26 febbraio 2021

1) Sistemi lineare
2) Trasformazioni lineari
3) Sottospazi vettoriali e sottospazi affini

Sistemi lineari

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \end{cases}$$

$$\begin{cases} x - y + 3z = 1 \\ x \quad\;\; - 4z = 0 \end{cases} \qquad C = \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array}\right)$$

A

B

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_n \end{cases}$$

$$\Updownarrow$$

$$\boxed{AX} = \boxed{B}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

1) Permutare le equazioni

1') Permutare le righe della matrice completa

$$C = (A|B)$$

2) Moltiplicare una equazione per $k \neq 0$.

2') Moltiplicare una riga di C per $k \neq 0$.

3) Aggiungere un multiplo di una equazione a un'altra equazione.

3') Aggiungere un multiplo di una riga a un'altra riga di C.

$$\begin{cases} x - y + 3z = 1 \\ x \quad\;\; - 4z = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array}\right)$$

$$\begin{cases} x - y + 3z = 1 \\ \quad\;\; y - 7z = -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \end{array}\right)$$

$$\begin{cases} x \quad\;\; -4z = 0 \\ \quad\;\; y - 7z = -1 \end{cases} \qquad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array}\right) \end{array}$$

$$\begin{cases} x - y + 3z = 1 \\ x \qquad - 4z = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array}\right)$$

$$\begin{cases} x - y + 3z = 1 \\ \qquad y - 7z = -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \end{array}\right)$$

$$\begin{cases} x \qquad - 4z = 0 \\ \qquad y - 7z = -1 \end{cases} \qquad \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array}\right) \end{array}$$

$$\begin{cases} x = 4t \\ y = 7t-1 \\ z = t \end{cases}$$

Soluzione parametrica del sistema lineare

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4t \\ 7t-1 \\ t \end{pmatrix} = \begin{pmatrix} 4t \\ 7t \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = t\begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = 2 \end{cases}$$

$$C = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 7 & 2 \end{array}\right)$$

$$C = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & -2 & 4 & 4 & -4 \end{array}\right)$$

$$C = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & u \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\begin{cases} x+2y-z+u=2 \\ x+y+z+3u=0 \\ 3x+4y+z+7u=2 \end{cases}$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 7 & 2 \end{array}\right)$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & -2 & 4 & 4 & -4 \end{array}\right)$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccccc} x & y & z & u & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+2y-z+u=2 \\ x+y+z+3u=0 \\ 3x+4y+z+7u=2 \end{cases}$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 3 & 4 & 1 & 7 & 2 \end{array}\right)$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & -2 & 4 & 4 & -4 \end{array}\right)$$

$$C=\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccccc} x & y & z & u & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = -3s - 5t - 2 \\ y = 2s + 2t + 2 \\ z = s \\ u = t \end{cases}$$

$$s\begin{pmatrix}-3\\2\\1\\0\end{pmatrix} + t\begin{pmatrix}-5\\2\\0\\1\end{pmatrix} + \begin{pmatrix}-2\\2\\0\\0\end{pmatrix}$$

$$\|$$

$$\begin{pmatrix}x\\y\\z\\u\end{pmatrix} = \begin{pmatrix}-3s\\2s\\s\\0\end{pmatrix} + \begin{pmatrix}-5t\\2t\\0\\t\end{pmatrix} + \begin{pmatrix}-2\\2\\0\\0\end{pmatrix}$$

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = \cancel{2}\ 4 \end{cases}$$

$$C = \left(\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

$$\begin{cases} x + 3z + 5u = -2 \\ y - 2z - 2u = 2 \end{cases}$$

$$\boxed{0 = 2}$$

Teorema di Rouché-Capelli

Il sistema lineare $AX=B$ è risolubile se e solo se

$$r(A)=r(A|B)$$

Oss: Se il sistema lineare $AX=B$ è risolubile, allora l'insieme delle soluzioni si può scrivere come un sottospazio vettoriale U di $\mathbb{R}^n$ traslato tramite una soluzione particolare del sistema lineare omogeneo associato.

Inoltre $\dim U = n - r(A)$

Trasformazioni lineari

Si dice che $f: V \to W$ è una trasformazione lineare se $\forall v_1, v_2 \in V$ e $\forall \alpha, \beta \in \mathbb{K}$ si ha

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin x$

$f(x) = x^2$

$f(x) = 1$

NON SONO LINEARI!

Rappresentazione matriciale delle trasformazioni lineari

$f: V \to W$

Prendiamo una base

$B_1 = (v_1, \ldots, v_n)$ di V

e una base

$B_2 = (w_1, \ldots, w_m)$ di W.

Consideriamo la matrice A definita nel modo seguente:

$$
A \quad
\begin{array}{c}
m \\ \text{righe}
\end{array}
\begin{array}{cccccc}
f(v_1) & f(v_2) & \cdots & f(v_i) & \cdots & f(v_n) \\
\hline
 & & & a_{1i} & & \\
 & & & \vdots & & \\
 & & & a_{mi} & &
\end{array}
$$

n colonne

$$\boxed{f(v_i) = a_{1i} w_1 + \cdots\cdots + a_{mi} w_m}$$

$v \in V \longleftrightarrow (x_1, \dots, x_n) \in \mathbb{R}^n$

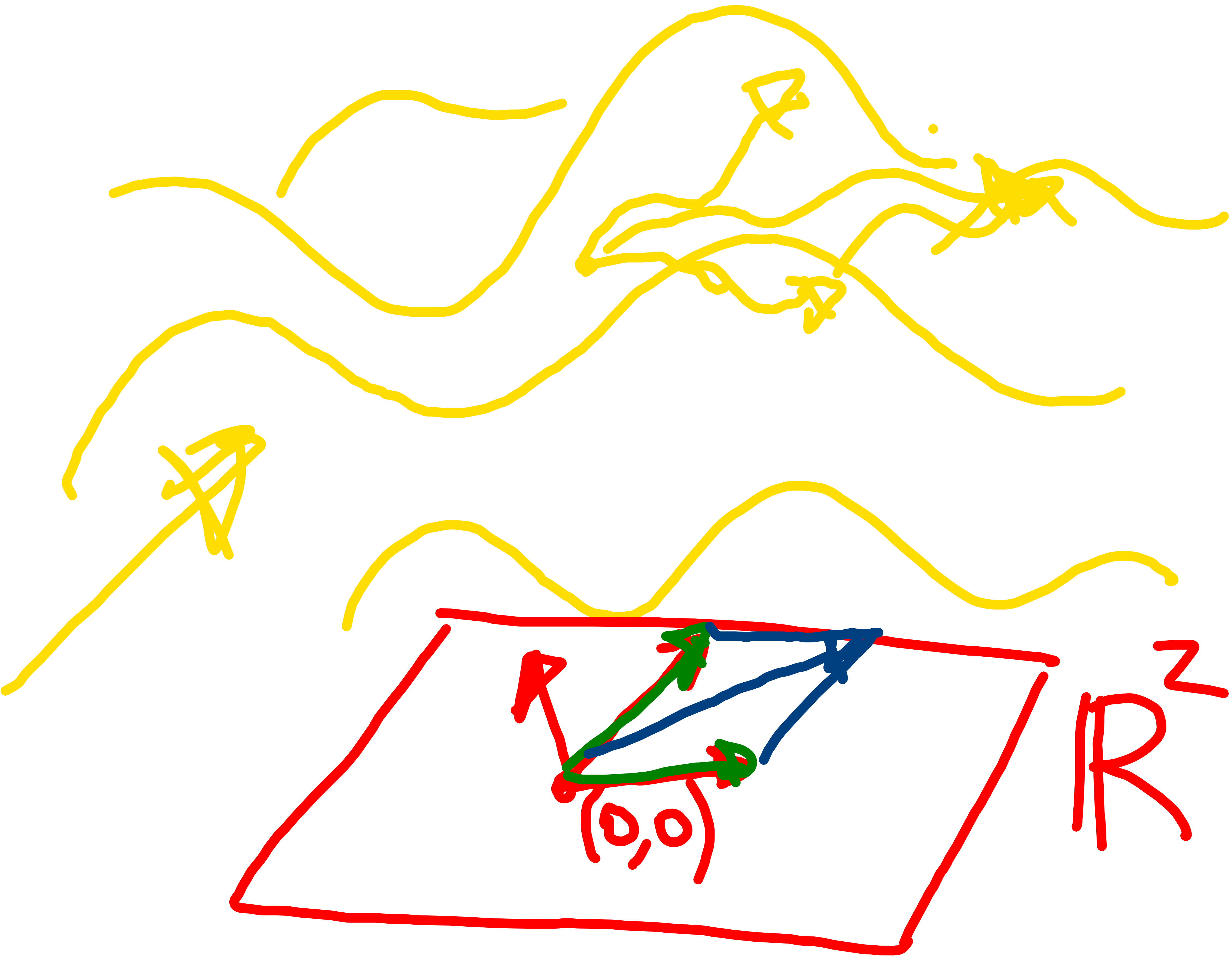
$v = x_1 v_1 + \dots + x_n v_n$

$w = f(v)$

$w = y_1 w_1 + \dots + y_m w_m \longleftrightarrow (y_1, \dots, y_m) \in \mathbb{R}^m$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boxed{A = M_{B_1 B_2}(f)}$$



(0,0)
$\mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad$ $f(x,y) = (x+2y,\, y,\, 3y)$

$B_1 = \left(\underset{v_1}{(1,1)}, \underset{v_2}{(1,0)} \right)$

$f(v_1) = (3,1,3)$

$f(v_2) = (1,0,0)$

$B_2 = \left((1,1,1), (1,1,0), (1,0,0) \right)$

$$\begin{array}{cc} f(v_1) & f(v_2) \end{array}$$

$$\underset{3\times 2}{A} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \\ 2 & 1 \end{pmatrix}$$

$(3,1,3) = x(1,1,1) +$
$y(1,1,0) +$
$z(1,0,0)$

$$\begin{cases} x+y+z=3 \\ x+y=1 \\ x=3 \end{cases}$$

$$\begin{cases} x=3 \\ y=-2 \\ z=3-x-y=3-3+2=2 \end{cases}$$

Domanda: se conosco $M_{B_1 B_2}(f)$ come posso ricavare $M_{B_1' B_2'}(f)$?

Risposta:

$$M_{B_1' B_2'}(f) = M_{B_2 B_2'}(id_W)\, M_{B_1 B_2}(f)\, M_{B_1' B_1}(id_V)$$

$id_V(v) = v$

$id_W(w) = w$

$f(g(f(v)))$

$h \circ g \circ f(v)$

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[rr, bend right, no head] & W \arrow[r, "g"] & U
\end{tikzcd}
$$

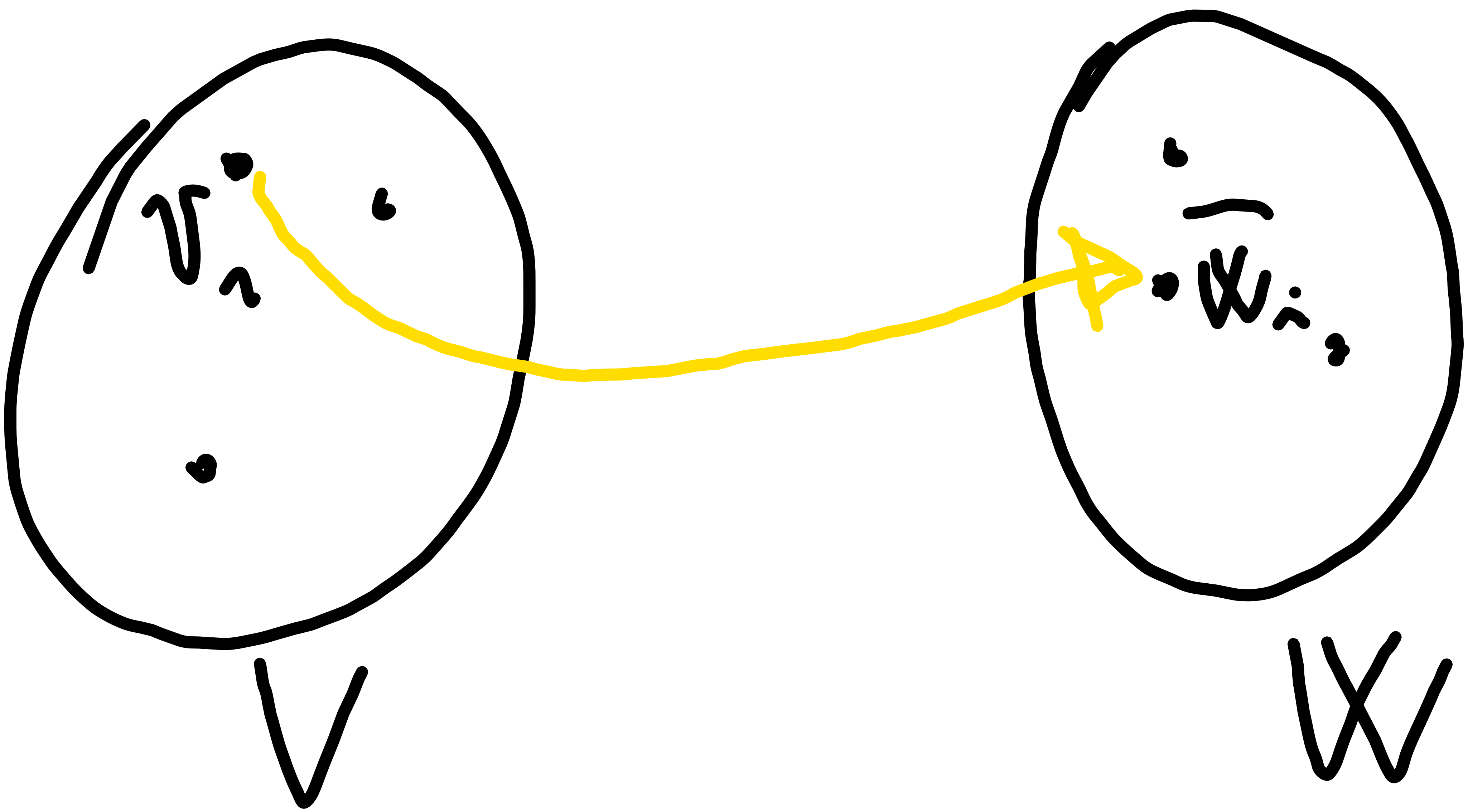
$$
B_V \qquad\qquad B_W \qquad\qquad B_U
$$

$$
M_{B_V B_U}(g \circ f) = M_{B_W B_U}(g)\, M_{B_V B_W}(f)
$$

Teorema fondamentale delle trasformazioni lineari

Siano V e W due s.v. sul campo $\mathbb{K}$. Sia $(v_1, \dots, v_n)$ una base ordinata di V e sia $(\overline{w}_1, \dots, \overline{w}_n)$ una n-upla ordinata di vettori di W.

$\exists!\ f: V \to W$ lineare t.c. $f(v_i) = \bar{w}_i \ \forall i.$



$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$B_1 = \big((1,1,1),(1,1,0),(1,0,0)\big)$$

$$T_2 = \big((1,1),(2,2),(3,3)\big)$$

$$f(1,1,1) = (1,1)$$

$$f(1,1,0) = (2,2)$$

$$f(1,0,0) = (3,3)$$

$$B_2 = \big((1,1),(1,0)\big)$$

$$A = M_{B_1 B_2}(f) = \begin{array}{ccc} f(1,1,1) & f(1,1,0) & f(1,0,0) \\ \left(\begin{matrix} 1 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 0 \end{matrix} & \left.\begin{matrix} 3 \\ 0 \end{matrix}\right) \\ (1,1) & (2,2) & (3,3) \end{array}$$

$$(1,1) = 1(1,1) + 0(1,0)$$

$B_1 = ((1,1,1),(1,1,0),(1,0,0))$

$B_2 = ((1,1),(1,0))$

$M_{B_1B_2}(f) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$B_1' = ((1,1,1),(0,1,1),(0,0,1))$

$B_2' = ((1,0),(0,1))$

$M_{B_1'B_2'}(f) = ?$

$M_{B_1'B_2'}(f) = M_{B_2B_2'}(\mathrm{id}) \, M_{B_1B_2}(f) \, M_{B_1'B_1}(\mathrm{id}_V)$

$$M_{B_1' B_1}(\mathrm{id}_V)\begin{pmatrix}x_1'\\ \vdots\\ x_n'\end{pmatrix}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$\mathrm{id}(v_1)\ \ldots\ \mathrm{id}(v_n)$

$$M_{B'_1 B_1}(\mathrm{id}_V) = \begin{array}{ccc} (1,1,1) & (0,1,1) & (0,0,1) \\ \end{array}$$

$$\left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right)$$

V_1

3×3

$$B_2 = \left((1,1), (1,0) \right)$$

$$B_2' = \underline{\underline{\left((1,0), (0,1) \right)}}$$

$$M_{B_2 B_2'}(id_V) = \begin{array}{cc} (1,1) & (1,0) \\ \end{array}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

2×2

$$M_{B_1' B_2'}(f) = \underset{2\times 2}{\begin{pmatrix}1&1\\1&0\end{pmatrix}}\underset{2\times 3}{\begin{pmatrix}1&2&3\\0&0&0\end{pmatrix}}\underset{3\times 3}{\begin{pmatrix}1&1&1\\0&0&-1\\0&-1&0\end{pmatrix}}$$

$$= \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}1&1&1\\0&0&-1\\0&-1&0\end{pmatrix} = \underset{2\times 3}{\begin{pmatrix}1&-2&-1\\1&-2&-1\end{pmatrix}}$$

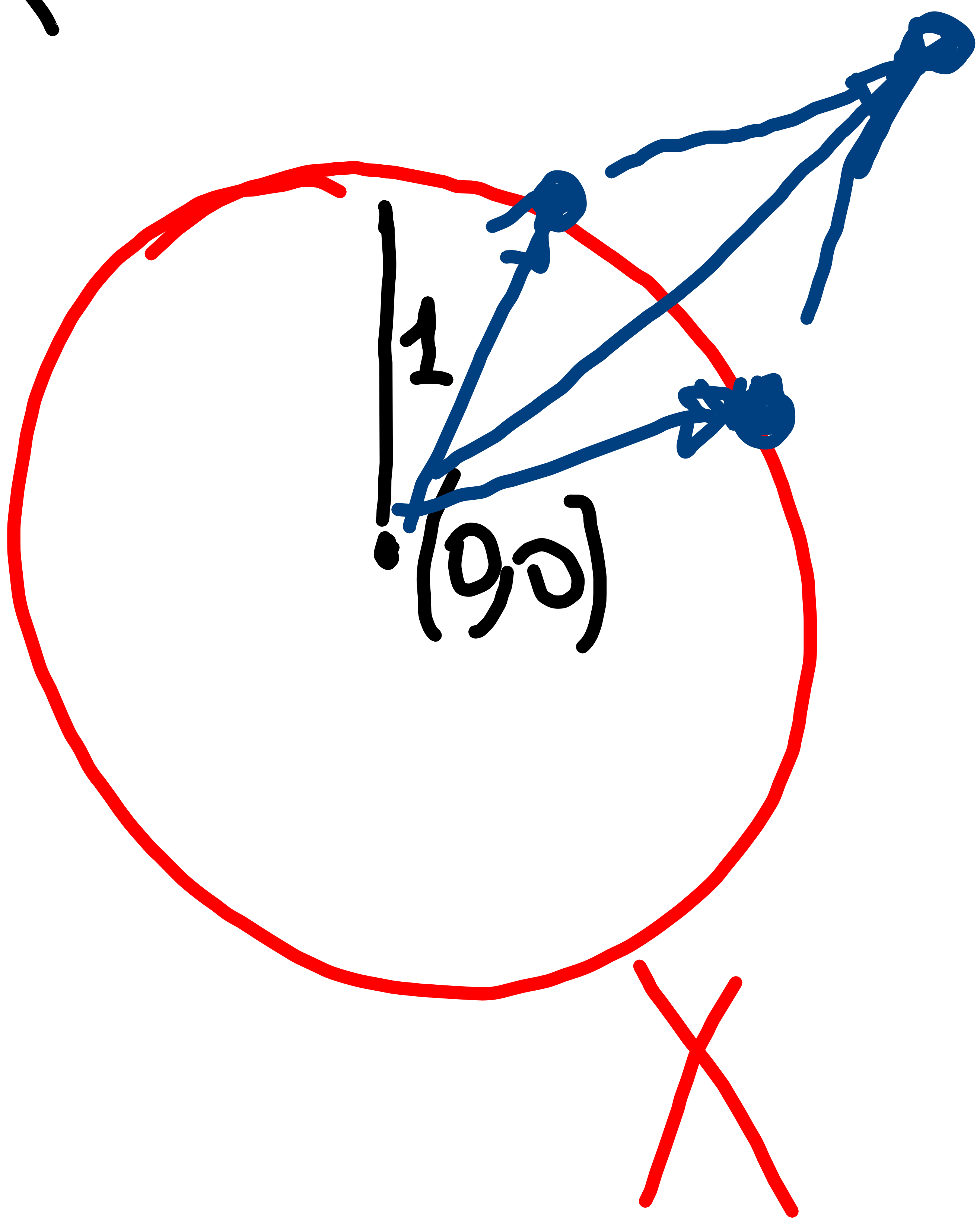
NB

$$M_{B_1 B_2}(\mathrm{id}) = \left(M_{B_2 B_1}(\mathrm{id})\right)^{-1}$$

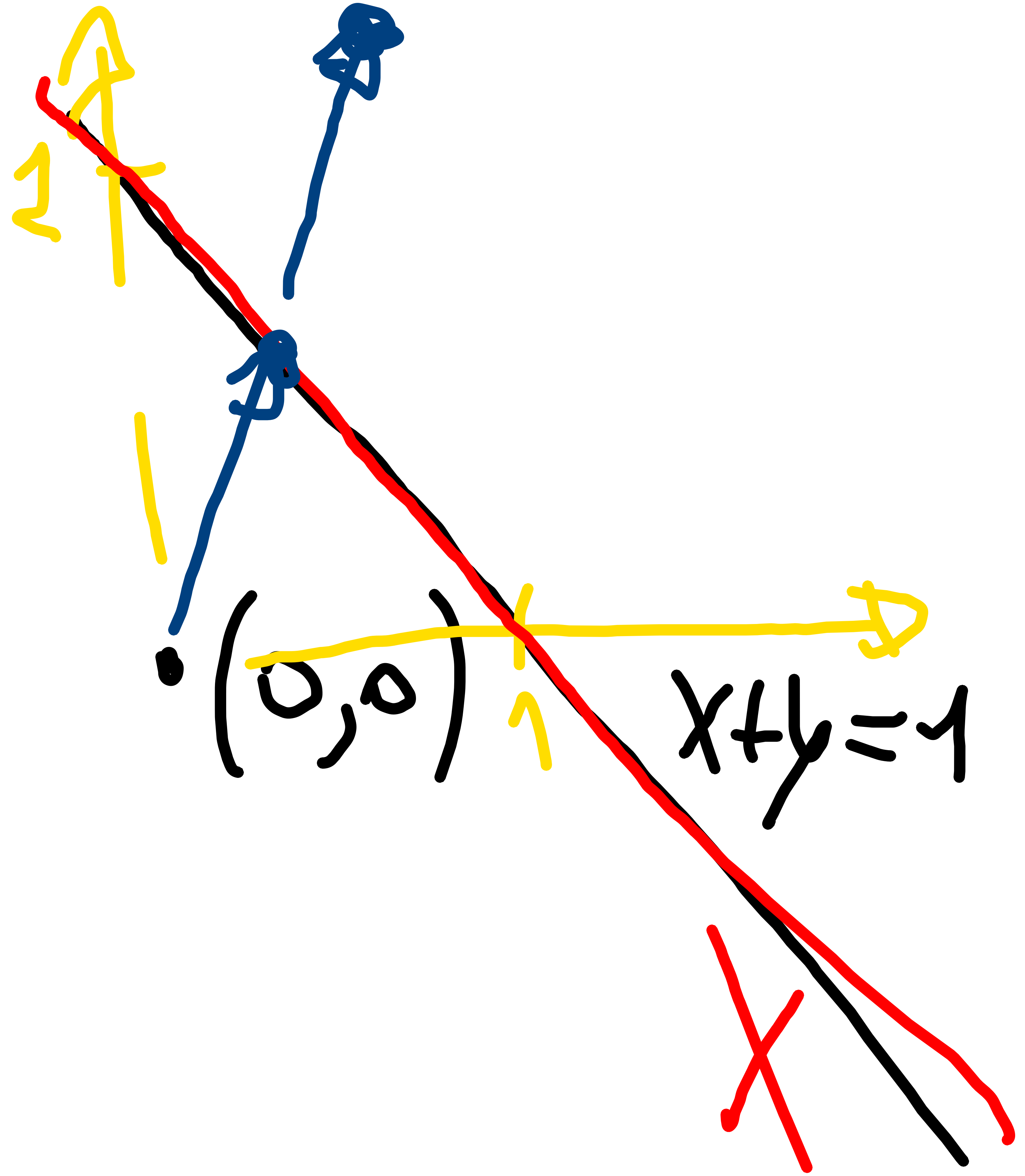
Sottospazi vettoriali e sottospazi affine.

Si dice che $X \subseteq V$ è un sottospazio vettoriale dello s.v. V se X è chiuso rispetto alle combinazioni lineari in V.

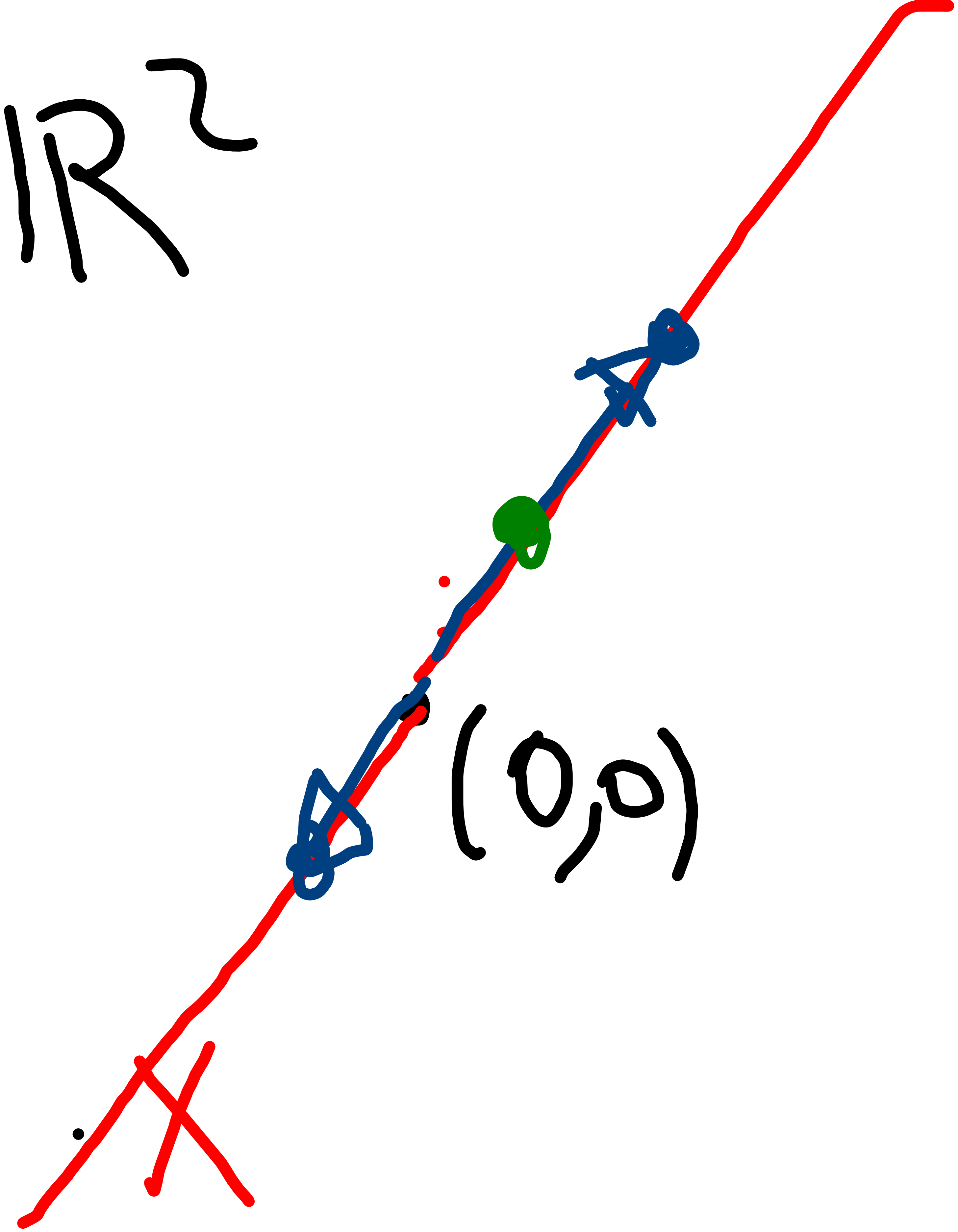
$V = \mathbb{R}^2$



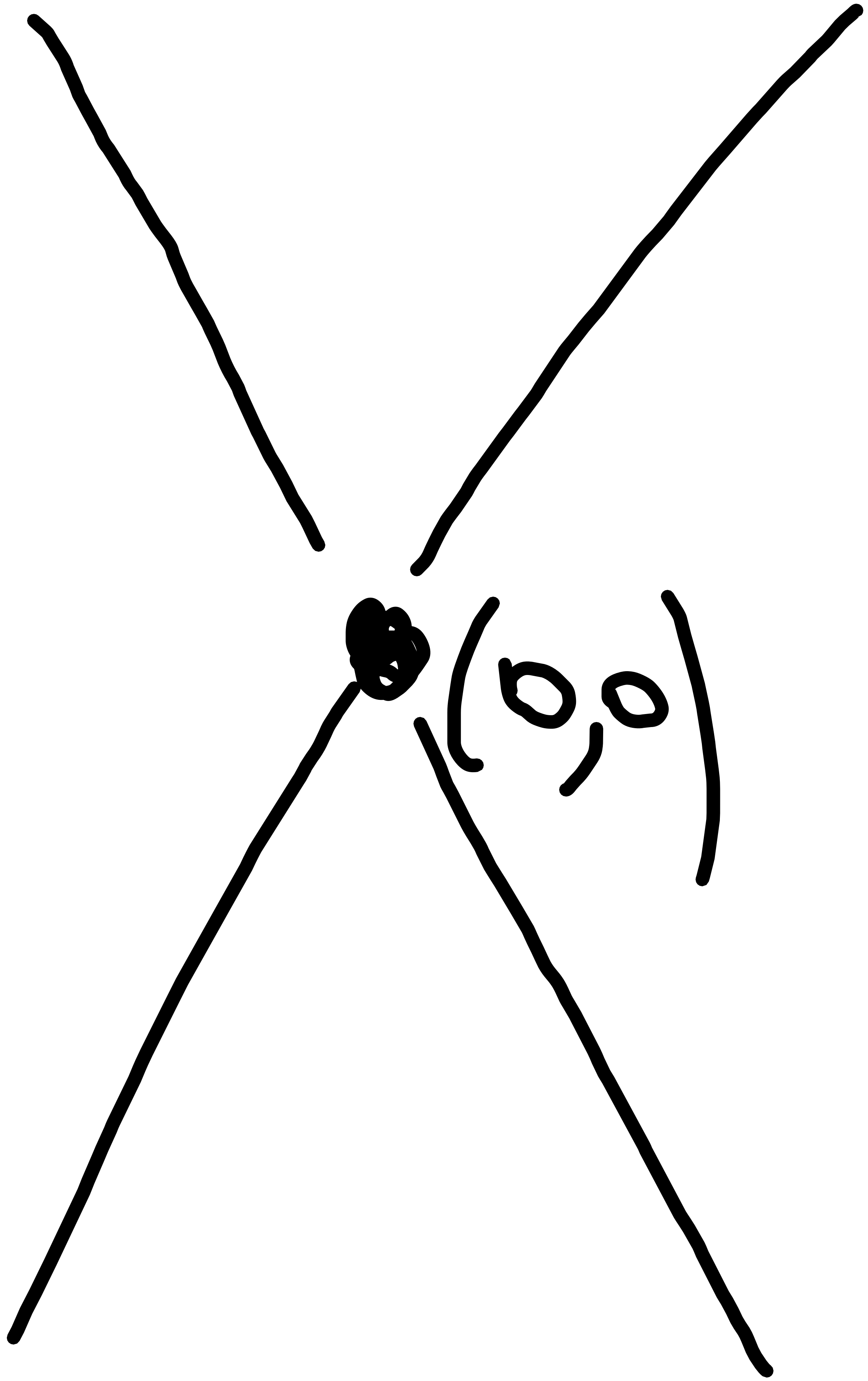
$V = \mathbb{R}^2$



$V = \mathbb{R}^2$



$V = \mathbb{R}^2$



$V = \mathbb{R}^3$

$\{(0,0,0)\}$

$\mathbb{R}^3$

rette per $(0,0,0)$

piani per $(0,0,0)$

Spazio affine

(A, φ) $\quad \varphi : \underline{A \times A} \to \underline{V}$

1) Se fisso $P \in A$, la funzione $\varphi(P, \cdot)$ è una biezione fra A e V.

2) $\varphi(P,Q) + \varphi(Q,R) = \varphi(P,R)$

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

$f(Q) := \varphi(\bar{P}, Q) \in V$

f deve essere biunivoca per ogni $\bar{P}$.

Un modo semplice per costruire uno spazio affine è il seguente.
Poniamo $A = V$ dove V è uno s.v.

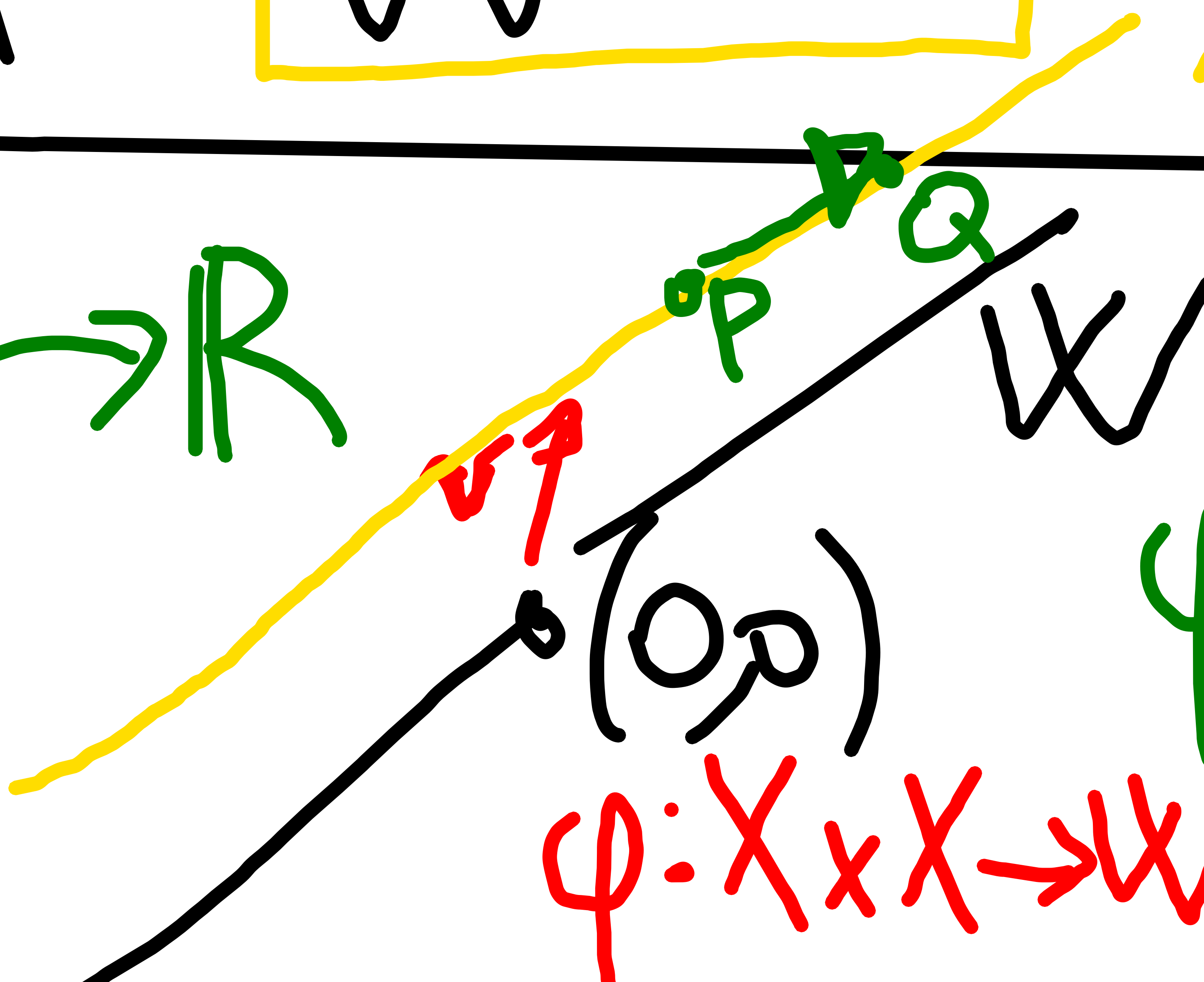
$$(A, \varphi) \qquad A = V$$

$$\varphi(\underset{\substack{\| \\ v}}{P}, \underset{\substack{\| \\ w}}{Q}) = Q - P = \underline{w - v}$$

$V \qquad X \leq V \qquad W \underset{\text{s.s.v.}}{\leq} V$

$X = \boxed{W + \bar{v}}$ è uno spazio affine.

$\varphi : X \to \mathbb{R}$



$\varphi : X \times X \to W$

$\varphi(P,Q) = Q - P = w_1 - w_2$

$P = w_1 + \bar{v}$

$Q = w_2 + \bar{v}$

$\mathbb{R}^2$ $\qquad A = \mathbb{R}^2$

$\overrightarrow{PQ}$ $\qquad \varphi(P, Q)$