \cdots \frac{\partial^{\eta_n}}{\partial x_n^{\eta_n}} G$
- $(\mathbf{x} - \boldsymbol{\mu})^{\boldsymbol{\eta}} = (x_1 - \mu_1)^{\eta_1} \ldots (x_n - \mu_n)^{\eta_n}$

The polynomial:

$$\sum_{|\eta|=p} \frac{(\partial^{\boldsymbol{\eta}} G)(\mathbf{m})}{\boldsymbol{\eta}!} (\mathbf{x}-\boldsymbol{\mu})^{\boldsymbol{\eta}} \tag{20}$$

is positive semidefinite for $p < 2k(\boldsymbol{\mu})$ and positive definite for $p = 2k(\boldsymbol{\mu})$.

The integer $k(\boldsymbol{\mu})$ is positive and is called the type of the minimum point $\boldsymbol{\mu}$. In particular if $k(\boldsymbol{\mu}) = 1$, that is the determinant of the Hessian matrix of $G(\mathbf{x})$ computed in the minimum point is not zero, around $\boldsymbol{\mu}$ we have:

$$G(\mathbf{x}) = G(\boldsymbol{\mu}) + \frac{1}{2}\langle \mathcal{H}_G(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}), (\mathbf{x}-\boldsymbol{\mu})\rangle + o\Bigg(\bigg\|\sum_{|\boldsymbol{\eta}|=2}(\mathbf{x}-\boldsymbol{\mu})^{\boldsymbol{\eta}}\bigg\|\Bigg) \tag{21}$$

where $\mathcal{H}_G(\boldsymbol{\mu})$ is the Hessian matrix computed in the minimum point $\boldsymbol{\mu}$.

We define the random vector $\mathbf{v}(k(\boldsymbol{\mu}))$:

$$\mathbf{v}(k(\boldsymbol{\mu})) = \left(\frac{S_1(\sigma) - N_1\mu_1}{(N_1)^{1-1/2k(\boldsymbol{\mu})}}, \ldots, \frac{S_n(\sigma) - N_n\mu_n}{(N_n)^{1-1/2k(\boldsymbol{\mu})}}\right) \tag{22}$$

In particolar as $k(\boldsymbol{\mu}) = 1$:

$$\mathbf{v}(1) = \left(\frac{S_1(\sigma) - N_1\mu_1}{\sqrt{N_1}}, \ldots, \frac{S_n(\sigma) - N_n\mu_n}{\sqrt{N_n}}\right) \tag{23}$$

Theorem 1. *Let $H_N = -Ng(m_1(\sigma), \ldots, m_n(\sigma))$ be the Hamiltonian where $g(m_1(\sigma), \ldots, m_n(\sigma))$ is a convex bounded function defined in (6). Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be the unique global minimum point of the function $G(\mathbf{x})$ given from (12). Let $k(\boldsymbol{\mu})$ the type of the minimum point.*

- *If $k(\boldsymbol{\mu}) = 1$ the asymptotic behaviour of the random vector $\mathbf{v}(1)$ defined in (23) as $N_1 \to \infty$, ..., $N_n \to \infty$, for fixed values of α_1, ..., α_n, is given by a normal multivariate distribution whose covariance matrix is:*

$$\widetilde{\boldsymbol{\chi}} = \begin{pmatrix} \frac{\partial \mu_1}{\partial h_1} & \sqrt{\frac{\partial \mu_1}{\partial h_2}\frac{\partial \mu_2}{\partial h_1}} & \cdots & \sqrt{\frac{\partial \mu_1}{\partial h_n}\frac{\partial \mu_n}{\partial h_1}} \\ \sqrt{\frac{\partial \mu_1}{\partial h_2}\frac{\partial \mu_2}{\partial h_1}} & \frac{\partial \mu_2}{\partial h_2} & \cdots & \sqrt{\frac{\partial \mu_2}{\partial h_n}\frac{\partial \mu_n}{\partial h_2}} \\ \vdots & \vdots & & \vdots \\ \sqrt{\frac{\partial \mu_1}{\partial h_n}\frac{\partial \mu_n}{\partial h_1}} & \sqrt{\frac{\partial \mu_2}{\partial h_n}\frac{\partial \mu_n}{\partial h_2}} & \cdots & \frac{\partial \mu_n}{\partial h_n} \end{pmatrix} \tag{24}$$

where $(\mu_1, \ldots, \mu_n)$ is the solution of the Mean Field Equations (10) corresponding to the minimum.

- *if $k(\boldsymbol{\mu}) > 1$ and the partial derivatives of order smaller then $2k(\boldsymbol{\mu})$ are equal to zero (homogenity hipothesis), that is*

$$\frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} = 0 \qquad \text{for} \quad 2 \leq |\eta| < 2k(\boldsymbol{\mu}) \tag{25}$$

the asymptotic behavior of the random vector $\mathbf{v}(k(\boldsymbol{\mu}))$ defined in (22) as $N_1 \to \infty$, $\ldots$, $N_n \to \infty$, for fixed values of $\alpha_1, \ldots, \alpha_n$, has density proportional to:

$$\exp\left[- \sum_{|\boldsymbol{\eta}|=2k(\boldsymbol{\mu})} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} \Big(\frac{\mathbf{x}}{\boldsymbol{\alpha}^{1/2k(\boldsymbol{\mu})}} \Big)^{\boldsymbol{\eta}} \right] \tag{26}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$

Theorem 2. *Let $H_N = -Ng(m_1(\sigma), \ldots, m_n(\sigma))$ be the Hamiltonian where $g(m_1(\sigma), \ldots, m_n(\sigma))$ is a convex bounded function defined in (6). Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be a global or a local minimum point of the function $G(\mathbf{x})$ given from (12). Let $k(\boldsymbol{\mu})$ the type of the minimum point. Then there exists $A > 0$ such that for all $a \in (0, A)$ when $(m_1(\sigma), \ldots, m_n(\sigma)) \in B^{(n)}(\boldsymbol{\mu}, a)$,*

1. *if $k(\boldsymbol{\mu}) = 1$ the asymptotic behavior of the random vector $\mathbf{v}(1)$ defined in (23) as $N_1 \to \infty$, $\ldots$, $N_n \to \infty$, for fixed values of $\alpha_1, \ldots, \alpha_n$, is given by a normal multivariate distribution whose covariance matrix is $\widetilde{\boldsymbol{\chi}}$ defined in (24).*

2. *If $k(\boldsymbol{\mu}) > 1$ and the partial derivatives of order smaller then $2k(\boldsymbol{\mu})$ are equal to zero (homogenity hipothesis), that is*

$$\frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} = 0 \qquad \textit{for} \quad 2 \le |\eta| < 2k(\boldsymbol{\mu}) \tag{27}$$

the asymptotic behavior of the random vector $\mathbf{v}(k(\boldsymbol{\mu}))$ defined in (22) as $N_1 \to \infty$, $\ldots$, $N_n \to \infty$, for fixed values of $\alpha_1, \ldots, \alpha_n$, has density proportional to:

$$\exp\Big[-\sum_{|\boldsymbol{\eta}|=2k(\boldsymbol{\mu})} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}\Big(\frac{\mathbf{x}}{\boldsymbol{\alpha}^{1/2k(\boldsymbol{\mu})}}\Big)^{\boldsymbol{\eta}}\Big] \tag{28}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$

2 Proofs

2.1 Proof of theorem 1

To prove this theorem we need the following two lemmas.

Lemma 1. *Suppose that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are independent random vectors such that $X \to \nu$.*
Then $Y \to \mu$ if and only if $X + Y \to \nu * \mu$

Lemma 2. *Given the random vector $(W_1, \ldots, W_n)$ which joint distribution is the normal multivariate*

$$\rho(\mathbf{x}) = \sqrt{\frac{\det \mathbf{A}}{(2\pi)^n}}\ \exp\Big[-\frac{1}{2}\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle\Big] \tag{29}$$

where $\mathbf{A} = \mathbf{D}_{\boldsymbol{\alpha}}\mathbf{J}\mathbf{D}_{\boldsymbol{\alpha}}$ is a matrix positive defined, if $(W_1, \ldots, W_n)$ is independent of $(S_1(\sigma), \ldots, S_n(\sigma))$ then for $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ the joint distribution of

$$\Big(\frac{W_1}{(N_1)^{1/2-\gamma}}, \ldots, \frac{W_n}{(N_n)^{1/2-\gamma}}\Big) + \Big(\frac{S_1 - N_1\mu_1}{(N_1)^{1-\gamma}}, \ldots, \frac{S_n - N_n\mu_n}{(N_n)^{1-\gamma}}\Big) \tag{30}$$

is given by

$$\frac{\exp\Big[-NG\Big(\frac{x_1}{N_1^{\gamma}}+\mu_1,\dots,\frac{x_n}{N_n^{\gamma}}+\mu_n\Big)\Big]dx_1\dots dx_n}{\int\exp\Big[-NG\Big(\frac{x_1}{N_1^{\gamma}}+\mu_1,\dots,\frac{x_n}{N_n^{\gamma}}+\mu_n\Big)\Big]dx_1\dots dx_n} \tag{31}$$

where $G(\mathbf{x})$ *is the fuction defined in (12).*

The proof of the lemma is in the appendix. We remark that as $\gamma < 1/2$, the random vector $(W_1,\dots,W_n)$ does not contribute to the limit of (31) as $N_1\to\infty$, $\dots$, $N_n\to\infty$. Taking $\gamma = 1/2k(\boldsymbol{\mu})$ by lemma (2) and lemma (1) we have to prove that, for any bounded continuous function $\psi(x_1,\dots,x_n)$

$$\frac{\int\exp\Big[-NG\Big(\frac{x_1}{N_1^{\gamma}}+\mu_1,\dots,\frac{x_n}{N_n^{\gamma}}+\mu_n\Big)\Big]\psi(x_1,\dots,x_n)dx_1\dots dx_n}{\int\exp\Big[-NG\Big(\frac{x_1}{N_1^{\gamma}}+\mu_1,\dots,\frac{x_n}{N_n^{\gamma}}+\mu_n\Big)\Big]dx_1\dots dx_n}$$

$$\to\frac{\int\exp\Big[-\sum_{|\boldsymbol{\eta}|=2k}\frac{(\partial_{x_1}^{\eta_1}\dots\partial_{x_n}^{\eta_n}G)(\boldsymbol{\mu})}{(2k)!}\Big(\frac{\mathbf{x}}{\boldsymbol{\alpha}^{\gamma}}\Big)^{\boldsymbol{\eta}}\Big]\psi(x_1,\dots,x_n)dx_1\dots dx_n}{\int\exp\Big[-\sum_{|\boldsymbol{\eta}|=2k}\frac{(\partial_{x_1}^{\eta_1}\dots\partial_{x_n}^{\eta_n}G)(\boldsymbol{\mu})}{(2k)!}\Big(\frac{\mathbf{x}}{\boldsymbol{\alpha}^{\gamma}}\Big)^{\boldsymbol{\eta}}\Big]dx_1\dots dx_n} \tag{32}$$

where $k=k(\boldsymbol{\mu})$. Defined the function $B(\mathbf{x},\boldsymbol{\mu}) = G(\mathbf{x}+\boldsymbol{\mu})-G(\boldsymbol{\mu})$ then there exists $\delta>0$ sufficiently small so that for $||\mathbf{x}||<\delta a$ with $a=\max\{\sqrt{N_1},\dots,\sqrt{N_n}\}$, as $N_1\to\infty$, $\dots$, $N_n\to\infty$

$$\begin{aligned}N\cdot B\Big(\frac{x_1}{N_1^{\gamma}},\dots,\frac{x_n}{N_n^{\gamma}},\mu_1,\dots,\mu_n\Big) &= \sum_{|\boldsymbol{\eta}|=2k}\frac{(\partial_{x_1}^{\eta_1}\dots\partial_{x_n}^{\eta_n}G)(\mu_1,\dots,\mu_n)}{(2k)!}\Big(\frac{x_1}{\alpha_1^{1/2k}}\Big)^{\eta_1}\cdots\Big(\frac{x_n}{\alpha_n^{1/2k}}\Big)^{\eta_n}\\ &\quad+o\Big(\Big|\Big|\frac{x_1}{\alpha_1^{1/2k}},\dots\frac{x_n}{\alpha_n^{1/2k}}\Big|\Big|^{\boldsymbol{\eta}}\Big)\end{aligned} \tag{33}$$

Defined $f=\min\{G(\mathbf{x})|\mathbf{x}\in\mathbb{R}^n\}$ for any closed (possibly unbounded) subset V of $\mathbb{R}^n$

which contains no global minima of $G(\mathbf{x})$ there exist $\epsilon > 0$, so that

$$e^{Nf} \int_V \exp[-NG(\mathbf{x})] d\mathbf{x} = O(e^{-N\epsilon}) \qquad N \to \infty \tag{34}$$

We pick $\delta > 0$ as in (33). By (34) there exists $\epsilon > 0$ so that

$$e^{Nf} \int\limits_{||\mathbf{x}|| \geq \delta a} \exp\Big[- NG\Big(\frac{x_1}{N_1^\gamma} + \mu_1, \ldots, \frac{x_n}{N_n^\gamma} + \mu_n\Big)\Big] \psi(x_1, \ldots, x_n) dx_1 \ldots dx_n$$
$$= O((N_1^\gamma + \cdots + N_n^\gamma) \exp[-N\epsilon]) \tag{35}$$

whereas by (33) and dominate convergence, we have that:

$$e^{Nf} \int\limits_{||\mathbf{x}|| < \delta a} \exp\Big[- NG\Big(\frac{x_1}{N_1^\gamma} + \mu_1, \ldots, \frac{x_n}{N_n^\gamma} + \mu_n\Big)\Big] \psi(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

$$= \int\limits_{||\mathbf{x}|| < \delta a} \exp\Big[- NB\Big(\frac{x_1}{N_1^\gamma} + \mu_1, \ldots, \frac{x_n}{N_n^\gamma} + \mu_n, \mu_1, \ldots, \mu_n\Big)\Big] \psi(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

$$= \int\limits_{||\mathbf{x}|| < \delta a} \exp\Big[- \sum_{|\eta| = 2k} \frac{(\partial_{x_1}^{\eta_1} \ldots \partial_{x_n}^{\eta_n} G)(\mu_1, \ldots, \mu_n)}{(2k)!} \frac{x_1^{\eta^1}}{\alpha_1^{\eta_1/2k}} \cdots \frac{x_n^{\eta^n}}{\alpha_n^{\eta_n/2k}}\Big] \psi(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

For $k = 1$ the (33) becomes:

$$N \cdot B\Big(\frac{x_1}{\sqrt{N_1}}, \ldots, \frac{x_n}{\sqrt{N_n}}, \mu_1, \ldots, \mu_n\Big) = \frac{1}{2}\Big(\sum_{l=1}^n \frac{\boldsymbol{\mathcal{H}}_{ll}}{\alpha_l} x_l^2 + \sum_{k,l} \frac{\boldsymbol{\mathcal{H}}_{lk}}{\sqrt{\alpha_l \alpha_k}} x_l x_k\Big)$$
$$+ o\Big(\Big|\Big|\frac{x_1}{\sqrt{N_1}}, \ldots, \frac{x_n}{\sqrt{N_n}}\Big|\Big|^3\Big) \tag{36}$$

In analogous way, for $||\mathbf{x}|| < \delta a$, we prove that for any bounded continuous function $\psi(x_1, \ldots, x_n)$:

$$\int \exp\Big[-NG\Big(\frac{x_1}{\sqrt{N_1}} + \mu_1, \ldots, \frac{x_n}{\sqrt{N_n}} + \mu_n\Big)\Big]\psi(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

$$\to \int \exp\Big[-\frac{1}{2}\langle \widetilde{\mathcal{H}}_G(\boldsymbol{\mu})\mathbf{x}, \mathbf{x}\rangle\Big]\psi(\mathbf{x})d\mathbf{x} \quad (37)$$

and:

$$\int \exp\Big[-NG\Big(\frac{x_1}{\sqrt{N_1}} + \mu_1, \ldots, \frac{x_n}{\sqrt{N_n}} + \mu_n\Big)\Big]dx_1 \ldots dx_n \to \frac{\sqrt{\det \widetilde{\boldsymbol{\mathcal{H}}}_G(\boldsymbol{\mu})}}{(2\pi)^{n/2}} \quad (38)$$

where $\widetilde{\boldsymbol{\mathcal{H}}}_G = \mathbf{D}_\alpha^{-1}\boldsymbol{\mathcal{H}}_G\mathbf{D}_\alpha^{-1}$. Hence to obtein the result we have to see that the distribution

$$\frac{\sqrt{\det \widetilde{\boldsymbol{\mathcal{H}}}_G(\boldsymbol{\mu})}}{(2\pi)^{n/2}} \exp\Big[-\frac{1}{2} < \widetilde{\boldsymbol{\mathcal{H}}}_G(\boldsymbol{\mu})\mathbf{x}, \mathbf{x} > \Big]d\mathbf{x} \quad (39)$$

is exactly the convolution of the distribution of the random vector $(W_1, \ldots, W_n)$ with a multivariate normal which covariance matrix is $\widetilde{\boldsymbol{\chi}}$.

The characteristic function of a random vector

$$(X_1, \ldots, X_n) \sim \frac{\sqrt{\det \mathbf{B}}}{(2\pi)^{n/2}} \exp[-1/2(\langle \mathbf{Bz}, \mathbf{z}\rangle)]$$

is

$$\phi(\boldsymbol{\lambda}) = \exp[-1/2\langle \mathbf{B}^{-1}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle]$$

Indicated with $\phi_{\mathbf{A}}(\boldsymbol{\lambda})$, $\phi_{\mathbf{v}}(\boldsymbol{\lambda})$ and $\phi_{\widetilde{\boldsymbol{\mathcal{H}}}_G}(\boldsymbol{\lambda})$ respectively the characteristic function of the random vectors $(W_1, \ldots, W_n)$, $\mathbf{v}(1)$ and of their sum we have:

$$\phi_{\widetilde{\boldsymbol{\mathcal{H}}}_G}(\boldsymbol{\lambda}) = \phi_{\mathbf{A}}(\boldsymbol{\lambda})\ \phi_{\mathbf{v}}(\boldsymbol{\lambda}) \quad (40)$$

So to determinate $\phi_{\mathbf{v}}(\boldsymbol{\lambda})$ we compute the matrix $\mathbf{A}^{-1}$ and $\widetilde{\boldsymbol{\mathcal{H}}}_G^{-1}$. Taking off $\mathbf{A}^{-1}$ from $\widetilde{\boldsymbol{\mathcal{H}}}_G^{-1}$ we obtein the matrix $\widetilde{\boldsymbol{\chi}}$.

2.2 Proof of theorem 2

To ease the notation, we set $k = k(\boldsymbol{\mu})$ and $\gamma = 1/2k(\boldsymbol{\mu})$. Given $k(\boldsymbol{\mu}) > 1$, we must find $A > 0$ such that for each $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ and any $a \in (0, A)$

$$\omega_N(\exp[i\mathbf{r}\mathbf{v}])\big|_E = \frac{\int_E \exp[i\mathbf{r}\mathbf{v}] \exp[-H_N(S_1, \ldots, S_n)] \prod d\rho(\sigma_j)}{\int_E \exp[-H_N(S_1, \ldots, S_n)] \prod d\rho(\sigma_j)} \tag{41}$$

where:

$$E = \left\{ \left| \frac{S_1}{N_1} - \mu_1 \right| \leq a \times \cdots \times \left| \frac{S_n}{N_n} - \mu_n \right| \leq a \right\} \tag{42}$$

and:

$$H_N(S_1, \ldots, S_n) = -\frac{1}{2} \left[\sum_{l=1}^{n} \frac{\alpha_l J_{ll}}{N_l} S_l^2 + \sum_{l \neq s} J_{ls} \sqrt{\frac{\alpha_l \alpha_s}{N_l N_s}} S_l S_s \right] - \sum_{l=1}^{n} h_l S_l \tag{43}$$

tends as $N_1 \to \infty, \ldots, N_n \to \infty$, for fixed values of $\alpha_1, \ldots, \alpha_n$ to

$$\frac{\displaystyle\int \exp[i\mathbf{r}\mathbf{w}] \exp\left[-\sum_{|\boldsymbol{\eta}|=2k} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} \left(\frac{\mathbf{w}}{\boldsymbol{\alpha}^{1/2k}} \right)^{\boldsymbol{\eta}} \right] d\mathbf{w}}{\displaystyle\int \exp\left[-\sum_{|\boldsymbol{\eta}|=2k} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!} \left(\frac{\mathbf{w}}{\boldsymbol{\alpha}^{1/2k}} \right)^{\boldsymbol{\eta}} \right] d\mathbf{w}}. \tag{44}$$

Given

$$\widetilde{H}_N(S_1, \ldots, S_n) = -\frac{1}{2} \left[\sum_{l=1}^{n} \alpha_l J_{ll} \left(\frac{S_l - N_l \mu_l}{\sqrt{N_l}} \right)^2 + \sum_{l \neq s} J_{ls} \sqrt{\alpha_l \alpha_s} \left(\frac{S_l - N_l \mu_l}{\sqrt{N_l}} \right) \left(\frac{S_s - N_s \mu_s}{\sqrt{N_s}} \right) \right]$$

we can rewrite (41) as:

$$\frac{\int_E \exp[i\mathbf{r}\mathbf{v}] \exp[-\widetilde{H}_N(S_1, \ldots, S_n)] \exp[\sum_{l=1}^{n} S_l (h_l + \sum_{s=1}^{n} J_{ls} \alpha_s \mu_s)] \prod d\rho(\sigma_j)}{\int_E \exp[-\widetilde{H}_N(S_1, \ldots, S_n)] \exp[\sum_{l=1}^{n} S_l (h_l + \sum_{s=1}^{n} J_{ls} \alpha_s \mu_s)] \prod d\rho(\sigma_j)} \tag{45}$$

Defining:

$$d\rho_j(x) = \frac{\exp[x(h_j + \sum_{s=1}^{n} J_{js} \alpha_s \mu_s)] d\rho(x)}{\int \exp[x(h_j + \sum_{s=1}^{n} J_{js} \alpha_s \mu_s)] d\rho(x)} \quad j = 1, \ldots, n \tag{46}$$

we have:

$$\omega_N(\exp[i\mathbf{r}\mathbf{v}])\big|_E = \frac{\int_E \exp[i\mathbf{r}\mathbf{v}] \exp[-\widetilde{H}_N(S_1, \ldots, S_n)] \prod_{j=1}^{n} \prod_{l \in P_j} d\rho_j(\sigma_l)}{\int_E \exp[-\widetilde{H}_N(S_1, \ldots, S_n)] \prod_{j=1}^{n} \prod_{l \in P_j} d\rho_j(\sigma_l)} \tag{47}$$

We introduce the random vector:

$$(U_1, \dots, U_n) = \left(\frac{S_1 - N_1\mu_1}{N_1}, \dots, \frac{S_n - N_n\mu_n}{N_n} \right) \tag{48}$$

and indicate by $d\nu(\mathbf{u})$ its distribution on $(\mathbb{R}^N, \prod_{j=1}^n \prod_{l\in P_j} d\rho_j(x_l))$. We can write:

$$\omega_N(\exp[i\mathbf{r}\mathbf{v}])\big|_E = \frac{\int_{||\mathbf{u}||\leq a} \exp[iN^\gamma \mathbf{r}\boldsymbol{\alpha}^\gamma \mathbf{u} + \frac{N}{2}\langle \mathbf{Q}\mathbf{u}, \mathbf{u}\rangle] d\nu(\mathbf{u})}{\int_{||\mathbf{u}||\leq a} \exp[\frac{N}{2}\langle \mathbf{Q}\mathbf{u}, \mathbf{u}\rangle] d\nu(\mathbf{u})} \tag{49}$$

Since

$$\exp\Big[\frac{N}{2}\langle \mathbf{Q}\mathbf{u}, \mathbf{u}\rangle\Big] = \sqrt{\frac{N \det \mathbf{Q}}{(2\pi)^n}} \int \exp\Big[-\frac{N}{2}\langle \mathbf{Q}\mathbf{w}, \mathbf{w}\rangle\Big] \exp\Big[N\langle \mathbf{Q}\mathbf{w}, \mathbf{u}\rangle\Big] d\mathbf{w} \tag{50}$$

After the semplification of the terms $\sqrt{N \det \mathbf{A}/(2\pi)^n}$ and the change of variable

$$w'_j = w_j + \frac{iN^{\gamma-1}}{\alpha_j \det \mathbf{J}} \xi_j(r_1, \dots, r_n) \qquad j = 1, \dots, n$$

where the functions $\xi_j(r_1, \dots, r_n)$ are properly choosen to semplify $\exp[iN^\gamma \mathbf{r}\boldsymbol{\alpha}^\gamma \mathbf{u}]$, we have:

$$\begin{aligned} &\omega_N(\exp[i\mathbf{r}\mathbf{v}])\big|_E \\ &= \frac{\exp\Big[\frac{N^{2\gamma-1}}{2(\det \mathbf{J})^2}\langle \mathbf{J}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle\Big] \int \exp\Big[-\frac{N}{2}\langle \mathbf{Q}\mathbf{w}, \mathbf{w}\rangle + iN^\gamma \mathbf{r}\boldsymbol{\alpha}^\gamma \mathbf{u}\Big] \int_{||\mathbf{u}||\leq a} \exp\Big[N\langle \mathbf{Q}\mathbf{w}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w}}{\int \exp\Big[-\frac{N}{2}\langle \mathbf{Q}\mathbf{w}, \mathbf{w}\rangle\Big] \int_{||\mathbf{u}||\leq a} \exp\Big[N\langle \mathbf{Q}\mathbf{w}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w}} \end{aligned} \tag{51}$$

Since $k > 1$ we have $2\gamma - 1 < 0$ and thus $\exp\Big[\frac{N^{2\gamma-1}}{2(\det \mathbf{J})^2}\langle \mathbf{J}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle\Big] \to 1$ as $N \to \infty$ for each $\mathbf{r} \in \mathbb{R}^n$. For the rest of the proof we need the following:

Lemma 3 (Transfer Principle)**.** *There exists $\widehat{B} > 0$ depending only on ρ such that for each $B \in (0, \widehat{B})$ and for each $a \in (0, B/2)$ and each $\mathbf{r} \in \mathbb{R}^n$, there exists $\delta = \delta(a, B) > 0$ such that as $N \to \infty$:*

$$\begin{aligned} &\int \exp\Big[-\frac{N}{2}\langle \mathbf{Q}\mathbf{w}, \mathbf{w}\rangle\Big] \exp[iN^\gamma \mathbf{r}\boldsymbol{\alpha}^\gamma \mathbf{w}] \int_{||\mathbf{u}||\leq a} \exp\Big[N\langle \mathbf{Q}\mathbf{w}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w} \\ &\quad = \int_{||\mathbf{w}||\leq B} \exp\Big[-\frac{N}{2}\langle \mathbf{Q}\mathbf{w}, \mathbf{w}\rangle\Big] \exp[iN^\gamma \mathbf{r}\boldsymbol{\alpha}^\gamma \mathbf{w}] \int \exp\Big[N\langle \mathbf{Q}\mathbf{w}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w} + O(e^{-N\delta}) \end{aligned} \tag{52}$$

For the proof of the transfer princeple see the appendix.

Once we have found $\widehat{B}$ we set A in the theorem equal to $\widehat{B}/2$. If $a \in (0, \widehat{B}/2)$, then at the price of an exponentially small error, the quotien of integrals in (51) can be replaced by a quotient of integrals using the formula (52) chosing the numerator with generic $\mathbf{r}$ and the denominator with $\mathbf{r}$ equal to the zero vector. Making the following change of variables:

$$w_j = \frac{w'_j}{(N\alpha_j)^\gamma} \quad j = 1, \ldots, n \tag{53}$$

we obtain:

$$\omega_N(\exp[i\mathbf{r}\mathbf{v}])\big|_E = \frac{\displaystyle\int_F \exp\Big[-\tfrac{N}{2}\langle \mathbf{Q}\tfrac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}, \tfrac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}\rangle\Big] \exp[i\mathbf{r}\mathbf{w}] \int \exp\Big[N\langle \mathbf{Q}\tfrac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w}}{\displaystyle\int_F \exp\Big[-\tfrac{N}{2}\langle \mathbf{Q}\tfrac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}, \tfrac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}\rangle\Big] \int \exp\Big[N\langle \mathbf{Q}\tfrac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) d\mathbf{w}} \tag{54}$$

where $F = \{|w_1| \leq BN_1^\gamma, \ldots |w_n| \leq BN_n^\gamma\}$.

We define the function:

$$\begin{aligned} \Phi(x_1, \ldots, x_n) &= \sum_{l=1}^n \alpha_l \ln\left[\cosh\left(\sum_{s=1}^n \alpha_s J_{ls}\, x_s + h_l\right)\right] \\ &= -\ln 2 + \frac{1}{2}\left(\sum_{l=1}^n \alpha_l^2\, J_{ll}\, x_l^2 + \sum_{l\neq s} \alpha_l\alpha_s\, J_{ls}\, x_l x_s\right) - G(x_1, \ldots, x_n) \end{aligned} \tag{55}$$

So we have:

$$\begin{aligned} &\exp\Big[-\frac{N}{2}\langle \mathbf{Q}\frac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}, \frac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}\rangle\Big] \int \exp\Big[N\langle \mathbf{Q}\frac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}, \mathbf{u}\rangle\Big] d\nu(\mathbf{u}) \\ &\quad = \exp\Bigg[-N\Bigg(\frac{1}{2}\sum_{l=1}^n \alpha_l^2\, J_{ll}\, \frac{w_l^2}{N_l^{2\gamma}} + \frac{1}{2}\sum_{l\neq s} \alpha_l\alpha_s\, J_{ls}\, \frac{w_l}{N_l^\gamma}\frac{w_s}{N_s^\gamma} \\ &\quad\quad - \Bigg(\Phi\Big(\frac{w_1}{N_1^\gamma} + \mu_1, \ldots, \frac{w_n}{N_n^\gamma} + \mu_n\Big) - \Phi(\mu_1, \ldots, \mu_n) - \sum_{l=1}^n \alpha_l^2\, J_{ll}\, \frac{w_l}{N_l^\gamma}\mu_l - \sum_{l\neq s} \alpha_l\alpha_s\, J_{ls}\, \frac{w_l}{N_l^\gamma}\mu_s\Bigg)\Bigg] \\ &\quad = \exp\Bigg[-N\Bigg(G\Big(\frac{w_1}{N_1^\gamma} + \mu_1, \ldots, \frac{w_n}{N_n^\gamma} + \mu_n\Big) - G(\mu_1, \ldots, \mu_n)\Bigg)\Bigg] \\ &\quad = \exp\Bigg[-\frac{N}{N^{|\boldsymbol{\eta}|\gamma}}\sum_{|\boldsymbol{\eta}|=2k} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}\Big(\frac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}\Big)^{\boldsymbol{\eta}} + NO\Bigg(\Bigg\|\sum_{|\boldsymbol{\eta}|=2k+1}\Big(\frac{\mathbf{w}}{(N\boldsymbol{\alpha})^\gamma}\Big)^{\boldsymbol{\eta}}\Bigg\|\Bigg)\Bigg] \end{aligned} \tag{56}$$

Hence, for each $\mathbf{w} \in \mathbb{R}^n$ the expression (56) as $N \to \infty$ tends to:

$$\exp\Bigg[-\sum_{|\boldsymbol{\eta}|=2k} \frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}\Big(\frac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}\Big)^{\boldsymbol{\eta}}\Bigg] \tag{57}$$

Let ϵ be any number which satisfies

$$\epsilon \in \left(0, \min_{|\eta|=2k}\left\{\frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}\right\}\right)$$

We may find $\widetilde{B} > 0$ such that

$$N\Bigg(\Phi\Big(\frac{w_1}{N_1^\gamma}+\mu_1,\dots,\frac{w_n}{N_n^\gamma}+\mu_n\Big)-\Phi(\mu_1,\dots,\mu_n)-\sum_{l=1}^{n}\alpha_l^2\, J_{ll}\,\frac{w_l}{N_l^\gamma}\mu_l-\sum_{l\neq s}\alpha_l\alpha_s\, J_{ls}\,\frac{w_l}{N_l^\gamma}\mu_s\Bigg)$$
$$\leq N\Bigg(\sum_{l=1}^{n}\alpha_l^2\, J_{ll}\,\frac{w_l^2}{N_l^{2\gamma}}+\sum_{l\neq s}\alpha_l\alpha_s\, J_{ls}\,\frac{w_l}{N_l^\gamma}\frac{w_s}{N_s^\gamma}\Bigg)-\sum_{|\boldsymbol{\eta}|=2k}\Bigg(\frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}-\epsilon\Bigg)\Big(\frac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}\Big)^{\boldsymbol{\eta}} \tag{58}$$

whenever $\left|\frac{w_j}{N_j^\gamma}\right| \leq \widetilde{B} \quad i=1,\dots,n$. Hence the last expression in (56) is bounded by

$$\exp\Bigg[-\sum_{|\boldsymbol{\eta}|=2k}\Bigg(\frac{(\partial^{\boldsymbol{\eta}} G)(\boldsymbol{\mu})}{\boldsymbol{\eta}!}-\epsilon\Bigg)\Big(\frac{\mathbf{w}}{\boldsymbol{\alpha}^\gamma}\Big)^{\boldsymbol{\eta}}\Bigg] \qquad \text{whenever } \left|\frac{w_j}{N_j^\gamma}\right| \leq \widetilde{B} \tag{59}$$

Setting $\overline{B} = \min\{\widehat{B}, \widetilde{B}\}$ we obtain the result by the dominated convergence theorem. This proves the statement 2 of the theorem.

For $k=1$ in analogous way we prove that as $N_1 \to \infty$, ..., $N_n \to \infty$, for fixed values of α_1, ..., α_n, the expression (41) converges to:

$$\frac{\exp\Big[\frac{1}{2}\langle(\mathbf{D}_{\boldsymbol{\alpha}}\mathbf{J}\mathbf{D}_{\boldsymbol{\alpha}})^{-1}\mathbf{r},\mathbf{r}\rangle\Big]\int\exp[i\mathbf{r}\mathbf{w}]\exp\Big[-\frac{1}{2}\langle\mathcal{H}_G(\boldsymbol{\mu})\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}},\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}}\rangle\Big]d\mathbf{w}}{\exp\Big[-\frac{1}{2}\langle\mathcal{H}_G(\boldsymbol{\mu})\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}},\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}}\rangle\Big]d\mathbf{w}} \tag{60}$$

In particular

$$\frac{\int\exp[i\mathbf{r}\mathbf{w}]\exp\Big[-\frac{1}{2}\langle\mathcal{H}_G(\boldsymbol{\mu})\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}},\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}}\rangle\Big]d\mathbf{w}}{\int\exp\Big[-\frac{1}{2}\langle\mathcal{H}_G(\boldsymbol{\mu})\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}},\frac{\mathbf{w}}{\sqrt{\boldsymbol{\alpha}}}\rangle\Big]d\mathbf{w}} = \exp\Big[-\frac{1}{2}\langle\widetilde{\mathcal{H}}_G^{-1}(\boldsymbol{\mu})\mathbf{r},\mathbf{r}\rangle\Big] \tag{61}$$

where $\widetilde{\mathcal{H}}_G = \mathbf{D}_\alpha^{-1}\mathcal{H}_G\mathbf{D}_\alpha^{-1}$. Taking off $(\mathbf{D}_\alpha\mathbf{J}\mathbf{D}_\alpha)^{-1}$ from $\widetilde{\mathcal{H}}_G^{-1}$ we obtain the matrix $\widetilde{\boldsymbol{\chi}}$. Hence the (41) tends to a normal multivariate distribution whose covariance matrix is $\widetilde{\boldsymbol{\chi}}$. This proves the statement 1 of the theorem.

3 Examples

We now analyze the case of two populations of same cardinality. The Hamiltonian

$$H_N(m_1,m_2) = -\frac{N}{8}(J_{11}m_1^2 + J_{22}m_2^2 + 2J_{12}m_1m_2 + 4h_1m_1 + 4h_2m_2) \tag{62}$$

is a convex function of the magnetizations if the redux interaction matrix $\mathbf{J}$

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix} \tag{63}$$

is positive definited, that is $J_{11} > 0$ and $J_{11}J_{22} - J_{12}^2 > 0$. The stationary points of the function G

$$\begin{aligned} G(x_1,x_2) = &\frac{1}{8}\Big(J_{11}x_1^2 + J_{22}x_2^2 + 2J_{12}\, x_1x_2\Big) \\ &- \frac{1}{2}\ln\Big[2\cosh(\frac{J_{11}}{2}x_1 + \frac{J_{12}}{2}x_2 + h_1)\Big] \\ &- \frac{1}{2}\ln\Big[2\cosh(\frac{J_{12}}{2}x_1 + \frac{J_{22}}{2}x_2 + h_2)\Big] \end{aligned} \tag{64}$$

are solutions (μ_1,μ_2) of the Mean Field Equations of the model:

$$\begin{cases} \mu_1 &= \tanh(\frac{J_{11}}{2}\mu_1 + \frac{J_{12}}{2}\mu_2 + h_1) \\ \mu_2 &= \tanh(\frac{J_{12}}{2}\mu_1 + \frac{J_{22}}{2}\mu_2 + h_2) \end{cases} \tag{65}$$

The Hessian matrix of the function G computed in a stationary point is:

$$\mathcal{H}_G = \frac{1}{8}\begin{pmatrix} 2J_{11} - J_{11}^2(1-\mu_1^2) - J_{12}^2(1-\mu_2^2) & 2J_{12} - J_{11}J_{12}(1-\mu_1^2) - J_{22}J_{12}(1-\mu_2^2) \\ 2J_{12} - J_{11}J_{12}(1-\mu_1^2) - J_{22}J_{12}(1-\mu_2^2) & 2J_{22} - J_{12}^2(1-\mu_1^2) - J_{22}^2(1-\mu_2^2) \end{pmatrix} \tag{66}$$

and its determinant is:

$$\det_{\mathcal{H}_G}(\mu_1,\mu_2) = \frac{\det\mathbf{J}}{64}[4 - 2J_{11}(1-\mu_1^2) - 2J_{22}(1-\mu_2^2) + \det\mathbf{J}(1-\mu_1^2)(1-\mu_2^2)] \tag{67}$$

So the stationary point (μ_1,μ_2) is a minimum of G of type $k=1$ if:

$$\begin{cases} (\mathcal{H}_G)_{11}(\mu_1,\mu_2) > 0 \\ \det_{\mathcal{H}_G}(\mu_1,\mu_2) > 0 \end{cases} \tag{68}$$

For example if we consider the particular case in which the external field h_1 and h_2 are equal to zero and the parameters J_{11} and J_{22} are the same. The stationary point $(0,0)$ verifies the conditions (68) if:

- $0 < J_{11} \leq 1$ and $-J_{11} < J_{12} < J_{11}$
- $1 < J_{11} < 2$ and $J_{11} - 2 < J_{12} < 2 - J_{11}$

To have a minimum point of type $k > 1$ the Hessian matrix $\mathcal{H}_G(\mu_1, \mu_2)$ must be equal to the matrix with zero elements. This means:

$$\begin{cases} J_{11} \geq 2 \\ J_{22} \geq 2 \\ J_{12} = 0 \\ \mu_1^2 = \dfrac{J_{11} - 2}{J_{11}} \\ \mu_2^2 = \dfrac{J_{22} - 2}{J_{22}} \end{cases} \tag{69}$$

Only if the third partial derivatives of G computed in (μ_1, μ_2) are equal to zero the point can be a minimum. This is verified if and only if $J_{11} = J_{22} = 2$. Hence $(\mu_1, \mu_2) = (0,0)$. Computing the partial derivatives of fourth order we can assert that this is a minimum of type $k = 2$.

4 Appendix

Proof of lemma (2). Given $\theta_1, \ldots, \theta_n$ real

$$P\left\{ \frac{W_1}{(N_1)^{1/2-\gamma}} + \frac{S_1 - N_1\mu_1}{(N_1)^{1-\gamma}} \leq \theta_1, \ \ldots, \ \frac{W_n}{(N_n)^{1/2-\gamma}} + \frac{S_n - N_n\mu_n}{(N_n)^{1-\gamma}} \leq \theta_n \right\} =$$

$$P\left\{ \sqrt{N_1}W_1 + S_1 \in E_1, \ \ldots, \ \sqrt{N_n}W_n + S_n \in E_n \right\} \tag{70}$$

where $E_l = \Big[-\infty,\ (N_l)^{1-\gamma}\theta_l + N_l\mu_l \Big]$
The distribution of $(\sqrt{N_1}\, W_1, \ldots, \sqrt{N_n}\, W_n)$ is

$$\widetilde{\rho}(\mathbf{x}) = \sqrt{\frac{\det \widetilde{\mathbf{A}}}{(2\pi)^n}}\ \exp\Big[-\frac{1}{2}\langle \widetilde{\mathbf{A}}\mathbf{x}, \mathbf{x}\rangle \Big] \tag{71}$$

where $\widetilde{\mathbf{A}} = 1/N\mathbf{J}$. From the definition of the matrix $\mathbf{A}$ follows that $\widetilde{\mathbf{A}}$ is positive defined. The joint distribution of the random vector $(S_1(\sigma), \ldots, S_n(\sigma))$ is:

$$\frac{e^{-H_N(x_1,\ldots,x_n)} d\rho^{*N_1}(x_1) * \cdots * \rho^{*N_n}(x_n)}{\int\limits_{\mathbb{R}^N} e^{-H_N(x_1,\ldots,x_n)} d\rho^{*N_1}(x_1) * \cdots * \rho^{*N_n}(x_n)} \tag{72}$$

where $d(\rho^{*N_1}(x_1) * \cdots * \rho^{*N_n}(x_n))$ denotes the N-fold convolution of the measure ρ with itself and $H_N(x_1, \ldots, x_n)$ is the Hamiltonian expressed as function of the sums of spins (43). The distribution of (30) is given by the convolution of the distribution (71) with the distribution (72), so we have:

$$P\Big\{ \sqrt{N_1}\, W_1 + S_1 \in E_1, \ldots, \sqrt{N_n}\, W_n + S_n \in E_n \Big\} =$$

$$= \frac{\sqrt{\det \widetilde{\mathbf{A}}}}{(2\pi)^{n/2} Z_N} \int_E dx_1 \ldots dx_n \int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\Big(\sum_{l=1}^n \frac{\alpha_l J_{ll}}{N_l}(x_l - t_l)^2 + \sum_{l \neq s} J_{ls}\sqrt{\frac{\alpha_l\alpha_s}{N_l N_s}}(x_l - t_l)(x_s - t_s) \Big) \Big]$$

$$\exp\Big[\frac{1}{2}\Big(\sum_{l=1}^n \frac{\alpha_l J_{ll}}{N_l} t_l^2 + \sum_{l \neq s} J_{ls}\sqrt{\frac{\alpha_l\alpha_s}{N_l N_s}} t_l t_s \Big) + \sum_{l=1}^n h_l t_l \Big] \rho^{*N_1}(dt_1) \ldots \rho^{*N_n}(dt_n)$$

$$= \frac{\sqrt{\det \widetilde{\mathbf{A}}}}{(2\pi)^{n/2} Z_N} \int_E dx_1 \ldots dx_n \exp\Big[-\frac{1}{2}\Big(\sum_{l=1}^n \frac{\alpha_l J_{ll}}{N_l} x_l^2 + \sum_{l \neq s} J_{ls}\sqrt{\frac{\alpha_l\alpha_s}{N_l N_s}} x_l x_s \Big) \Big]$$

$$\prod_{l=1}^n \int_{\mathbb{R}} \exp\Big[t_l \Big(\sum_{k=1}^n J_{lk}\sqrt{\frac{\alpha_l\alpha_k}{N_l N_k}}\ x_k + h_l \Big) \Big] \rho^{*N_l}(dt_l) \tag{73}$$

where $E = E_1 \times E_2 \times \cdots \times E_n$. If we make the following change of variables:

$$s_l = \frac{x_l - N_l \mu_l}{(N_l)^{1-\gamma_l}} \tag{74}$$

we obtain for (73):

$$\frac{\sqrt{\det \widetilde{\mathbf{A}}}\ (N_1)^{1/2-\gamma} \dots (N_n)^{1/2-\gamma}}{(2\pi)^{n/2} Z_N} \int_{-\infty}^{\theta_1} \cdots \int_{-\infty}^{\theta_n} ds_1 \dots ds_n \ \exp\Big[-\frac{N}{2} \Big(\sum_{l=1}^{n} \alpha_l^2 J_{ll} \Big(\frac{s_l}{N_l^{\ \gamma}} + \mu_l \Big)^2 +$$

$$\sum_{l \neq k} \alpha_l \alpha_k J_{lk} \Big(\frac{s_l}{N_l^{\ \gamma}} + \mu_l \Big) \Big(\frac{s_k}{N_k^{\ \gamma}} + \mu_k \Big) \Big) + \sum_{l=1}^{n} N_l \ln \Big[2 \cosh \Big(\sum_{k=1}^{n} \alpha_k J_{lk} \Big(\frac{s_k}{N_k^{\ \gamma}} + \mu_k \Big) + h_l \Big) \Big] \Big]$$

$$= \frac{\sqrt{\det \widetilde{\mathbf{A}}\ N_1 \dots N_n}}{(2\pi)^{n/2} Z_N} \int_{-\infty}^{\theta_1} \cdots \int_{-\infty}^{\theta_n} \exp\Big[-NG\Big(\frac{s_1}{N_1^{\ \gamma}} + \mu_1, \dots, \frac{s_n}{N_n^{\ \gamma}} + \mu_n \Big) \Big] ds_1 \dots ds_n$$

Taking $\theta_1 \to \infty, \dots, \theta_n \to \infty$ gives an equation for Z_N which when substituted back yiealds the result. The integral in the last expression is finite by (14). $\square$

Proof of Transfer Principle. We shall find $\widehat{B}$ such that for each $B \in (0, \widehat{B})$ and each $a \in (0, B/2)$ there exists $\delta = \delta(a, B)$ such that as $N \to \infty$

$$\int_{||\mathbf{w}||>B} \exp\Big[-\frac{N}{2} \langle \mathbf{Q}\mathbf{w}, \mathbf{w} \rangle \Big] \int_{||\mathbf{u}|| \leq a} \exp\Big[N \langle \mathbf{Q}\mathbf{w}, \mathbf{u} \rangle \Big] d\nu(\mathbf{u}) d\mathbf{w} = O(e^{-N\delta}) \tag{75}$$

and

$$\int_{||\mathbf{w}|| \leq B} \exp\Big[-\frac{N}{2} \langle \mathbf{Q}\mathbf{w}, \mathbf{w} \rangle \Big] \int_{||\mathbf{u}||>a} \exp\Big[N \langle \mathbf{Q}\mathbf{w}, \mathbf{u} \rangle \Big] d\nu(\mathbf{u}) d\mathbf{w} = O(e^{-N\delta}) \tag{76}$$

For any $B > 0$, any $a \in (0, B/2)$, the left-hand side of (75) is bounded by:

$$\int_{||\mathbf{w}|| \geq B} \exp\Big[-N \langle \mathbf{Q}\mathbf{w}, \frac{\mathbf{w}}{2} - \mathbf{a} \rangle \Big] d\mathbf{w} \leq \int_{||\mathbf{w}|| \geq B} \exp\Big[-N \langle \mathbf{Q}\mathbf{w}, \frac{\mathbf{B}}{2} - \mathbf{a} \rangle \Big] d\mathbf{w} = O(e^{-N\delta_1}) \tag{77}$$

with $\delta_1 = \langle \mathbf{Q}\mathbf{B}, \frac{\mathbf{B}}{2} - \mathbf{a} \rangle$, where $\mathbf{a} = (a, \dots, a)$ and $\mathbf{B} = (B, \dots, B)$. To prove (76) we introduce the Legendre transformation Φ^* of the function Φ:

$$\Phi^*(\mathbf{y}) = \sup_{\mathbf{x} \in \mathbb{R}^n} \{ \langle \mathbf{y}, \mathbf{x} \rangle - \Phi(\mathbf{x}) \} \qquad \mathbf{y} \in \mathbb{R}^n \tag{78}$$

and the function $\widetilde{\Phi}^*(\mathbf{y}) = \Phi^*(\boldsymbol{\alpha}\,\mathbf{.}\,\mathbf{I}(\mathbf{y}))$ where $\mathbf{I}(\mathbf{y}) = \Big(\sum_{i=1}^{n} \alpha_i J_{1i}\, y_i, \ldots, \sum_{i=1}^{n} \alpha_i J_{ni}\, y_i\Big)$ and we give the following:

Lemma 4. *The function Φ^* is convex, finite and smooth on a certain open (possibly unbounded) set I containing $\boldsymbol{\mu}$. $\Phi^* = +\infty$ on $\bar{I}^C$, and Φ^* is strictly convex on I.*
For any $\mathbf{u} = (u_1, \ldots, u_n)$ with $u_1, \ldots, u_n$ positive

$$P\{U_1 > u_1, \ldots, U_1 > u_n\} \leq \exp[-N(\widetilde{\Phi}^*(\boldsymbol{\mu}+\mathbf{u}) - \widetilde{\Phi}^*(\boldsymbol{\mu}) - \nabla\widetilde{\Phi}^*(\boldsymbol{\mu})\,\mathbf{.}\,\mathbf{u})] \tag{79}$$

There exists a number $u_0 > 0$ such that for all $u_j \in (0, u_0)$

$$\frac{\partial\widetilde{\Phi}^*}{\partial x_j}(\boldsymbol{\mu}+\mathbf{u}) - \frac{\partial\widetilde{\Phi}^*}{\partial x_j}(\boldsymbol{\mu}) = \alpha_j \mathbf{I}_j(\mathbf{u}) + \zeta_j(\mathbf{u}) \tag{80}$$

with $\zeta_j(\mathbf{u}) > 0$.

The left-hand side of (76) is bounded by

$$\frac{\pi^{n/2}B^n}{\Gamma(n/2+1)} \sup_{|\mathbf{w}|\leq B} \int_{|\mathbf{u}|>a} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{u}\rangle\Big)\right] d\nu(\mathbf{u}) \tag{81}$$

Integrating by parts, we have:

$$\begin{aligned}
&\sup_{|\mathbf{w}|\leq B} \int_{|\mathbf{u}|>a} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{u}\rangle\Big)\right] d\nu(\mathbf{u}) \\
&\quad\leq \sup_{|\mathbf{w}|\leq B} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{a}\rangle\Big)\right] P\{U_1 > a, \ldots, U_n > a\} \\
&\qquad + N \sup_{|\mathbf{w}|\leq B} \langle\mathbf{Q}\mathbf{w},\mathbf{1}\rangle \int_{|\mathbf{u}|>a} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{u}\rangle\Big)\right] P\{U_1 > u_1, \ldots, U_n > u_n\} d\mathbf{u}
\end{aligned} \tag{82}$$

Using (79) we bound $P\{U_1 > u_1, \ldots, U_n > u_n\}$ where $u_1, \ldots, u_n \geq a$. The last term in (82) is bounded by:

$$\begin{aligned}
&N \sup_{|\mathbf{w}|\leq B} \langle\mathbf{Q}\mathbf{w},\mathbf{1}\rangle \int_{a<|\mathbf{u}|<u_0} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{u}\rangle\frac{1}{2}\langle\mathbf{Q}\mathbf{u},\mathbf{u}\rangle + \langle\boldsymbol{\theta},\mathbf{1}\rangle\Big)\right] d\mathbf{u} \\
&\quad + N \sup_{|\mathbf{w}|\leq B} \langle\mathbf{Q}\mathbf{w},\mathbf{1}\rangle \int_{|\mathbf{u}|>u_0} \exp\left[-N\Big(\frac{1}{2}\langle\mathbf{Q}\mathbf{w},\mathbf{w}\rangle - \langle\mathbf{Q}\mathbf{w},\mathbf{u}\rangle + \langle\mathbf{u},\boldsymbol{\vartheta}\rangle\Big)\right] d\mathbf{u} \\
&\quad = O(e^{-N\delta_2})
\end{aligned} \tag{83}$$

where $\delta_2 = \min\{\langle\boldsymbol{\theta},\mathbf{1}\rangle/2, \langle\mathbf{u_0}, \boldsymbol{\vartheta}-\mathbf{B}\rangle\}$. The term in (82) involving $P\{U_1 > a, \ldots, U_n > a\}$ is handled similarly. We have thus proved (75) and (76) with $\delta = \min\{\delta_1, \delta_2\}$ $\square$

5 Conclusions and outlooks

In this paper we have generalized to multi-species Curie-Weiss models the study of the normalized sums of spins and their limiting distributions. We worked under a condition of convexity of the reduced interaction matrix which allows us to use the Ellis-Newman method. The theorems presented in this work obtain a complete classification of the distribution when the first non vanishing partial derivatives are all the same order (homogeneity hypothesis). The extension to non convex interactions or the complete classification of the limiting distribution beyond the homogeneity hypothesis will be subject of further investigation.

References

[BCG03] A. Bianchi, P. Contucci, and C. Giardina, *Thermodynamic Limit for Mean-Field Spin Models*, MPEJ (2003).

[CG07] P. Contucci and S. Ghirlanda, *Modeling society with statistical mechanics: an application to cultural contact and immigration*, Quality and Quantity **41** (2007), no. 4, 569–578.

[CS87] N.R. Chaganty and J. Sethuraman, *Limit theorems in the area of large deviations for some dependent random variables*, The Annals of Probability **15** (1987), no. 2, 628–645.

[EN78a] R.S. Ellis and C.M. Newman, *Limit theorems for sums of dependent random variables occurring in statistical mechanics*, Probability Theory and Related Fields **44** (1978), no. 2, 117–139.

[EN78b] ______, *The statistics of Curie-Weiss models*, Journal of Statistical Physics **19** (1978), no. 2, 149–161.

[ENR80] R.S. Ellis, C.M. Newman, and J.S. Rosen, *Limit theorems for sums of dependent random variables occurring in statistical mechanics*, Probability Theory and Related Fields **51** (1980), no. 2, 153–169.

[GBC08] I. Gallo, A. Barra, and P. Contucci, *parameter evaluation of a simple mean-field model of social interaction*, Arxiv preprint arXiv:0810.3029 (2008).

[GC08] I. Gallo and P. Contucci, *Bipartite Mean Field Spin Systems*, Existence and Solution, MPEJ **14** (2008).

[McF01] D. McFadden, *Economic choices*, American Economic Review **91** (2001), no. 3, 351–378.