

26 febbraio 2021

- 1) Sistemi lineare
- 2) Trasformazioni lineari
- 3) Sottospazi vettoriali e
sottospazi affini

Sistemi lineari

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \end{cases}$$

$$\begin{cases} \underline{x - y + 3z = 1} \\ x - 4z = 0 \end{cases}$$

$$C = \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array} \right)$$

A B

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\boxed{AX} = \boxed{B} \quad \leftarrow$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

1) Permutare le equazioni

1') Permutare le righe della
matrice completa

$$C = (A|B)$$

2) Moltiplicare una equazione
per $k \neq 0$.

2') Moltiplicare una sigla
di C per $k \neq 0$.

3) Aggiungere un multiplo
di una equazione a
un'altra equazione.

3') Aggiungere un multiplo
di una riga a un'altra riga
di C.

$$\left\{ \begin{array}{l} x - y + 3z = 1 \\ x - 4z = 0 \end{array} \right. \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array} \right)$$

$$\left\{ \begin{array}{l} x - y + 3z = 1 \\ y - 7z = -1 \end{array} \right. \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \end{array} \right)$$

$$\left\{ \begin{array}{l} x \\ y \end{array} \right. \left\{ \begin{array}{l} -4z = 0 \\ -7z = -1 \end{array} \right. \quad \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array} \right)$$

The image shows a handwritten solution for a system of linear equations. It consists of three stages of the system and its corresponding augmented matrix.

Stage 1: The system is $\begin{cases} x - y + 3z = 1 \\ x - 4z = 0 \end{cases}$. The augmented matrix is $\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array} \right)$.

Stage 2: The system is $\begin{cases} x - y + 3z = 1 \\ y - 7z = -1 \end{cases}$. The augmented matrix is $\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \end{array} \right)$.

Stage 3: The system is $\left\{ \begin{array}{l} x \\ y \end{array} \right. \left\{ \begin{array}{l} -4z = 0 \\ -7z = -1 \end{array} \right.$. The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array} \right)$.

Annotations include yellow circles around the leading ones (1) in the first column of the first two rows of each matrix, and the second column of the second two rows. A yellow box highlights the first row of the second matrix. A yellow line is drawn under the second row of the second matrix. Red 'x' and 'y' labels are placed above the first and second columns of the third matrix, respectively. A red 'z' label is placed above the third column of the third matrix. A yellow arrow points to the second row of the second matrix.

$$\begin{array}{l}
 \rightarrow \left\{ \begin{array}{l} x - y + 3z = 1 \\ x - 4z = 0 \end{array} \right. \\
 \left\{ \begin{array}{l} x - y + 3z = 1 \\ y - 7z = -1 \\ x - 4z = 0 \\ y - 7z = -1 \end{array} \right.
 \end{array}
 \quad \left| \quad
 \begin{array}{l}
 \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array} \right) \\
 \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \end{array} \right) \\
 \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array} \right)
 \end{array}
 \right.$$

The image shows a handwritten solution for a system of linear equations. The system is presented in three stages, with the augmented matrix shown to the right of each system. The equations are:

$$\begin{cases} x - y + 3z = 1 \\ x - 4z = 0 \end{cases}$$
 The augmented matrix for this system is:

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 1 & 0 & -4 & 0 \end{array} \right)$$
 The second system is:

$$\begin{cases} x - y + 3z = 1 \\ y - 7z = -1 \\ x - 4z = 0 \\ y - 7z = -1 \end{cases}$$
 The augmented matrix for this system is:

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & -7 & -1 \\ 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array} \right)$$
 The third system is:

$$\begin{cases} x - 4z = 0 \\ y - 7z = -1 \end{cases}$$
 The augmented matrix for this system is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -7 & -1 \end{array} \right)$$
 The solution is indicated by the final augmented matrix, which is in row echelon form. The variables x and y are circled in yellow, and the constants 0 and -1 are also circled in yellow. The variables x , y , and z are labeled in red below the corresponding columns of the final matrix.

$$\begin{cases} x = 4t \\ y = 7t - 1 \\ z = t \end{cases}$$

Soluzione
parametrica del
sistema
lineare



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4t \\ 7t - 1 \\ t \end{pmatrix} = \begin{pmatrix} 4t \\ 7t \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = t \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = 2 \end{cases}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 1 & 1 & 1 & 3 & | & 0 \\ 3 & 4 & 1 & 7 & | & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & -2 & 4 & 4 & | & -4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 5 & | & -2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

→

$$\begin{pmatrix} 1 & 0 & 3 & 5 & | & -2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

x y z u

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = 2 \end{cases}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 1 & 1 & 1 & 3 & | & 0 \\ 3 & 4 & 1 & 7 & | & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & -2 & 4 & 4 & | & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 5 & | & -2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

→

x y z u

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = 2 \end{cases}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 1 & 1 & 1 & 3 & | & 0 \\ 3 & 4 & 1 & 7 & | & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & -2 & 4 & 4 & | & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 5 & | & -2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 1 & | & 2 \\ 0 & -1 & 2 & 2 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

→

x y z u

$$\begin{cases} x = -3s - 5t - 2 \\ y = 2s + 2t + 2 \\ z = s \\ u = t \end{cases} \quad \left| \quad s \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right.$$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} -3s \\ 2s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ 2t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y - z + u = 2 \\ x + y + z + 3u = 0 \\ 3x + 4y + z + 7u = 4 \end{cases}$$

$$C = \begin{pmatrix} 1 & 0 & 3 & 5 & | & -2 \\ 0 & 1 & -2 & -2 & | & 2 \\ 0 & 0 & 0 & 0 & | & 2 \end{pmatrix}$$

$$\begin{cases} x + 3z + 5u = -2 \\ y - 2z - 2u = 2 \end{cases}$$

$$\boxed{0 = 2}$$

Teorema di Rouché-Capelli
Il sistema lineare $AX=B$
è risolubile se e solo se

$$\underline{r(A) = r(AB)}$$

Oss: Se il sistema lineare $AX=B$ è risolubile, allora l'insieme delle soluzioni

si può scrivere come un sottospazio vettoriale U di \mathbb{R}^n traslato tramite una soluzione particolare del sistema

lineare omogenee
assunto.

Inoltre $\dim U = n - r(A)$

Trasformazioni lineari

Si dice che $f: V \rightarrow W$

è una trasformazione

lineare se $\forall v_1, v_2 \in V$ e

$\forall \alpha, \beta \in K$ si ha

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\left. \begin{array}{l} f(x) = \sin x \\ f(x) = x^2 \\ f(x) = 1 \end{array} \right\}$$

NON
SONO
LINEARI!

Rappresentazione
matriciale delle
trasformazioni lineari

$$f: V \rightarrow W$$

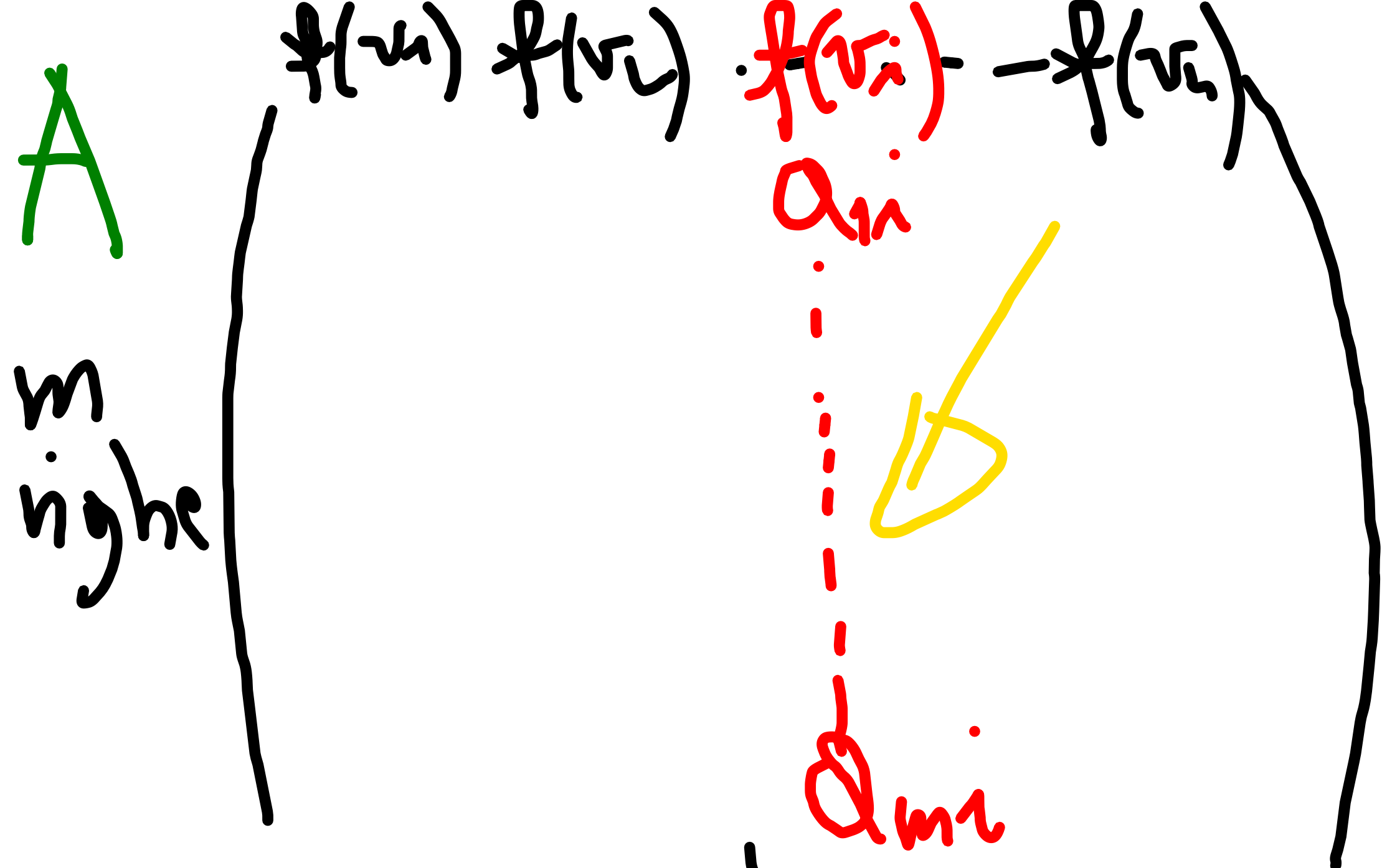
Prendiamo una base

$$B_1 = (v_1, \dots, v_n) \text{ di } V$$

e una base

$$B_2 = (w_1, \dots, w_m) \text{ di } W.$$

Consideriamo la matrice
A definita nel modo
seguinte:



h columns

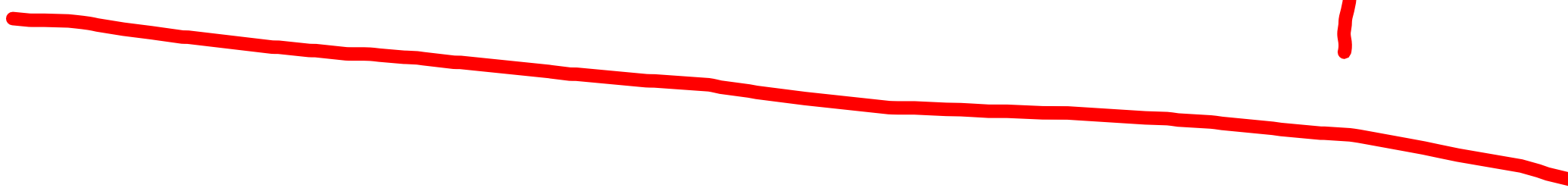
$$f(x_i) = a_{1i}x_1 + \dots + a_{mi}x_m$$

$$\underbrace{v}_{\text{circled}} \in V \longleftrightarrow (x_1, \dots, x_n) \in \mathbb{R}^n$$
$$v = x_1 v_1 + \dots + x_n v_n$$

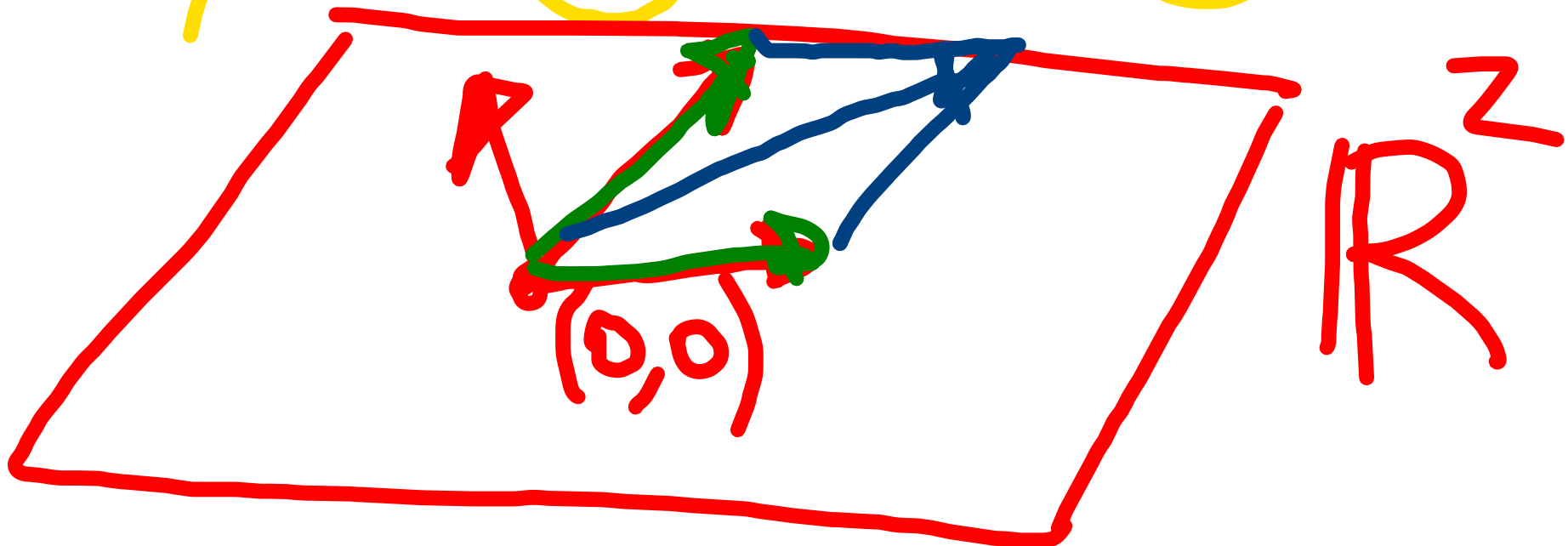
$$w = f(v)$$

$$w = y_1 w_1 + \dots + y_m w_m \longleftrightarrow (y_1, \dots, y_m) \in \mathbb{R}^m$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$



$$A = M_{B_1 B_2} (A)$$



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$f(x, y) = (x + 2y, y, 3y)$$

$$B_1 = \left(\underset{v_1}{(1, 1)}, \underset{v_2}{(1, 0)} \right)$$

$$f(v_1) = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$
$$f(v_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$B_2 = \left(\overset{f(v_1)}{(1, 1, 1)}, \overset{f(v_2)}{(1, 1, 0)}, (1, 0, 0) \right)$$

$$A = \begin{pmatrix} 3 & 0 \\ -2 & 0 \\ 2 & 1 \end{pmatrix}$$

3x2

$$(3, 1, 3) = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + y + z = 3 \\ x + y = 1 \\ x = 3 \end{cases}$$

$$\begin{cases} x = 3 \\ y = -2 \end{cases}$$

$$z = 3 - x - y = 3 - 3 + 2 = 2$$

Domanda: se conosco
 $M, B_1, B_2(\mathcal{F})$ come posso
invalutare $M, B'_1, B'_2(\mathcal{F})$?

Risposta:



$$id_v(v) = v$$
$$id_w(w) = w$$

$$h(g(f(v)))$$

↑ ↑ ↑

$$h \circ g \circ f(v)$$

$$V \xrightarrow{f} W \xrightarrow{g} U$$

$$B_V \quad B_W \quad B_U$$

$$M_{B_U B_V}(g \circ f) = M_{B_U B_W}(g) M_{B_W B_V}(f)$$

Teorema fondamentale delle
trasformazioni lineari

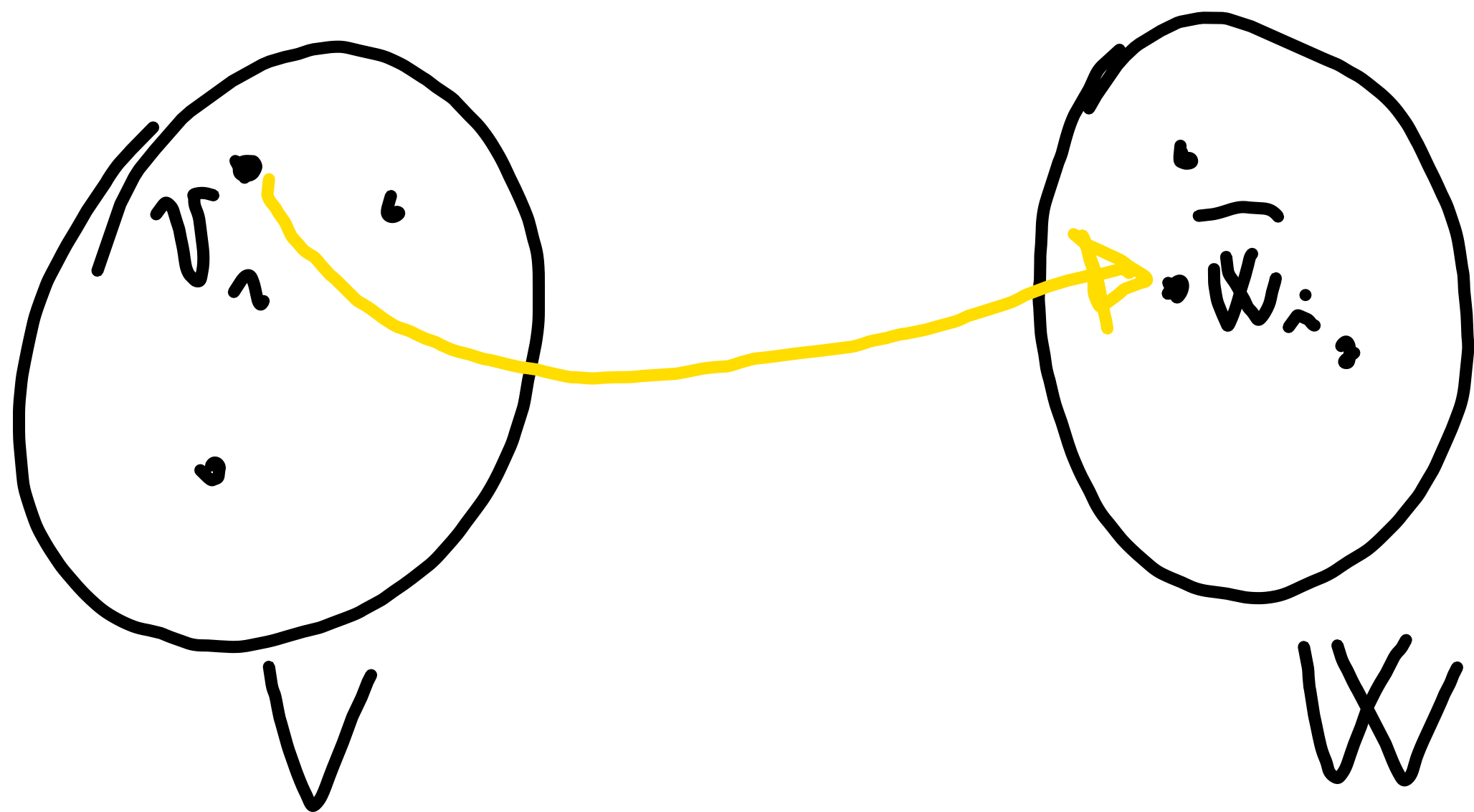
Siano V e W due s.v. su

campo K . Sia (v_1, \dots, v_n)

una base ordinata di V .

Sia (w_1, \dots, w_m) una m -upla
ordinata di vettori di W .

$\exists!$ $f: V \rightarrow W$ linear
t.c. $f(v_i) = \bar{w}_i \quad \forall i.$



$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$B_1 = ((1,1,1), (1,1,0), (1,0,0))$$

$$I_2 = ((1,1), (2,2), (3,3))$$

$$f(1,1,1) = (1,1)$$

$$f(1,0,0) = (3,3)$$

$$f(1,1,0) = (2,2)$$

$$B_2 = ((1,1), (1,0))$$

$$A = M_{B_1 B_2}(f) = \begin{pmatrix} f(1,1) & f(1,1,0) & f(1,0,0) \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$(1,1) = 1(1,1) + 0(1,0)$

$(1,1)$ $(2,2)$ $(3,3)$

$$B_1 = ((1,1,1), (1,1,0), (1,0,0)) \quad \bigg| \quad M_{B_1 B_2}(A) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = ((1,1), (1,0))$$

$$B'_1 = ((1,1,1), (0,1,1), (0,0,1))$$

$$B'_2 = ((1,0), (0,1))$$

$$M_{B'_1 B'_2}(A) = ?$$

$$M_{B'_1 B'_2}(A) = M_{B_2 B_1}(\text{id}) M_{B_1 B_2}(A) M_{B_1 B_2}(\text{id}_V)$$

$$\begin{pmatrix} M & B_1' & B_1 \\ \text{id}(v_1) & \cdot & \text{id}(v_2) \end{pmatrix} \begin{pmatrix} \downarrow \\ \text{id}(v_2) \\ \uparrow \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underline{M B_1' B_1} (\text{id}_V) = \begin{pmatrix} (1,1,1) & (0,1,1) & (0,0,1) \\ 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

3×3

$$B_2 = ((1,1), (1,0))$$

$$B_2' = \underline{\underline{((1,0), (0,1))}}$$

$$M_{B_2 B_2'}(\text{id}_W) = \begin{pmatrix} (1,1) & (1,0) \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

2x2

$$\begin{aligned}
 M_{B_1' B_2'}(\varphi) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & -1 \end{pmatrix}
 \end{aligned}$$

2×2 2×3 3×3
 2×3

NB

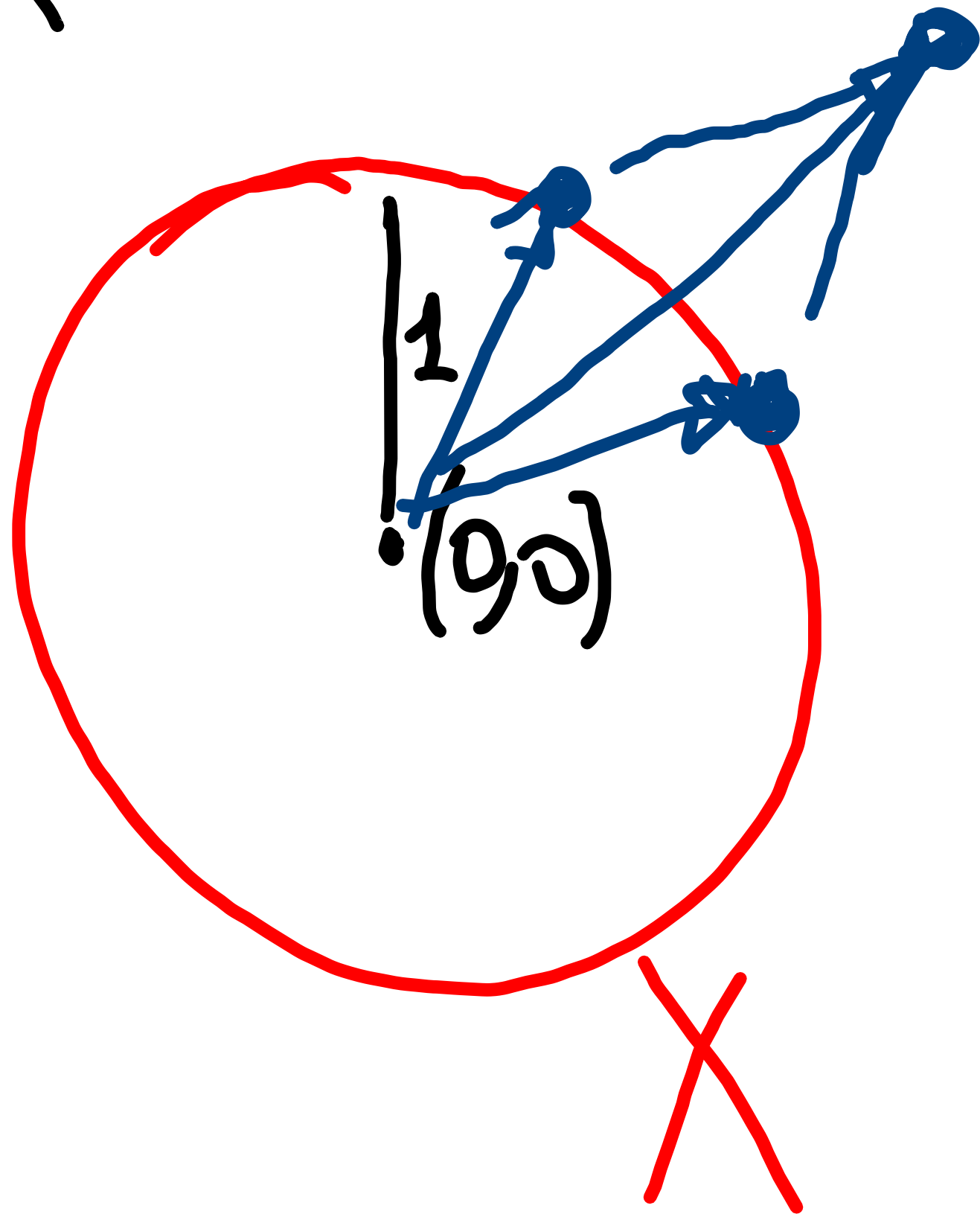
$$M_{B_1 B_2}(\text{id}) = \left(M_{B_2 B_1}(\text{id}) \right)^{-1}$$



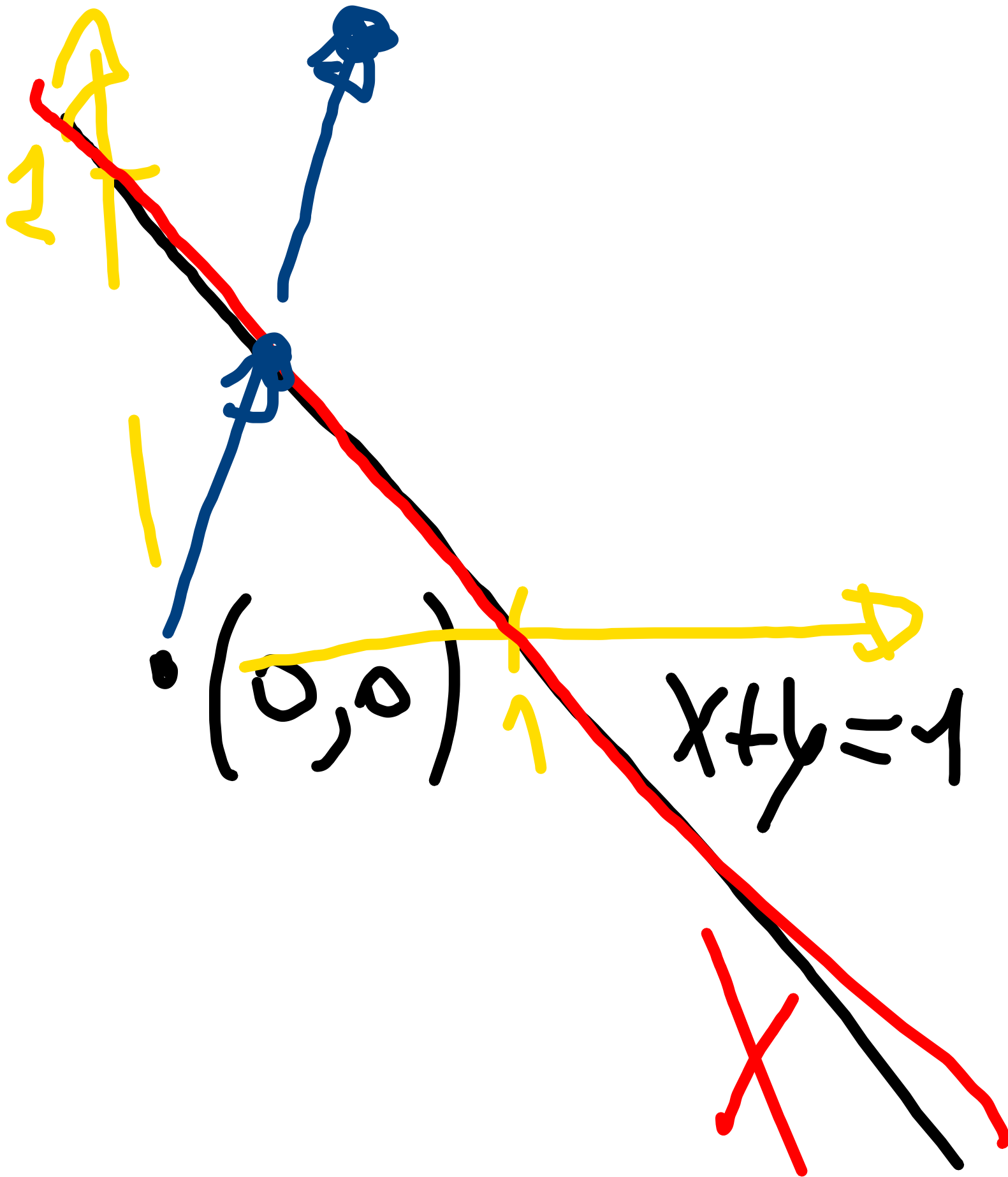
Sottospazi vettoriali
e sottospazi affini.

Si dice che $X \subseteq V$ è
un sottospazio vettoriale
dello s.v. V se X è
chiuso rispetto alle
combinazioni lineari in V .

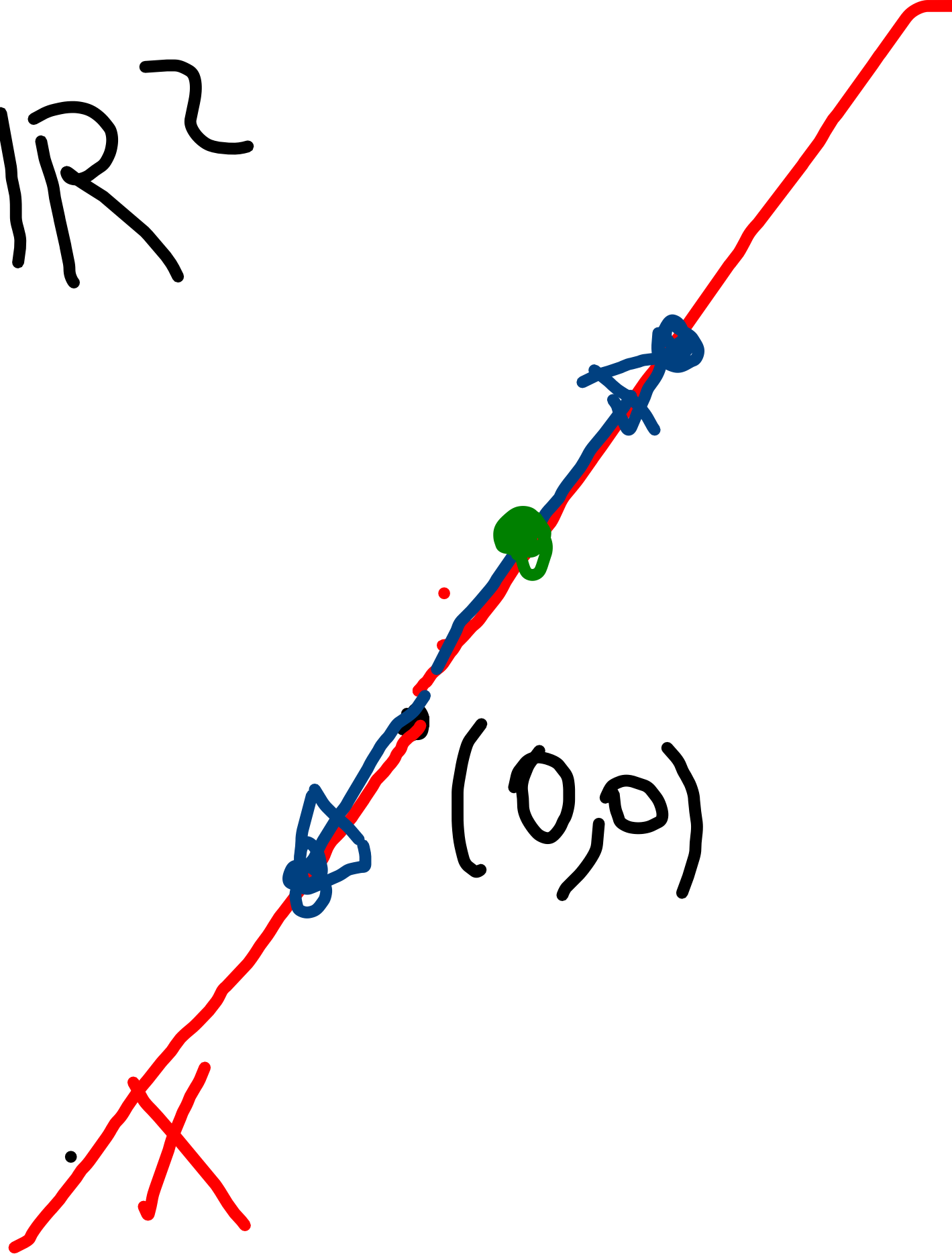
$$V = \mathbb{R}^2$$



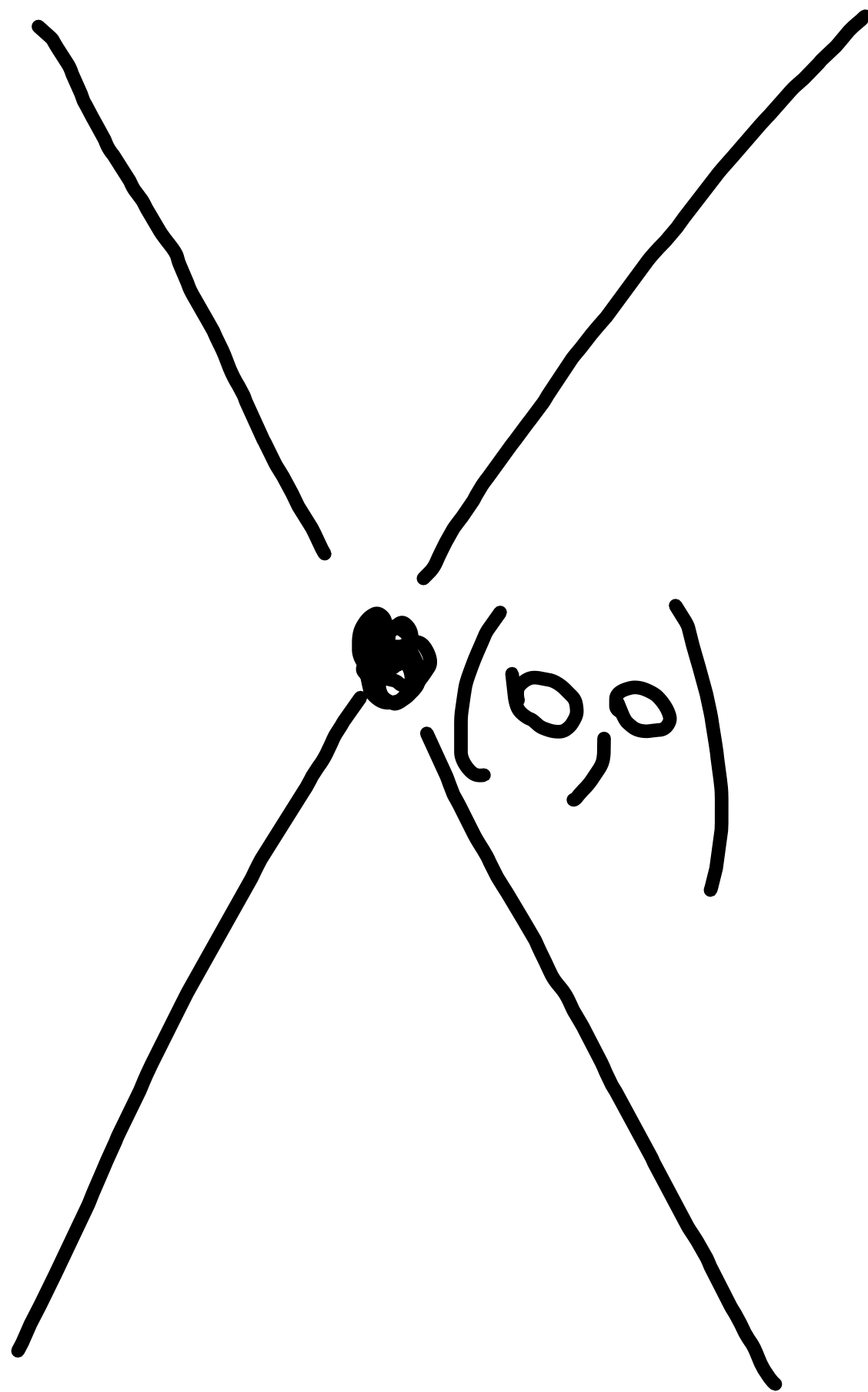
$$V = \mathbb{R}^2$$



$$V = \mathbb{R}^2$$



$$V = \mathbb{R}^2$$



$$V = \mathbb{R}^3$$

$$\{(0,0,0)\}$$

$$\mathbb{R}^3$$

rette per $(0,0,0)$

piani per $(0,0,0)$

Spazio affine

$$(A, \varphi) \quad \varphi: \underline{A \times A} \rightarrow \underline{V}$$

1) Se fissa $P \in A$, la funzione $\varphi(P, \cdot)$ è una biiezione fra A e V .

$$2) \varphi(P, Q) + \varphi(Q, R) = \varphi(P, R)$$

$$P \xrightarrow{\quad} Q \xrightarrow{\quad} R \equiv P \xrightarrow{\quad} R$$

$$f(Q) := \varphi(P, Q) \in V$$

f deve essere biunivoca
per ogni P .

Un modo semplice per
costruire uno spazio
affine è il seguente.

Prendiamo $A = \mathbb{V}$ dove
 \mathbb{V} è uno S.V.

(A, φ)

$A = V$

$$\varphi \left(\underset{\cong}{P}, \underset{\cong}{Q} \right) = Q - P = \underline{\underline{W - 25}}$$

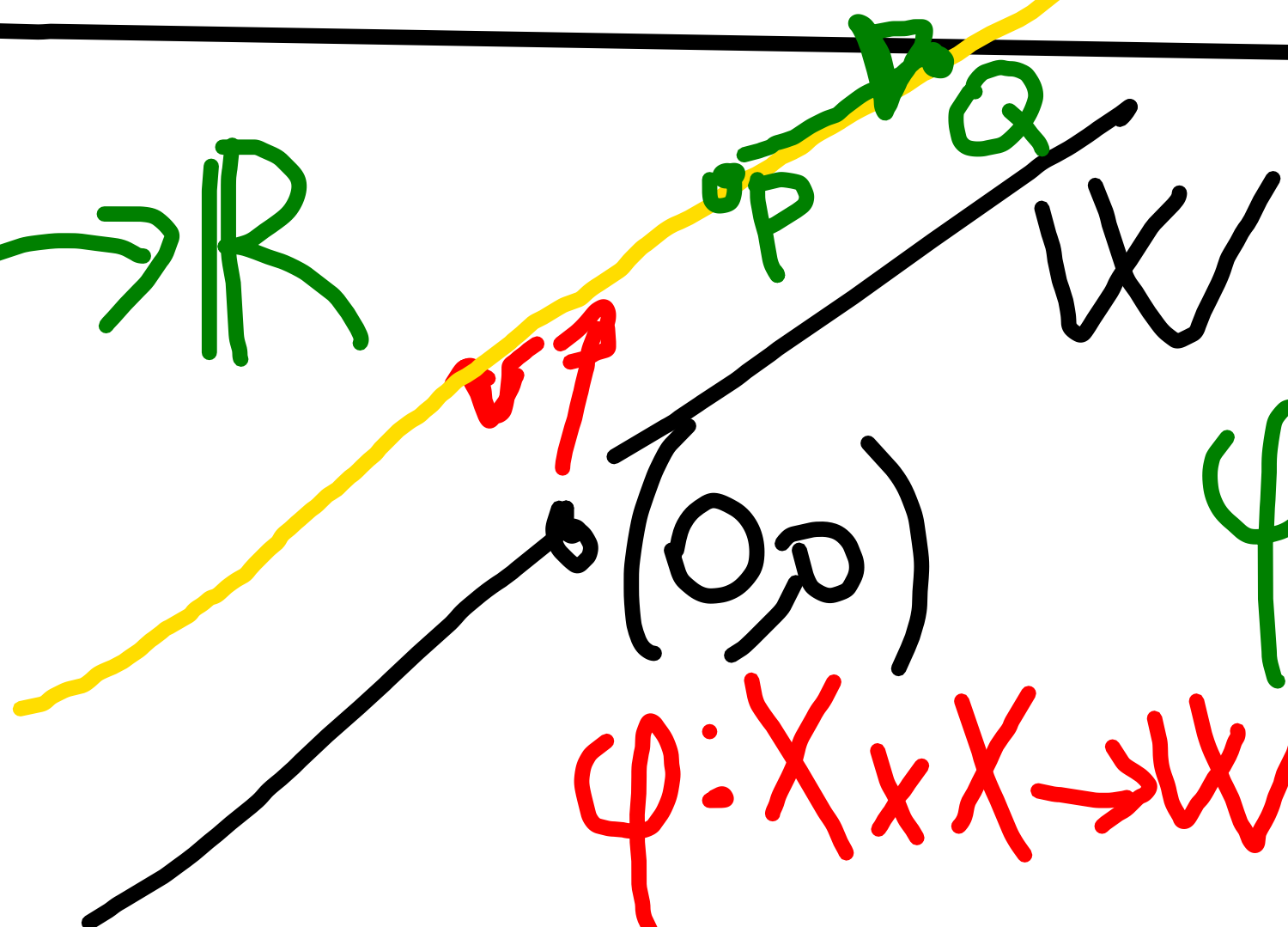
$$V \quad X \subseteq V \quad W \subseteq V$$

S.S.V.

$$X = W + \bar{v}$$

è un spazio affine.
X

$$\varphi: X \rightarrow \mathbb{R}$$



$$\varphi(P, Q) = Q - P$$

$$\frac{w_1 - w_2}{1}$$

$$\varphi: X \times X \rightarrow W$$

$$P = w_1 + \bar{v}$$

$$Q = w_2 + \bar{v}$$

\mathbb{R}^2

$A \subseteq \mathbb{R}^2$

\xrightarrow{PQ}

$\varphi(P, Q)$

o

o

