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Size Functions from a Categorical Viewpoint ⋆

FRANCESCA CAGLIARI, MASSIMO FERRI and PAOLA POZZI
Dipartimento di Matematica, Università, Piazza di Porta S. Donato, 5, I-40126 Bologna, Italy. e-mail: {cagliari,ferri,pozzi}@dm.unibo.it

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Abstract. A new categorical approach to size functions is given. Using this point of view, it is shown that size functions of a Morse map, $f\colon \mathbf{M} \to \Re$ can be computed through the 0-dimensional homology. This result is extended to the homology of arbitrary degree in order to obtain new invariants of the shape of the graph of the given map.

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1. Introduction: A Brief Recall of Size Functions

This paper is devoted to a theoretical extension of a mathematical transform, the ‘size function’, which we use extensively in pattern recognition. Since the theory is still young and not so widespread, the general reader is not necessarily acquainted with it, so we start by stating what size functions are, and why we are willing to extend them.

Size functions are a simple, but effective tool for automatic recognition: they become particularly useful when no standard, geometric templates are available. Examples of application are tree-leaves [16], hand-drawn sketches [3], monograms [5], hand-written characters [6], white blood cells [7], and the sign alphabet [15]. We are currently experimenting with size functions in the recognition of human profiles, melanomas, and sounds.

Let us recall very briefly the definition of a size function. See [12] for an extended survey, or also [10]. Consider a real function $f\colon \mathbf{M} \to \Re$ defined on a subset $\mathcal{M}$ of a Euclidean space. The *size function* of the pair $(\mathbf{M}, f)$ is a function $\ell_{\mathbf{M}}\colon \Re^2 \to \mathbf{N} \cup \{\infty\}$. For each pair $(u, v) \in \Re^2$, consider the set of points on which f is worth $\leqslant u$. Two such points are then considered to be equivalent if they either coincide or can be connected in $\mathbf{M}$ by a path on whose points f is worth $\leqslant v$. Then $\ell_{\mathbf{M}}(u, v)$ counts the equivalence classes so obtained. See Figure 1 for an example with the distance from the center of mass as the measuring function.

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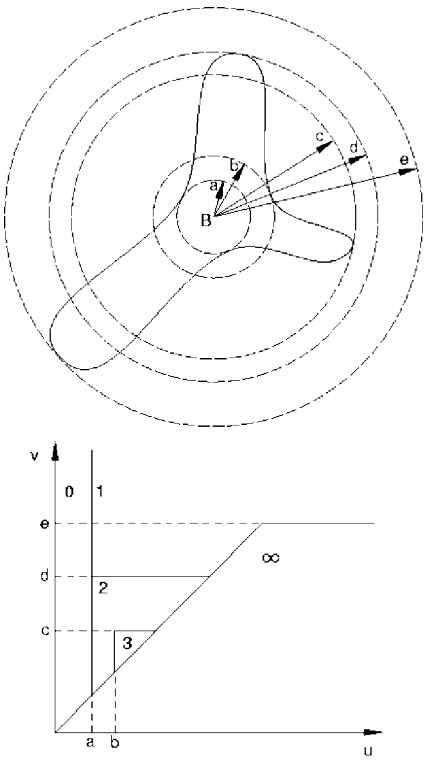


Figure 1. A curve, and its size function, relative to distance from center of mass.

Since size functions inherit the same invariance as the measuring functions they are defined on (e.g. invariance under rotations), it is of practical use to compare the size functions of, say, two images instead of the images themselves after guessing the correct transformation (e.g. rotation). The choice of the measuring function is a crucial element in that it is the way for an expert to select the relevant features for the recognition problem under study. This is already a categorical issue, since the stress is on morphisms rather than on objects. For an example, see Figure 2, where the contour of a leaf and its image under a similitude are shown, together with the respective size functions, computed with respect to the measuring function: normalized distance from center of mass.

Of course, a discrete version of the theory is needed (and exists [8]), since in practice one has to work with discrete objects (mostly bitmap images). But we stick to the principle that the theoretical background should be in a continuous domain, as bitmaps – and the like – are just discretizations of continuous models of reality.

A 'natural' discretization comes from Morse Theory: If M is a closed manifold and the measuring function f is a Morse function, then the corresponding size function is completely determined by some critical points of f (see [9]): The corre-

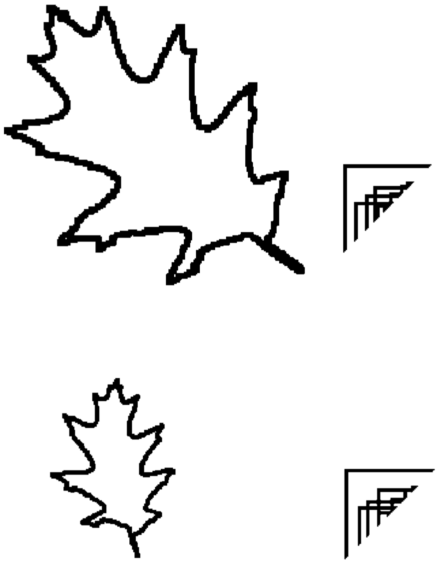


Figure 2. A leaf contour, and its image under a similitude, with the respective size functions.

sponding critical values yield co-ordinates of the ‘cornerpoints’ and ‘cornerlines’, sufficient to reconstruct the whole size function [11].

A natural question is: Why should M be a manifold and f be a Morse function? How does this fit in with concrete settings? For example, a character ‘a’ can be modelled as a one-dimensional subspace of the plane, but then is not a manifold. It can be considered to be a two-dimensional ‘blob’ and, in this case, it can be modelled as a manifold but with nonempty boundary.

True, but in the latter model we can adopt its boundary as a one-dimensional (nonconnected) manifold M; this is exactly what we generally do (see Figure 3). As for f being a Morse function, we just note that most measuring functions which come from experience, are continuous, so they can be approximated by Morse functions with substantial conservation of the size function [9, Prop. 1.1].

Since long we have guessed that size functions may conceal a deeper structure. An immediate remark is that size functions only take into account the cardinalities of the images of equivalence classes under inclusion, and not their ‘story’. This is corrected here in categorical terms by considering morphisms and not just their images. The ‘size functor’ we define in Section 2, turns out to be not only a more adequate and elegant shape descriptor, but also a more informative one (as the example of Figure 4 shows). This approach reveals that size functions actually address only the 0-homology of a family of subspaces of the considered manifold. The size functor extends the study to all homology, so recovering the meaning of critical points, which were dismissed as ‘inessential’ in the previous setting.

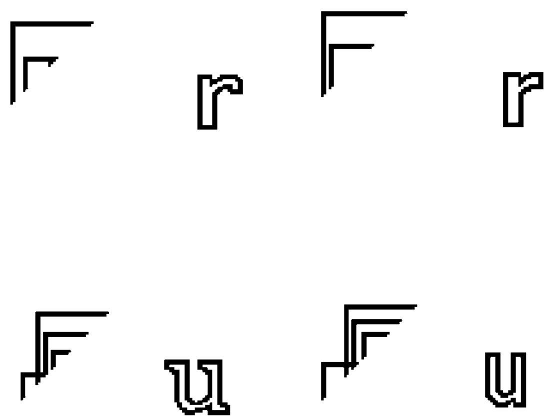


Figure 3. Two letters ‘r’ and two letters ‘u’ of different fonts, with the respective size functions, relative to the distance from the center of mass.

The choice of natural representatives is formalized in the coreflection of Section 3, and is – in a sense – an extension of the notion of cornerpoint [11]. Section 4 analyzes the advantage of the size functor with respect to size functions; Section 5 is concerned with the problems related to actual computation: A discrete version of the size functor is beyond of the scope of the present paper, but will eventually follow in order to make practical computations possible.

We want to point out that this article contains no new results in algebraic topology nor in category theory, but just a new way to use these ‘pure’ subjects in the concrete field of pattern recognition.

2. Size Functor

For the definitions and notions not explicitly given here, we refer to [14] for the topological and differential notions, to [13] for homology theory, and to [2] for the categorical setting.

Throughout this paper, $\mathbf{M}$ is a compact, smooth, n-dimensional submanifold of the m-dimensional real space E^m. A *measuring function* is an arbitrary continuous map $f\colon \mathbf{M} \to \Re$. If f is a C^∞-function whose critical points are nondegenerate (i.e: with nonvanishing Hessian), f is called a *Morse measuring function*. We recall that a Morse function has only a finite number of critical points.

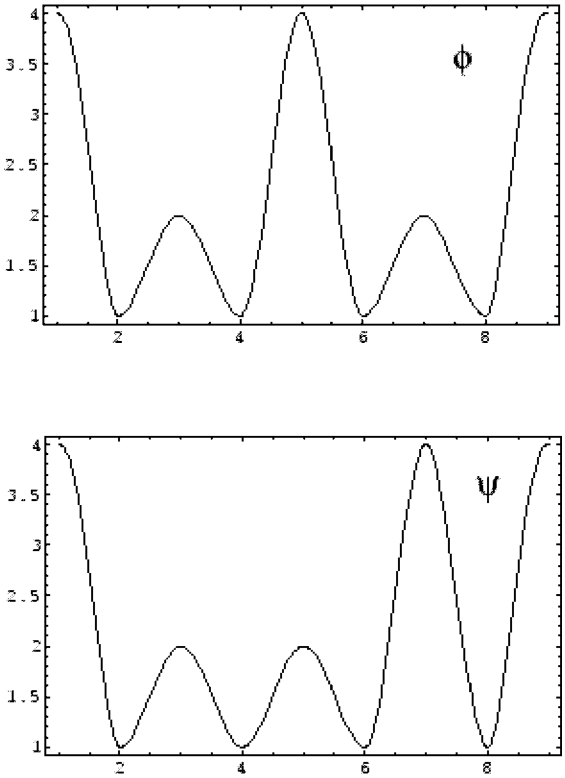


Figure 4. An example of measuring functions distinguished by the size functor but not by the size function.

We denote by $C(x)$ the set of all points of **M**, where the value of the measuring function f does not exceed x, that is

$$C(x) = \{p \in \mathbf{M} \mid f(p) \leqslant x\}.$$

Moreover, **Ab** denotes the category of Abelian groups and **Rord** the linear order category associated to the linear ordered set of real numbers. More precisely, the objects of **Rord** are the real numbers, and given two real numbers x and y, there is a morphism from x to y (denoted by k_{xy}) if and only if $x \leqslant y$.

Fun(**Rord**, **Ab**) denotes the category whose objects are the functors from **Rord** to **Ab**, and the morphisms are the natural trasformations. This category, by [2] (2.15.1, Vol. 1), has sums on any set of objects. This means that if we consider a set of functors from **Rord** to **Ab**, it is possible to get their sum: This is the functor which assigns to each x the sum of the Abelian groups associated to x by each functor of the considered set.

We denote by H_i (resp. H) the i-dimensional homology functor (resp. homology functor) from the category of topological spaces and continuous maps to the category of Abelian groups.

In this section, for any continuous measuring function $f\colon \mathbf{M} \to \Re$, we define $n+1$ functors whose sum in Fun(**Rord**, **Ab**) is the *size functor*.

DEFINITION 2.1. For any $i = 0, \ldots, n$, we define a functor F_i from **Rord** to **Ab** as follows: for each $x \in \Re$, $F_i(x) = H_i(C(x))$; if $x \leqslant y$, j_{xy} is the inclusion from $C(x)$ into $C(y)$; set $F_i(k_{xy}) = H_i(j_{xy})$. $F = \bigoplus_{i=0}^{m} F_i$ in Fun(**Rord**, **Ab**). F_i and F are called the *i-size functor* and *size functor*, respectively.

THEOREM 2.2. *Let $f\colon \mathbf{M} \to \Re$ be a Morse function and x and y be real values $(x \leqslant y)$, with no critical value in the interval $(x, y]$. Then $C(x)$ is a deformation retract of $C(y)$, and the inclusion of $C(x)$ into $C(y)$ is a homotopy equivalence.*

Proof. Propositions 3.1, 3.4 in [14]. □

COROLLARY 2.3. *For each real value x, there is $y > x$ such that $F_i(k_{xy})$ is an isomorphism of Abelian groups for each i.*

Proof. The map f has only a finite number of critical points, so we can choose y in such a way that there are no critical points in the interval $(x, y]$, and apply Theorem 3.2. □

From Corollary 3.3 we can see that if we study the homology of $C(x)$ by letting x increase, we face a change only when we meet a critical point, so there is only a finite number of discontinuities in the homology groups. The crossing of a critical point does not change the whole homology, but at most two homology groups, whose degree depends on the index of the critical point. To see this we use again a result of [14].

From now on, we assume that x and y are two real numbers such that the interval $[x, y]$ contains the image of only one critical point p of index λ.

THEOREM 2.4. *The space obtained via the adjunction of a λ-cell to $C(x)$ is a deformation retract of $C(y)$.*

Proof. [14, Theorem 3.1]. □

So, when we attach a λ-cell to the space $C(x)$ by a map g, we obtain a space with the homotopy type of $C(y)$. That is, the space M can be built, from a homotopic viewpoint, as a CW-complex (finite when f is a Morse function).

We recall, with adapted notation:

THEOREM 2.5 [13, Cor. 19.16–19.18].

(1) *If* $i \neq \lambda$, $i \neq \lambda - 1$ *then* $\tilde{H}_i(C(x)) \cong \tilde{H}_i(C(y))$.
(2) $\tilde{H}_{\lambda-1}(C(y)) \cong \tilde{H}_{\lambda-1}(C(x))/\operatorname{Im}(\tilde{H}_{\lambda-1}(g))$.
(3) *The sequence* $0 \to \tilde{H}_\lambda(C(x)) \to \tilde{H}_\lambda(C(y)) \to \operatorname{Ker} \tilde{H}_{\lambda-1}(g) \to 0$ *is exact and split, i.e.* $\tilde{H}_\lambda(C(y)) \cong \tilde{H}_\lambda(C(x)) \oplus \operatorname{Ker} \tilde{H}_{\lambda-1}(g)$.

Point (1) of Theorem 3.5 shows that the attachment of a λ-cell changes, at most, the λ or the $(\lambda - 1)$-dimensional homology, so that if $i \neq \lambda$ and $i \neq \lambda - 1$, $F_i(x) \cong F_i(y)$. As a consequence, we are allowed to give the following definition:

DEFINITION 2.6. A critical value x for a Morse function f is *i-essential* if $F_i(k_{zx})$ is not an isomorphism, where $z \leqslant x$ and there are no critical values in the interval $[z, x)$.

From Theorem 3.5, we can also deduce that any critical value is essential for at least one i. Moreover, if we consider the attachment of a λ-cell, we can observe:

Remark 2.7. (1) Every critical point of index λ modifies the homology group at degree either λ or $\lambda - 1$.

(2) If $\mathrm{Im}(\tilde{H}_{\lambda-1}(g))$ is infinite, then $\tilde{H}_\lambda(C(y))$ is isomorphic to $\tilde{H}_\lambda(C(x))$ while the rank of $\tilde{H}_{\lambda-1}(C(y))$ is one less than the one of $\tilde{H}_{\lambda-1}(C(x))$.

(3) If $\mathrm{Im}(\tilde{H}_{\lambda-1}(g))$ is the zero-group, then $\tilde{H}_{\lambda-1}(C(y))$ is isomorphic to $\tilde{H}_{\lambda-1}(C(x))$ while the rank of $\tilde{H}_\lambda(C(y))$ is one more than the one of $\tilde{H}_\lambda(C(x))$.

(4) If $\mathrm{Im}(\tilde{H}_{\lambda-1}(g))$ is finite, then the rank of $\tilde{H}_\lambda(C(y))$ is one more than the one of $\tilde{H}_\lambda(C(x))$ while $\tilde{H}_{\lambda-1}(C(y))$ is the quotient of $\tilde{H}_{\lambda-1}(C(x))$ over the image of $\tilde{H}_{\lambda-1}(g)$.

3. Coreflection Induced by the Size Functor

In this section we show that just a finite set of real numbers is necessary to describe the functor F_i (resp. F).

We will see that the behavior of F_i (resp. F) is the same on any object in a suitable half-closed real interval. So we will be able to reduce the study of the functor just to the minima of these intervals.

The categorical tool to describe this fact is the coreflection $\mathbf{C}_i^f$ (resp. $\mathbf{C}^f$) defined on the image of F_i (resp. F). The composition $C_i F_i$ (resp. CF) describes the behavior of the i-homology (resp. homology) pointing out when it changes.

Moreover, as in homology theory, the whole homology can be seen as the sum of each i-homology, that is CF is the sum of the $C_i F_i$ (this sum, of course, cannot be performed in **Ab** but must be considered in the category of functors from **Rord** to **Ab**).

First of all, we can observe that the image of F_i (resp. F) is a subcategory of **Ab** since F_i (resp. F) is injective on the objects.

DEFINITION 3.1. For each i let $\mathbf{C}_i^f$ (resp. $\mathbf{C}^f$) be the full subcategory of $\mathrm{Im}\, F_i$ (resp. $\mathrm{Im}\, F$) whose objects are $F_i(\alpha)$ (resp. $F(\alpha)$) where α is an i-essential critical value (resp. a critical value) for f.

PROPOSITION 3.2. $\mathbf{C}_i^f$ *(resp.* $\mathbf{C}^f$*) is a coreflective subcategory of* Im F_i *(resp.* Im F).

Proof. Given $F_i(x)$, the coreflection is given by

$$c_i\colon F_i(x) \leftarrow F_i(\alpha)$$
$$c\colon F(x) \leftarrow F(\alpha)$$

where α is the first i-essential critical value (resp. the first critical value) less than or equal to x and c_i and c are the unique morphisms in Im F_i (resp. Im F) between the considered objects. It can be observed that by Theorem 3.2 and by definition both these arrows are always isomorphisms in **Ab**. □

Remark 3.3. If we denote by C_i (resp. C) the coreflection functor, we can consider the composition C_iF_i (resp. CF): **Rord** $\to$ **Ab** and it can be proved, by lengthy but straightforward verification, that $CF = \oplus C_iF_i$ in Fun(**Rord**, **Ab**) (i.e. they are isomorphic via a natural transformation).

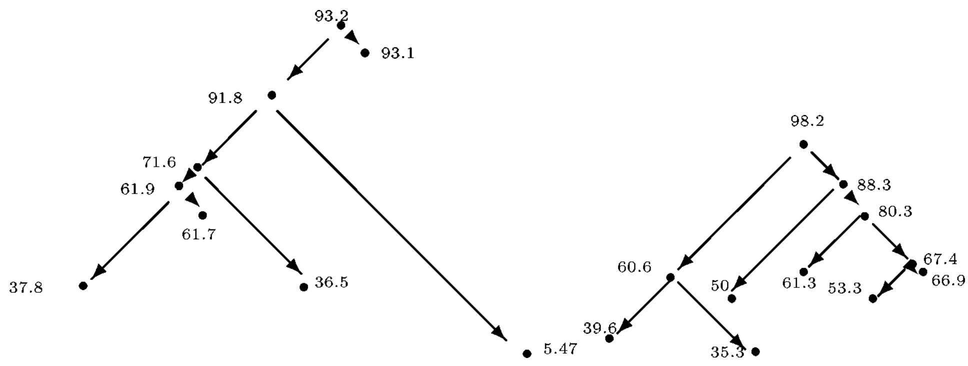
The previous proposition allows us to study only a finite number of slices of the manifold, in fact $\mathbf{C}_i^f$ (resp. $\mathbf{C}^f$) are finite categories. Moreover, dealing with categories and functors, we do not need to know the behavior of each morphism in $\mathbf{C}_i^f$ (resp. $\mathbf{C}^f$) but just the morphism between two subsequent critical points, since the others can be derived by composition; this may turn out to be a big advantage for applications.

4. The Functors B, F_0 and Size Functions

Given two real numbers x, y, with $x \leqslant y$ we can consider the set of arc-connected components of $C(x)$ and $C(y)$, and the map i_{xy} induced on them by the inclusion. This way we obtain a functor B from **Rord** to **Set** (the category of sets and functions). We have that the functor F_0 is the composition of B with the free functor from **Set** to **Ab**. These two functors B and F_0 are theoretically equivalent, so, since B is easier to be described and studied, it is preferable to F_0.

The classical definition of 'size function' as it is given in [12] and sketched in the Introduction of this paper, can be seen as the cardinality of the image of i_{xy}. The functor B (hence F_0) is a finer tool, in comparison with the size function. In fact it is possible to distinguish two measuring functions with B (or F_0) which are not distinguishable by size functions: See Figure 4, where the end-points of the domain segment are to be identified. In this example, the 'natural size distance' – as defined in [12] – of two functions ϕ and ψ defined on the same manifold (a circle) is far from vanishing.

The functor B, in comparison to F_0, has an advantage: it can be described by trees. In fact, for any measuring function f, there is a coreflective subcategory $\mathbf{D}^f$ of Im B analogous to the one defined in Section 3 for the functor F_0. $\mathbf{D}^f$ can be seen as an oriented tree:



(a) (b)

Figure 5. (a) The H_O tree of letter 'r' of Figure 3 (right); (b) the H_O tree of letter 'u' of Figure 3 (right).

- the vertices are the points of sets in $\mathbf{D}^f$, i.e. a vertex v_x is a connected component of $C(x)$ where x is a 0-critical point for f,
- there is an arc from v_y to v_x ($x \leqslant y$) if there is a map h in $\mathbf{D}^f$ such that $hv_x = v_y$.

We can simplify this tree, by inductively deleting vertices with outdegree and indegree equal to one, and substituting the incident arcs with just one arc representing composition. The tree so obtained is the H_O tree of f (Figure 5). From this tree we can obtain the cornerpoints used to compute the size function associated to f as described in [4].

An advantage of the functor F versus size functions is the following. If, for instance, we consider two different manifolds $\mathbf{M}$: a sphere and a torus (or even a segment and circle, by a suitable extension of the theory) with some two measuring functions, the functor F_0 and hence the size function does not necessarily distinguish between these two maps, while F_1 does it for certain. This is obvious, since F_0 depends only on the arcwise connected components of the slices of the manifold, while the F_i together are sensitive to the general topological structure of the same slices.

5. Implementation Problems

Effective applications require, of course, effective implementation. This is available for computation of size functions: It is based on a steepest descent method. Actually, the algorithm is discrete, and the input is a graph with vertices labelled by real numbers; this graph is meant to represent the manifold, the vertices being sampled points of it. Minima (i.e. critical points of index 0) are detected as vertices, and critical points of index 1 are detected as pairs of adjacent vertices pointing to different minima. 0-essential critical points of index 1 are selected by comparison of labels (i.e. of values of the measuring function).

The functor F_0 can then be completely recovered by a suitable adaptation of the quoted algorithm, but also F_1 can, since the previously discarded critical points of index 1 are just the 1-essential ones, and the critical points of index 2 can easily be spotted too. Note that for the most important case in current applications, i.e. a bitmap image, the domain manifold is just a rectangle, so that H_2 is trivial for all submanifolds and F_2 is meaningless. Therefore, automatic computation of F for grey tone images seems to be very near in the future.

Quite different problems are posed by higher-degree functors. Note that this is far from being a purely academic task. In fact, we have worked so long with measuring functions defined on single points of one- or two-dimensional manifolds, but a sensible progress will be the use of measuring functions defined on k-tuples of points, so on points of a repeated topological product of the manifold by itself. So F_i with $i > 1$ will presumably be rich of precious information even in the case of images. Computation of higher degree F_i's will be a major goal for the algorithmic component of our team.

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