$$E^1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow[\mathrm{II}+\mathrm{I}]{} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$$

$$\xrightarrow[\mathrm{II}-\mathrm{I}]{} \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$$

$\mathrm{II}-\mathrm{I}$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \mathrm{II} \leftarrow -\mathrm{II}$$

$$\beta_0 = \alpha^0 - \gamma_0 - \gamma_{-1} = 2 - 1 - 0 = 1$$
$$\beta_1 = \alpha^1 - \gamma_1 - \gamma_0 = 3 - 2 - 1 = 0$$
$$\beta_2 = \alpha^2 - \gamma_2 - \gamma_1 = 2 - 0 - 2 = 0$$

$$H_0(\mathbb{R}P^2) \simeq \mathbb{Z} \quad H_1(\mathbb{R}P^2) \simeq \mathbb{Z}_2 \quad H_2(\mathbb{R}P^2) = 0$$

$\vdots$

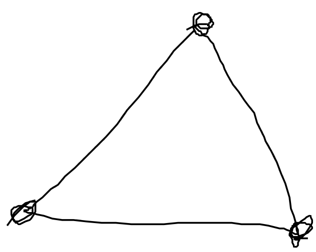
$$-\beta_3 = \left(\alpha^3 - \cancel{\gamma_3} - \cancel{\gamma_2}\right)$$

$$+\beta_2 = \alpha^2 - \cancel{\gamma_2} - \cancel{\gamma_1}$$

$$-\beta_1 = \left(\alpha^1 - \cancel{\gamma_1} - \cancel{\gamma_0}\right)$$

$$+\beta_0 = \alpha^0 - \cancel{\gamma_0} - \cancel{\gamma_{-1}}$$

S^1

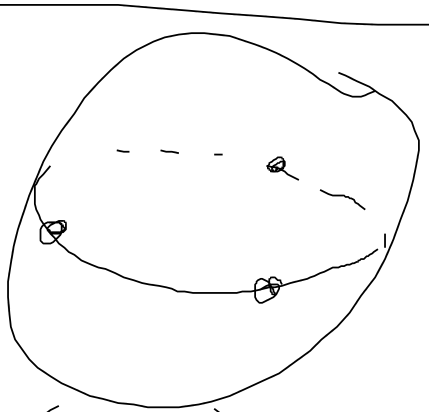


$\alpha^0 = 3$

$\alpha^1 = 3$

$$\chi(S^1) = 3 - 3 = 0$$

S^2

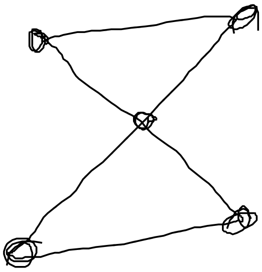


$\alpha^0 = 3$

$\alpha^1 = 3$

$\alpha^2 = 2$

$$\chi(S^2) = 3 - 3 + 2 = 2$$



$$\alpha^0 = 5$$

$$\alpha^1 = 6$$

$$\chi(8) = 5 - 6 = -1$$

$$m \longmapsto [m]$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

$$n \longmapsto 2n$$

$$
\begin{tikzcd}
0 \arrow[r] & C'_{k+1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & C_{k+1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & C''_{k+1} \arrow[r] \arrow[d, "\partial''"'] & 0 \\
0 \arrow[r] & C'_{k} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & C_{k} \arrow[r, "\beta"] \arrow[d, "\partial"'] & C''_{k} \arrow[r] \arrow[d, "\partial''"'] & 0 \\
0 \arrow[r] & C'_{k-1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & C_{k-1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & C''_{k-1} \arrow[r] & 0 \\
0 \arrow[r] & C'_{k-2} \arrow[r, "\alpha"] & C_{k-2} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & c \arrow[r, mapsto] \arrow[d, mapsto] & z'' \arrow[d, mapsto] \\
c' \arrow[r, mapsto] \arrow[d, mapsto] & \partial c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
0 \arrow[r, mapsto] & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & C'_{k+1} \arrow[r, "\alpha"] \arrow[d, "\partial'"] & C_{k+1} \arrow[r, "\beta"] \arrow[d, "\partial"] & C''_{k+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{k} \arrow[r, "\alpha"] \arrow[d, "\partial'"] & C_{k} \arrow[r, "\beta"] \arrow[d, "\partial"] & C''_{k} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{k-1} \arrow[r, "\alpha"] & C_{k-1} \arrow[r, "\beta"] & C''_{k-1} \arrow[r] & 0
\end{tikzcd}
$$

$d \mapsto d''$, $d' \mapsto c + \alpha d' + \partial d \mapsto z'' + \partial'' d''$, $c \mapsto z''$, $\partial' d' \mapsto \partial\alpha d'$, $c' \mapsto \partial c$

$$c_1 = c + \alpha d' + \partial d$$

$$\partial c_1 = \partial(c + \alpha d' + \partial d) = \partial c + \partial\alpha d' + \cancel{\partial\partial d} =$$

$$= \partial c + \alpha\partial' d' = \alpha c' + \alpha \partial' d' =$$

$$= \alpha(c' + \partial' d')$$

$$\{\alpha^{-1}\partial\beta^{-1}(z'' + \partial'' d'')\} = \{c' + \partial' d'\} =$$

$$\{\alpha^{-1}\partial\beta^{-1}(z'')\} \qquad = \{c'\}$$

$$\partial_*\{z''\}$$