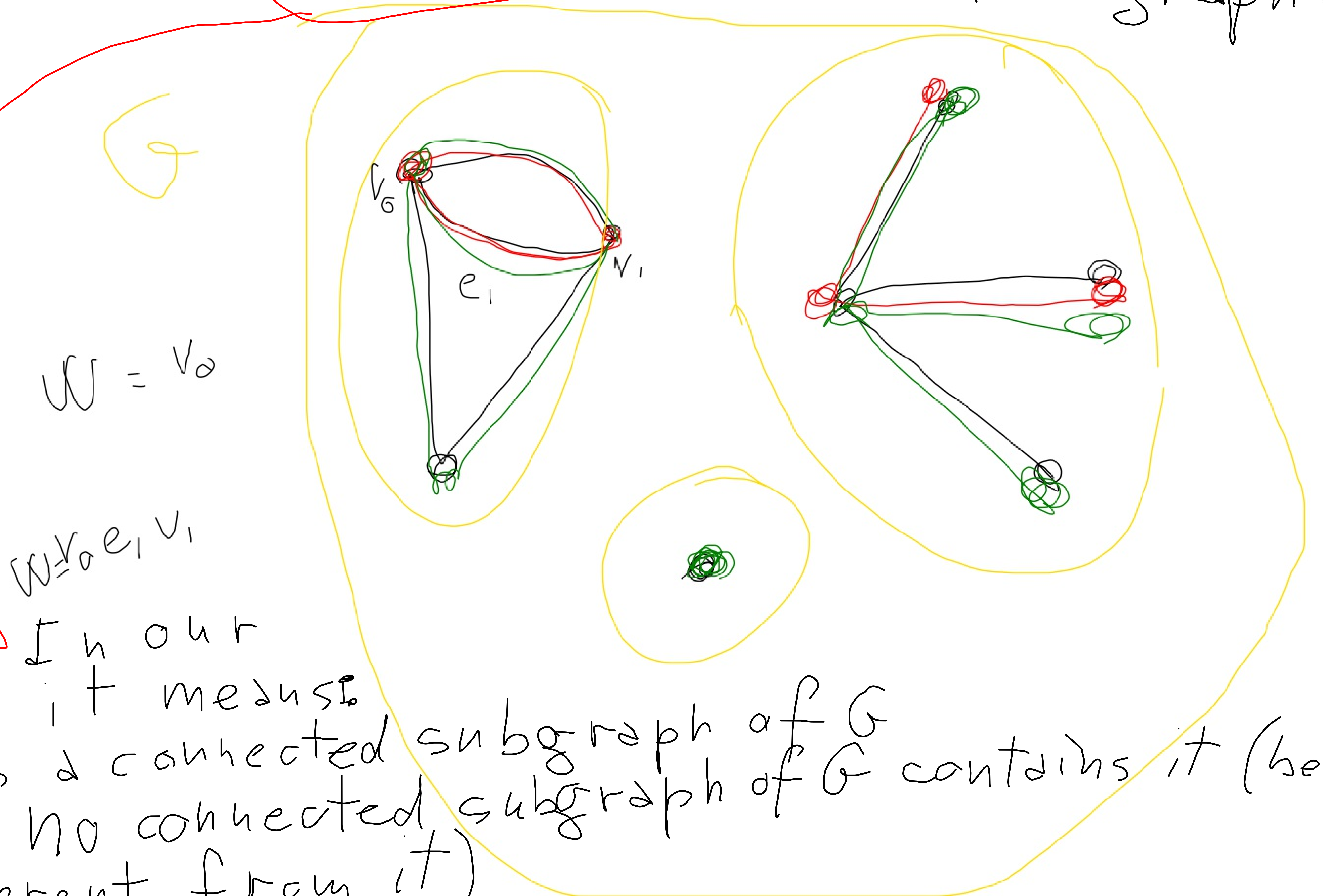
PROP — A component of a graph G turns out to be a maximal connected subgraph of G



In our case it means: it is a connected subgraph of G and no connected subgraph of G contains it (being different from it)

Equivalence relation $\mathcal{R}$ on a set X:

1) reflective:

$\forall x \in X \quad x \mathcal{R} x$

2) symmetrical:

$\forall x, y \in X \quad x \mathcal{R} y \Leftrightarrow y \mathcal{R} x$

3) Transitive:

$\forall x, y, z \in X$

$x \mathcal{R} y \wedge y \mathcal{R} z \Longrightarrow x \mathcal{R} z$

Assume that we have an equivalence relation $\mathcal{R}$ on X. Then for each $x \in X$ we can define

$$[x]_{\mathcal{R}} = \{ y \in X \mid x \mathcal{R} y \}$$

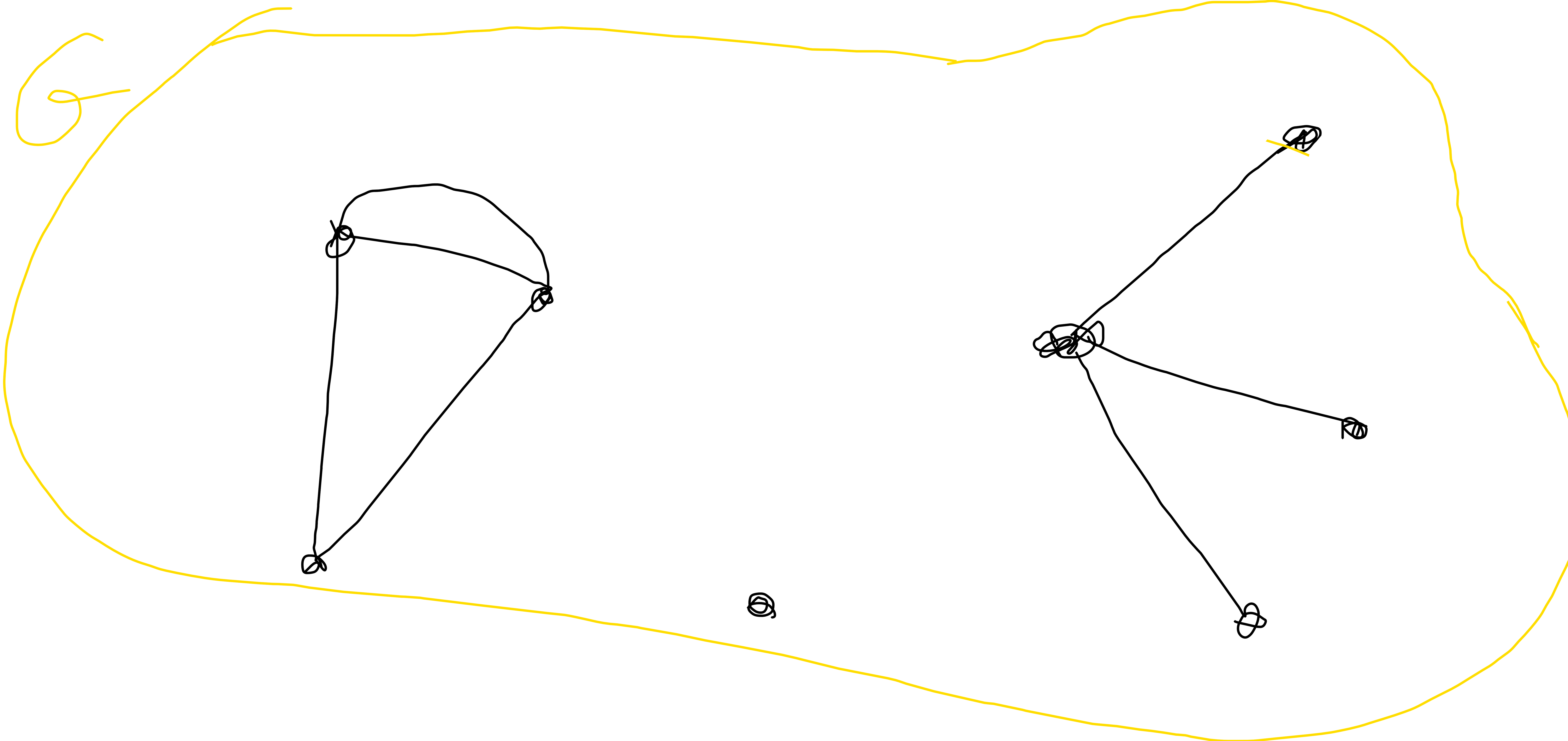
(Remark: if $y \in [x]_{\mathcal{R}}$, then also $x \in [y]_{\mathcal{R}}$, so $[x]_{\mathcal{R}} = [y]_{\mathcal{R}}$)

PROP - The set of equivalence classes is a partition of X

A partition of a set X is a set of subsets $X_1, \dots, X_n$ of X such that:

1) if $X_i \neq X_j$ then $X_i \cap X_j = \emptyset$

2) $\bigcup_{i=1}^{n} X_i = X$


G

Claim $\delta \leq 2\varepsilon/\nu \leq \Delta$

$V = \{v_1, v_2, \ldots, v_\nu\}$

$$\delta + \leq d(v_1) + \leq \Delta +$$

$$\delta + \leq d(v_2) + \leq \Delta +$$

$$\vdots$$

$$\delta \leq d(v_\nu) \leq \Delta +$$

$$\nu\delta \leq \sum_{i=1}^{\nu} d(v_i) \leq \nu\Delta$$

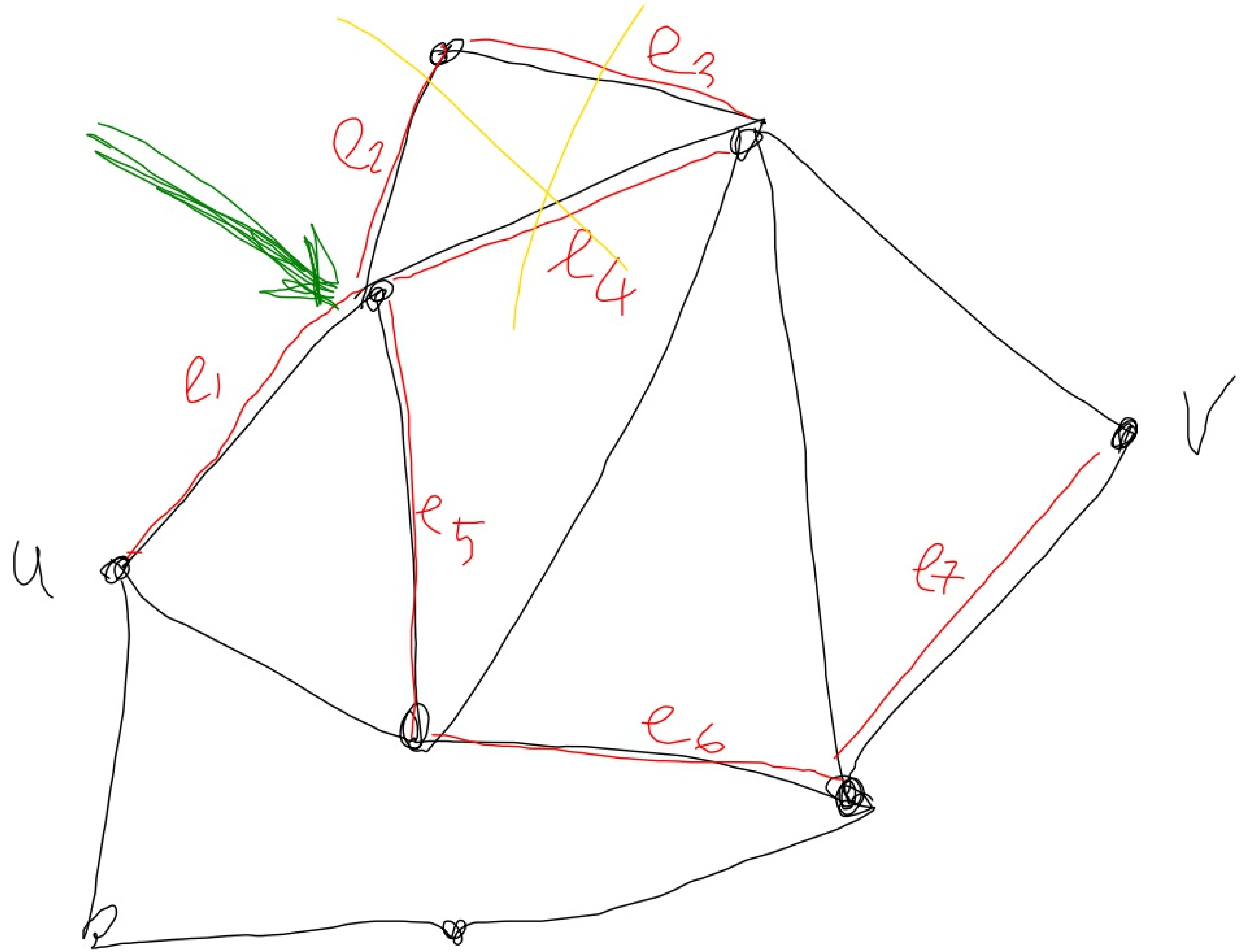
$$\sum_{i=1}^{\nu} d(v_i) = 2\varepsilon$$

$$\nu\delta \leq 2\varepsilon \leq \nu\Delta$$

$\frac{1}{\nu}$

$$\delta \leq \frac{2\varepsilon}{\nu} \leq \Delta$$

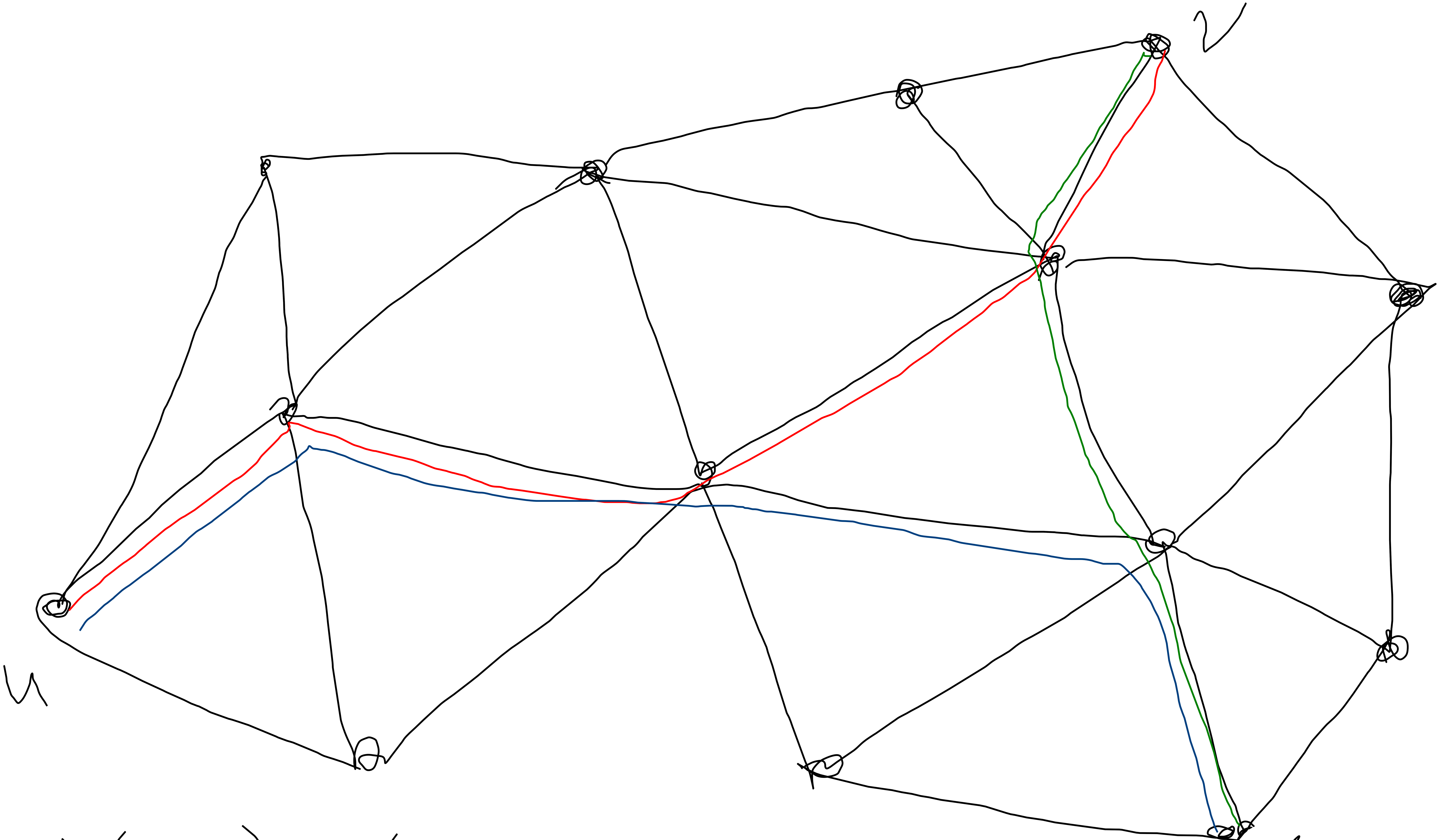
(3, 6, 5, 2, 2, 7, 1)



Metric space: a pair (X, d) where $d: X \times X \to \mathbb{R}$ so that $\forall\, x, y, z \in X$

1) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$
2) $d(x,y) = d(y,x)$
3) $d(x,z) \leq d(x,y) + d(y,z)$

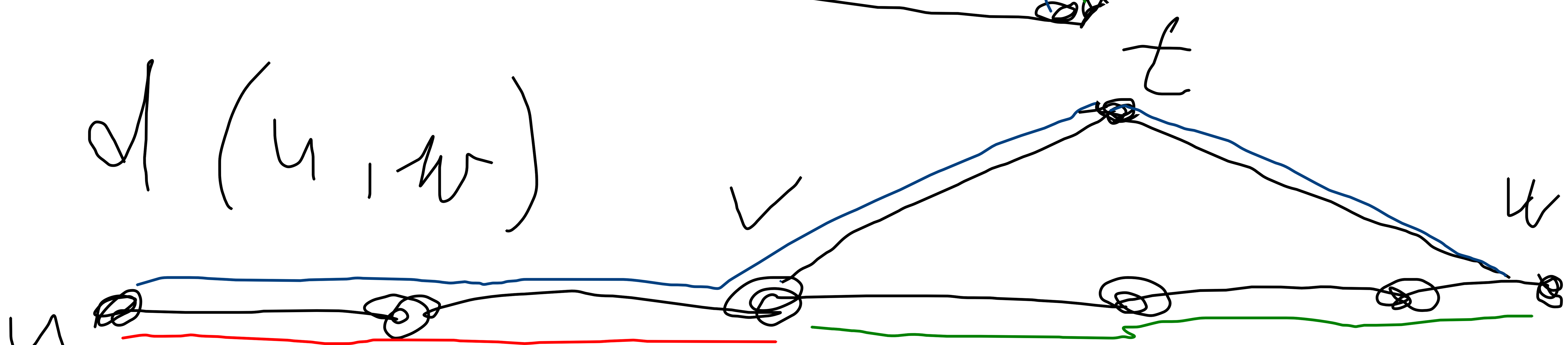
d is called a **distance**



$d(u,v) = 4$

$d(v,w) = 3$

$d(u,w)$



Proof by induction

It is a proof technique for statements about "something" which has a "size" k ($k \in \mathbb{N}$)

Example: call P_k the number of permutations of k objects.

Thm: $P_k = k!$

Two parts:

1) Inductive premise:
prove the statement for a low k
(generally k = 1)

2) Inductive step:
assume the inductive hypothesis:
"the statement holds for size n-1"
and prove the inductive thesis:
"the statement holds for size n"

I use induction for our example

Inductive premise:

How much is P_1? That is how many permutations are there of 1 element?

Answer: $P_1 = 1 = 1!$

Inductive step

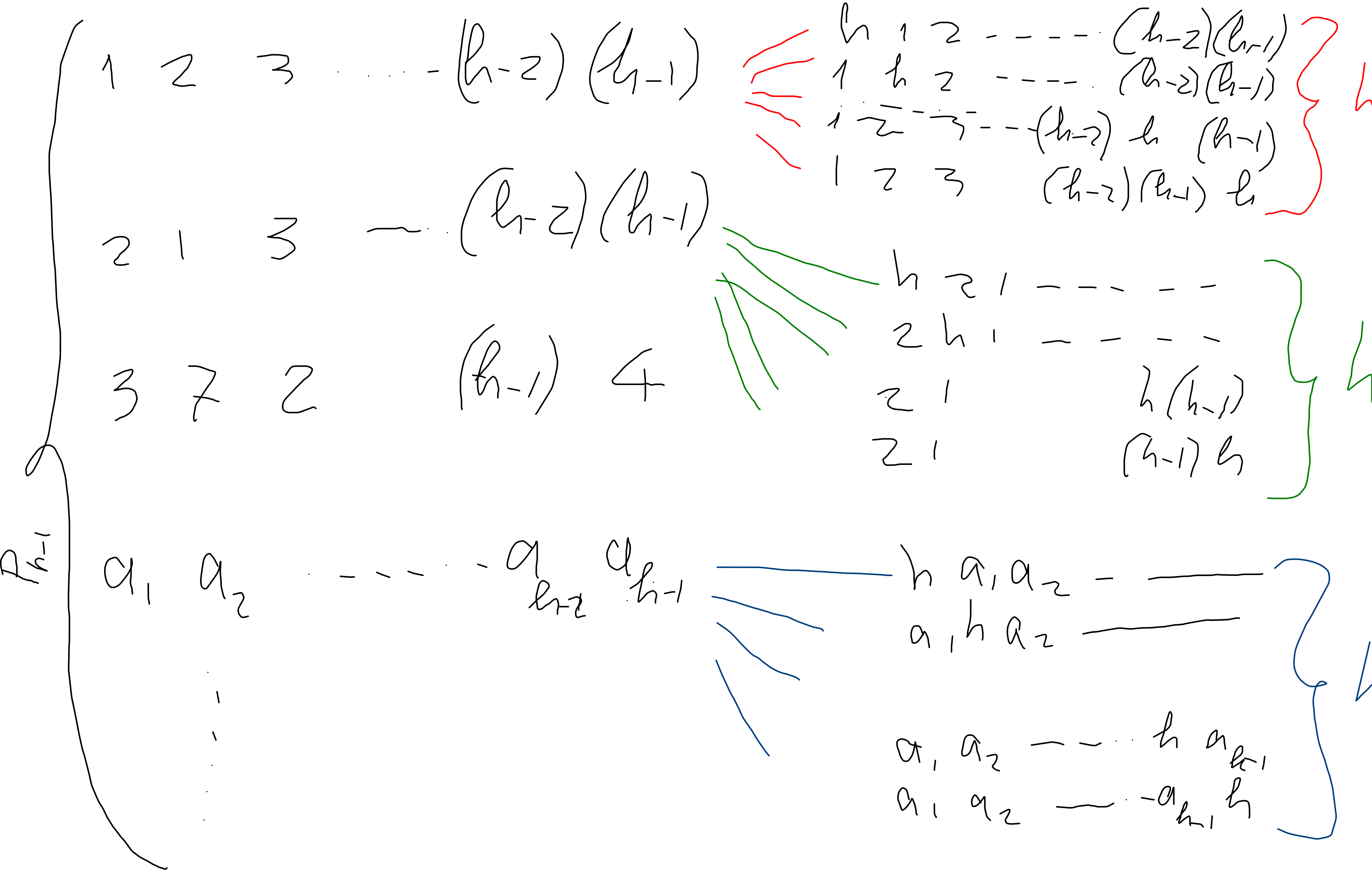
Inductive hypothesis: $P_{n-1} = (n-1)!$

Inductive thesis: $P_n = n!$

Assume that we have our n objects which are the first n natural numbers $1, 2, \ldots, n$.

Take n away for the moment.

List all P_{n-1} permutations of $1, 2, \ldots, n-1$:



1 2 3 ... (n-2) (n-1)
2 1 3 ... (n-2) (n-1)
3 7 2 (n-1) 4
a1 a2 ... a n-2 a n-1
P n-1
n 1 2 ... (n-2) (n-1)
1 n 2 ... (n-2) (n-1)
1 2 3 ... (n-2) n (n-1)
1 2 3 (n-2) (n-1) n
n
n 2 1 ...
2 n 1 ...
2 1 n (n-1)
2 1 (n-1) n
n
n a1 a2 ...
a1 n a2 ...
a1 a2 ... n a n-1
a1 a2 ... a n-1 n
n

So what can I say of P_n?

$$P_n = n P_{n-1}$$

Now I use the inductive hypothesis:

$$P_{n-1} = (n-1)!$$

Then we have

$$P_n = n P_{n-1} = n \cdot \left((n-1)! \right) = n!$$

$\square$

Let G be a graph with $\delta \geq k$. Then each vertex of G is the origin of at least one path of at least length k.

Inductive premise:

If $\delta \geq 1$ then each vertex is the origin of at least one path of length at least one.

Proof: Take any $v \in V$, $d(v) \geq 1$, so there is at least one edge e having ends v, v'. vev' is a path of length 1 starting in v.

Inductive step:

Ind. hyp: if $\delta \geq h-1 \geq 1$, then each vertex is the origin of at least one path of length at least $h-1$.

Ind. th: if $\delta \geq h \geq 2$, then each vertex is the origin of at least one path of length at least h

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Proof -

G a graph with $\delta \geq h \geq 2$. Let v be any vertex of G. $d(v) \geq \delta \geq h \geq 2$.

Then there is at least a vertex $v' \neq v$, adjacent to v (i.e. joined to it by an edge).

Let now $G' = G - v$

Necessarily:

1) $v' \in G'$

2) $\delta(G') \geq n-1$

Then the inductive hypothesis applies!

So v' is the origin of at least one path in G' of length at least $n-1$:

$v' = v_1$ e_1 v_2 e_2 $\cdots\cdots$ e_{h-1} v_h

But then, going back to G, we have a path

v e v_1 e_1 v_2 $\cdots\cdots$ e_{h-1} v_h

(with $v_1 = v'$)

which starts in V and has length (at least) h.

G
G'
v'
v
e
v1
e1
v2
e2
v3
δ(G) ≥ 3
δ(G') = 2
v1'
v e v1 e1 v2 e2 v3
length 2
length 3
d(v) ≥ 3
(=)