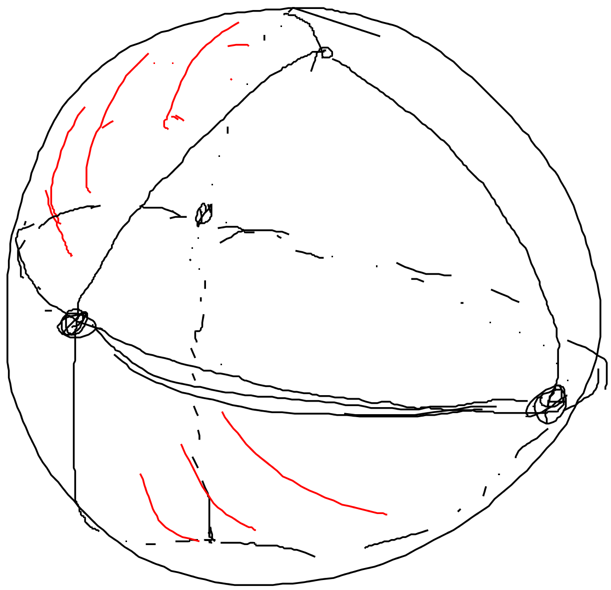
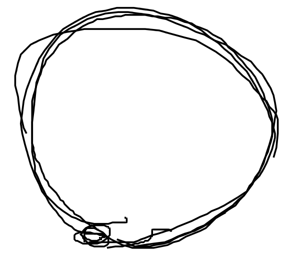
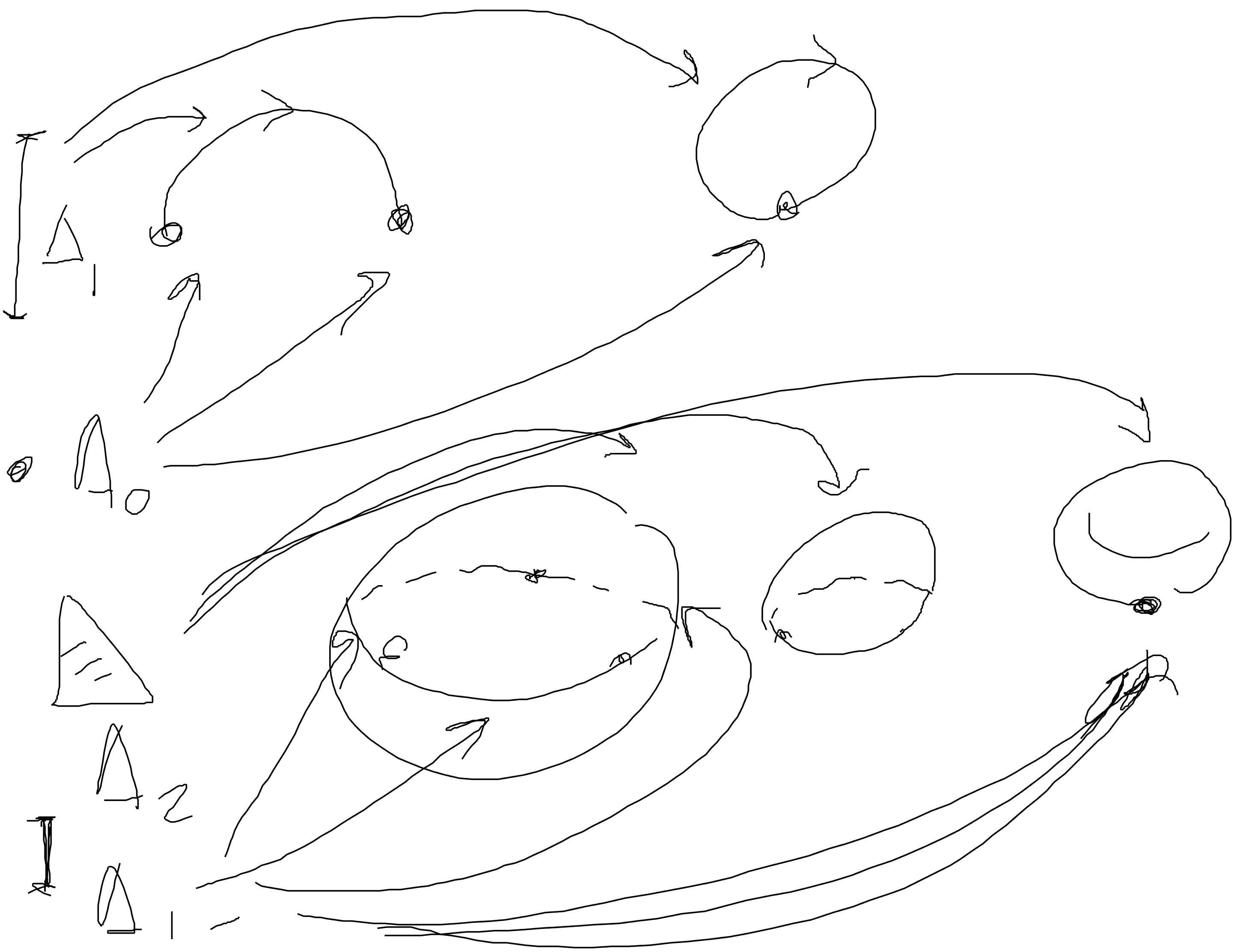


L(3,1)





$$0 \to \mathbb{Z} \xrightarrow{\alpha = (\cdot 2)} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0$$

$$1 \longmapsto \bar{1}$$

$$1 \longmapsto 2 \qquad\qquad \bar{1} \longmapsto 0$$

$$\otimes \, \mathbb{Z}_2$$

$$\operatorname{Ker}(\alpha \otimes 1_{\mathbb{Z}_2}) \qquad \bar{1} \longmapsto \bar{1}$$

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{\alpha \otimes 1_{\mathbb{Z}_2}} \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$

$$\bar{1} \longmapsto \bar{1}$$

$$\bar{1} \longmapsto \bar{0} \qquad\qquad \bar{1} \longmapsto 0$$

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$$

$$\otimes \, L$$

$$0 \to \operatorname{Ker}(\alpha \otimes 1_L) \to F \otimes L \xrightarrow{\alpha \otimes 1_L} G \otimes L \xrightarrow{\beta \otimes 1_L} H \otimes L \to 0$$

$\operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_p)$

$$0 \to 0 \to \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z} \to 0$$

$$1 \mapsto 1$$

$$\otimes \mathbb{Z}_p$$

$$0 \to 0 \to \mathbb{Z}_p \xrightarrow{1_{\mathbb{Z}_p}} \mathbb{Z}_p \to 0$$

$$[1]_p \mapsto [1]_p$$

$$\operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_p) = 0$$

$\operatorname{Tor}(\mathbb{Z}_q, \mathbb{Z}_q)$

$$1 \mapsto q$$

$$0 \to \mathbb{Z} \xrightarrow{\cdot q} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_q \to 0$$

$$1 \mapsto [1]_q$$

$$\otimes \mathbb{Z}_q$$

$$[1]_q \mapsto [0]_q$$

$$0 \to \mathbb{Z}_q \to \mathbb{Z}_q \longrightarrow \mathbb{Z}_q \xrightarrow{1_{\mathbb{Z}_q}} \mathbb{Z}_q \to 0$$

$$[1]_q \to [1]_q \qquad [1]_q \mapsto [1]_q$$

$$\operatorname{Tor}(\mathbb{Z}_q, \mathbb{Z}_q) = \mathbb{Z}_q$$

$$\operatorname{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) \qquad (p,q) = \operatorname{MCD}(p,q)$$

$$m \longmapsto pm$$

$$0 \to \mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_p \to 0$$

$$n \longmapsto [n]_p$$

$$\otimes \mathbb{Z}_q$$

$$[m]_q \longmapsto [pm]_q$$

$$0 \to \mathbb{Z}_{(p,q)} \longrightarrow \mathbb{Z}_q \longrightarrow \mathbb{Z}_q \longrightarrow \mathbb{Z}_{(p,q)} \longrightarrow 0$$

$$[r]_{(p,q)} \longmapsto \left[\frac{q r}{(p,q)}\right]_q \qquad [n]_q \longmapsto [n]_{(p,q)}$$

$$\operatorname{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_{(p,q)}$$

$$m \longmapsto 6m$$

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}_6 \to 0$$

$$n \longmapsto [n]_6$$

$$\otimes \mathbb{Z}_{15}$$

$$[m]_{15} \longmapsto [6m]_{15}$$

$$0 \to \mathbb{Z}_3 \longrightarrow \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_3 \to 0$$

$$[r]_3 \longmapsto [5r]_{15} \qquad [n]_{15} \longmapsto [n]_3$$

$$\operatorname{Ker}(\alpha \otimes 1_{\mathbb{Z}_{15}}) = \{[0]_{15}, [5]_{15}, [10]_{15}\}$$

$$\operatorname{Tor}(\mathbb{Z}_6, \mathbb{Z}_{15}) \cong \mathbb{Z}_3$$

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

$\operatorname{Im}\gamma = \operatorname{Ker}\delta$

$B/\operatorname{Ker}\beta$

$$0 \to \operatorname{Coker}\alpha \to C \to \operatorname{Ker}\delta \to 0$$

$$\to C_{k+1} \xrightarrow{\partial} C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial}$$

$$\to Z_{k+1} \xrightarrow{\partial=0} Z_k \xrightarrow{\partial=0} Z_{k-1} \to$$

$$\begin{tikzcd}
\arrow[r] & B_{k+1} \arrow[r, "\partial=0"] \arrow[d, equal] & B_k \arrow[r, "\partial=0"] \arrow[d, equal] & B_{k-1} \arrow[r, "\partial=0"] \arrow[d, equal] & {} \\
\arrow[r] & \overline{B}_{k+2} \arrow[r, "0"] & \overline{B}_{k+1} \arrow[r, "0"] & \overline{B}_k \arrow[r, "0"] & {}
\end{tikzcd}$$

$$0 \longrightarrow Z \longrightarrow C \longrightarrow B \longrightarrow 0$$

$$\beta = \partial$$

$$c \longmapsto \partial c$$

$$0 \longrightarrow Z_k \xrightarrow{\alpha} C_k \longrightarrow \overline{B}_k \longrightarrow 0$$

$$z \longmapsto z$$

$$\overline{B}_k = B_{k-1}$$

$$\oplus G$$

$$0 \longrightarrow Z_k \otimes G \xrightarrow{\alpha} C_k \otimes G \xrightarrow{\beta} \overline{B}_k \otimes G \longrightarrow 0$$

$$\overline{B}_k \otimes G = B_{k-1} \otimes G$$

$$H_k(Z;G) \xrightarrow{\alpha_*} H_k(C;G) \xrightarrow{\beta_*} H_k(B;G) \xrightarrow{\partial_*} H_{k-1}(Z;G)$$

$$\xrightarrow{0} Z_{k+1} \xrightarrow{0} Z_k \xrightarrow{0} Z_{k-1} \xrightarrow{0}$$

$$\to Z_{k+1}\otimes G \xrightarrow[\;=\,0\;]{0\otimes 1_G} Z_k\otimes G \xrightarrow[\;=\,0\;]{0\otimes 1_G} Z_{k-1}\otimes G$$

$$\mathrm{Ker} = Z_k\otimes G$$

$$\mathrm{Im} = 0$$

$$H_k(Z\otimes G) = Z_k\otimes G$$

$$\bar{\bar{B}}_{k+1} \xrightarrow{0} \bar{B}_k \xrightarrow{0} \bar{B}_{k-1}$$

$$\otimes G$$

$$B_k\otimes G \xrightarrow{0} B_{k-1}\otimes G \xrightarrow{0} B_{k-2}\otimes G$$

$$\mathrm{Ker} = B_{k-1}\otimes G$$

$$\mathrm{Im} = 0$$

$$H_k(B\otimes G) = B_{k-1}\otimes G$$

$$H_{k+1}(B;G) \to H_k(Z;G) \xrightarrow{\alpha_*} H_k(C;G) \xrightarrow{\beta_*} H_k(B;G) \xrightarrow{\partial_*} H_{k-1}(Z;G)$$

$$B_k \otimes G \xrightarrow{\gamma_k \otimes 1_G} Z_k \otimes G \qquad B_{k-1} \otimes G \xrightarrow{\gamma_{k-1} \otimes 1_G} Z_{k-1} \otimes G$$

$$0 \to \operatorname{Coker}(\gamma_k \otimes 1_G) \longrightarrow H_k(C;G) \longrightarrow \operatorname{Ker}(\gamma_{k-1} \otimes 1_G) \to 0$$