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On some generalisations of Meyers-Serrin

theorem

(Su alcune generalizzazioni del teorema di
Meyers-Serrin)

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Abstract

We present a generalisation of Meyers-Serrin theorem, in which we replace the standard weak derivatives in open subsets of $\mathbb{R}^m$ with finite families of linear differential operators defined on smooth sections of vector bundles on a (not necessarily compact) manifold X.

Sunto

Presentiamo una generalizzazione del teorema di Meyers-Serrin, in cui sostituiamo le derivate deboli in sottoinsiemi aperti di $\mathbb{R}^m$ con famiglie finite di operatori differenziali lineari, definiti su sezioni regolari di fibrati vettoriali su una varietà (non necessariamente compatta) X.

Let us denote by $W^{m,p}(\Omega)$ the class of elements f in $L^p(\Omega)$ whose derivatives $\partial^\alpha f$ of order less or equal than m (in the sense of distributions) belong to $L^p(\Omega)$. $W^{m,p}(\Omega)$ becomes a Banach space if it is equipped with its natural norm

$$(1) \qquad \|f\|_{W^{m,p}(\Omega)} := \Big(\sum_{|\alpha| \le m} \|\partial^\alpha f\|_{L^p(\Omega)}^p \Big)^{1/p}.$$

Let us also denote by $H^{m,p}(\Omega)$ the closure of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ in $W^{m,p}(\Omega)$. Then, in a quite famous paper ([4]), N.G. Meyers and J. Serrin proved that

$$H^{m,p}(\Omega) = W^{m,p}(\Omega).$$

This identity holds for every open subset Ω of $\mathbb{R}^n$, regardless of the regularity of its boundary. Observe that, as the inclusion $H^{m,p}(\Omega) \subseteq W^{m,p}(\Omega)$ is obvious, Meyers-Serrin theorem can be also stated by saying that $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

The main aim of this seminar is to illustrate a generalisation of this result that we have recently obtained (see [2]), which is applicable to many different situations. A first possible generalisation of the class of spaces $W^{m,p}(\Omega)$ could be obtained in the following way: take a finite family of linear partial differential operators $\mathcal{P} = \{P_1, \ldots, P_s\}$ with smooth coefficients in Ω. Then we could consider the class $W^{\mathcal{P},p}(\Omega)$ of elements f in $L^p(\Omega)$ such that $P_j f$ (in the sense of distributions) belongs to $L^p(\Omega)$ for each $j \in \{1, \ldots, s\}$. It is straightforward to check that $W^{\mathcal{P},p}(\Omega)$ is a Banach space if it is equipped with the norm

$$\|f\|_{W^{\mathcal{P},p}(\Omega)} := \Big(\|f\|_{L^p(\Omega)}^p + \sum_{j=1}^s \|P_j f\|_{L^p(\Omega)}^p \Big)^{1/p}.$$

More generally, we could consider also differential operators in smooth manifolds, not only defined in spaces of scalar functions, but also defined on sections of vector bundles. So we begin by recalling some well known definitions (see, for example, [5]): let X be a smooth abstract $m-$ dimensional manifold without boundary. We always assume that its topology has a countable basis of open sets, in order to have at disposal the tool of partitions of unity. Suppose that (E, π_E) and (F, π_F) are vector bundles on X. This means that:

(i) E is a smooth manifold, $\pi_E \in C^\infty(E; X)$;

(ii) $E_x := \pi^{-1}(\{x\})$ has the structure of an l_E- dimensional vector space on $\mathbb{C}$ $(l_E \in \mathbb{N}_0)$;

(iii) $\forall x_0 \in X$ there exists a local chart (Φ, U) in X around x_0 (U open in X, $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^m$ diffeomorphism) and a smooth diffeomorphism $\Psi_E : \pi_E^{-1}(U) \to \Phi(U) \times \mathbb{C}^{l_E}$, such that, for $x \in U$, if $v \in E_x$,

$$\Psi_E(v) = (\Phi(x), L_E(x)v)$$

with $L_E(x)$ linear isomorphism between E_x and $\mathbb{C}^{l_E}$.

We denote by $\Gamma_{C^\infty}(X, E)$ the class of smooth sections with values in E, that is, the class of elements $f \in C^\infty(X, E)$ such that $f(x) \in E_x$ $\forall x \in X$, and with $\Gamma_{C_0^\infty}(X, E)$ the class of elements in $\Gamma_{C^\infty}(X, E)$ with compact support. Given $\phi : \Phi(U) \to C^{l_E}$, we can consider the following section $f_\phi : U \to E$:

$$(2) \qquad\qquad f_\phi(x) = L_E(x)^{-1}(\phi(\Phi(x))).$$

If $\phi \in C_0^\infty(\Phi(U); \mathbb{C}^{l_E})$, then f_ϕ, extended with zero outside U, belongs to $\Gamma_{C_0^\infty}(X, E)$.

Let P be a linear mapping from $\Gamma_{C^\infty}(X, E)$ to $\Gamma_{C^\infty}(X, F)$. We define the following operator P_Φ in $C_0^\infty(\Phi(U), \mathbb{C}^{l_E})$, with values in $C^\infty(\Phi(U), \mathbb{C}^{l_F})$:

$$(3) \qquad\qquad P_\Phi(\phi)(y) := L_F(\Phi^{-1}(y))P(f_\phi)(\Phi^{-1}(y)).$$

We say that P is a differential operator of order less or equal to k if, for every system (Φ, U, L_E, L_F), P_Φ is a differential operator in $\Phi(U)$, that is,

$$P_\Phi(\phi)(y) = \sum_{|\alpha| \leq k} A_\alpha(y) D_y^\alpha \phi(y),$$

with $A_\alpha(y)$ $l_F \times l_E-$matrix, depending smoothly on y. We say that P is elliptic of order k if P_Φ is elliptic for every (Φ, U), that is, if $l_E = l_F$ and the matrix $\sum_{|\alpha|=k} \xi^\alpha A_\alpha(y)$ is invertible $\forall y \in \Phi(U)$, $\forall \xi \in \mathbb{R}^m \setminus \{0\}$.

Suppose now that the manifold X is equipped with a smooth density μ. We recall that this means that a positive Borel measure μ is fixed in X, so that, $\forall(\Phi, U)$ local chart in X there exist $\mu_\Phi \in C^\infty(\Phi(U))$, positive, such that, if A is a Borel subset of U

$$\mu(A) = \int_{\Phi(A)} \mu_\Phi(y)dy.$$

Let (E, π_E) be a vector bundle on X. We fix a Hermitian structure (smoothly varying inner product $(\cdot, \cdot)_{E_x}$ in each E_x). Let $p \in [1, \infty)$. We denote by $\Gamma_{L_\mu^p}(X, E)$ the class of

measurable section of E, i.e., measurable functions $f : X \to E$, such that $f(x) \in E_x$ for almost every x and

$$\|f\|^p_{\Gamma_{L^p_\mu}(X,E)} := \int_X \|f(x)\|^p_{E_x} d\mu(x) < \infty.$$

As usual, we do not distinguish sections coinciding almost everywhere.

Let P be a differential operator between the vector bundles (E, π_E) and (F, π_F), both equipped with Hermitian structures (of course, this means that P maps $\Gamma_{C^\infty}(X, E)$ into $\Gamma_{C^\infty}(X, F)$). Then one can show (see [6], Proposition IV.2.8) that there exists a unique differential operator P^* (the adjoint of P) between (F, π_F) and (E, π_E) such that, $\forall f \in \Gamma_{C^\infty}(X, E)$, $\forall g \in \Gamma_{C^\infty_0}(X, F)$

$$\int_X (Pf(x), g(x))_{F_x} \mu(dx) = \int_X (f(x), P^*g(x))_{E_x} \mu(dx).$$

By simple computation, one can draw the following local expression for P^*: if $f = f_\phi$ as in (2) and P_Φ is the operator defined in (3), one has, for $x \in U$,

$$(4) \qquad\qquad P^*g(x) = \frac{1}{\mu_\Phi(\Phi(x))} L_E(x)^* P_\Phi^*(M\psi)(\Phi(x)),$$

with P_Φ^* adjoint of P_Φ,

$$(5) \qquad\qquad \psi(y) = L_F(\Phi^{-1}(y))g(\Phi^{-1}(y)), (y \in \Phi(U)),$$

$$(6) \qquad\qquad M(y) = \mu_\Phi(y) L_F(\Phi^{-1}(y))^{-1*} L_F(\Phi^{-1}(y))^{-1}.$$

This suggests the possibility of defining Pf in cases when f is not smooth. So we denote by $\Gamma_{L^1_{loc}}(X, E)$ the class of locally summable sections. It is not difficult to see that this class does not depend on μ, as far as we limit ourselves to considering smooth densities. We can give the following definition:

Definition 1. *Let P be a differential operator between the bundles (E, π_E) and (F, π_F), both equipped with Hermitian structures. Let $f \in \Gamma_{L^1_{loc}}(X, E)$ and $h \in \Gamma_{L^1_{loc}}(X, F)$. We write $Pf = h$ if, $\forall g \in \Gamma_{C^\infty_0}(X, F)$,*

$$\int_X (h(x), g(x))_{F_x} \mu(dx) = \int_X (f(x), P^*g(x))_{E_x} \mu(dx).$$

One can show that, although P^* depends on the Hermitian structures and μ, this definition of Pf is independent of them.

Lemma 1. *Using the notations (2) and (3), if $f = f_\phi \in \Gamma_{L^p_{loc}}(U, E)$ and $Pf \in \Gamma_{L^p_{loc}}(U, E)$ in the sense of Definition 1, $P_\Phi \phi \in L^p_{loc}(\Phi(U), \mathbb{C}^{l_F})$, where, of course, $P_\Phi \phi$ has a meaning in the sense of distributions.*

Proof. Let $g \in \Gamma_{C^\infty_0}(U, F)$. We set $\psi(y) := L_F(\Phi^{-1}(y))g(\Phi^{-1}(y))$. Of course, $\psi \in C^\infty_0(\Phi(U), \mathbb{C}^{l_F})$. Employing (4), we have

$$\int_U (f(x), P^*g(x))_{E_x} \mu(dx) \;=\; \int_{\Phi(U)} (\phi(y), P^*_\Phi(M\psi)(y))_{\mathbb{C}^{l_E}} dy$$

$$= (M(y)^* P_\Phi \phi(y), \psi(y)),$$

with $(\cdot, \cdot)$ standing for the duality $(\mathcal{D}'(\Phi(U), \mathbb{C}^{l_F}), C^\infty_0(\Phi(U), \mathbb{C}^{l_F}))$. On the other hand,

$$\int_U (Pf(x), g(x))_{F_x} \mu(dx) = \int_{\Phi(U)} (\mu_\Phi(y) L_F(\Phi^{-1}(y))^{-1*} Pf(\Phi^{-1}(y)), \psi(y))_{\mathbb{C}^{l_F}} dy,$$

so that

$$P_\Phi \phi(y) = \mu_\Phi(y) M(y)^{*-1} L_F(\Phi^{-1}(y))^{-1*} Pf(\Phi^{-1}(y)) \in L^p_{loc}(\Phi(U); \mathbb{C}^{l_F}).$$

$$\square$$

Finally, let $f \in \Gamma_{L^1_{loc}}(X, E)$. We write $f \in \Gamma_{W^{n,p}_{loc}}(X, E)$ if, for all systems (Φ, U, L_E), $y \to L_E(\Phi^{-1}(y))f(\Phi^{-1}(y))$ is in $W^{n,p}_{loc}(\phi(U); \mathbb{C}^{l_E})$. This definition also does not depend on μ and the Hermitian structures.

After these preliminaries, we introduce our assumptions:

(A1) X is a smooth $m-$ dimensional manifold without boundary, with a countable basis of open sets. μ is a fixed smooth density in X.

(A2) $(E, \pi_E), (F_1, \pi_{F_1}), \ldots, (F_s, \pi_{F_s})$ are vector bundles on X. Each of them is equipped with a Hermitian structure.

For simplicity we write $E, F_1, \ldots, F_s$ instead of $(E, \pi_E), (F_1, \pi_{F_1}), \ldots, (F_s, \pi_{F_s})$.

(A3) For each $j \in \{1, \ldots, s\}$ P_j is a differential operator between E and F_j.

Suppose that (A1)-(A3) are satisfied. We define, $\forall p \in [1, \infty)$,

$$(7) \qquad \Gamma_{W^{P,p}_\mu}(X, E) := \{f \in \Gamma_{L^p_\mu}(X, E) : P_j f \in \Gamma_{L^p_\mu}(X, F_j) \;\; \forall j \in \{1, \ldots, s\}\}.$$

It is straightforward to check that $\Gamma_{W_\mu^{\mathcal{P},p}}(X,E)$ becomes a Banach space if it is equipped with the norm

$$(8) \qquad \|f\|_{\Gamma_{W_\mu^{\mathcal{P},p}}(X,E)} := (\|f\|_{\Gamma_{L_\mu^p}(X,E)}^p + \sum_{j=1}^s \|P_j f\|_{\Gamma_{L_\mu^p}(X,F_j)}^p)^{1/p}.$$

Now we are able to state the following generalisations of Meyers-Serrin theorem:

Theorem 1. *Suppose that (A1)-(A3) are fulfilled and that the operators $P_1, \ldots, P_s$ are of order less or equal than k, with $k \in \mathbb{N}$. Then, if $W_\mu^{\mathcal{P},p}(X,E) \subseteq \Gamma_{W_{loc}^{k-1,p}}(X,E)$, $\Gamma_{C^\infty}(X,E) \cap W_\mu^{\mathcal{P},p}(X,E)$ is dense in $W_\mu^{\mathcal{P},p}(X,E)$.*

We show some examples and applications. The first result immediately follows from Theorem 1.

Corollary 1. *Suppose that (A1)-(A3) hold and the operaters $P_1, \ldots, P_s$ are of order not exceeding one. Then $\Gamma_{C^\infty}(X,E) \cap W_\mu^{\mathcal{P},p}(X,E)$ is dense in $W_\mu^{\mathcal{P},p}(X,E)$.*

A less obvious result is the following:

Proposition 1. *Suppose that (A1)-(A3) hold. Moreover, P_s is elliptic of order k and $P_1, \ldots, P_{s-1}$ are of order less or equal than $k+1$ in case $p \in (1,\infty)$, of order less or equal than k in case $p = 1$. Then $\Gamma_{C^\infty}(X,E) \cap W_\mu^{\mathcal{P},p}(X,E)$ is dense in $W_\mu^{\mathcal{P},p}(X,E)$.*

Proof. If $p \in (1,\infty)$, it is a consequence of standard elliptic theory in $\mathbb{R}^m$ and Lemma 1 that from $u \in \Gamma_{L_\mu^p}(X,E)$ and $P_s u \in \Gamma_{L_\mu^p}(X,E)$ it follows $u \in \Gamma_{W_{loc}^{k,p}}(X,E)$ (see [5], Theorem 10.3.6). In case $p = 1$, it is proved in [2] that, if P is an $l \times l-$elliptic system of order k in $\mathbb{R}^m$, $u \in L_{loc}^1(\mathbb{R}^m; \mathbb{C}^l)$ and $Pu \in L_{loc}^1(\mathbb{R}^m; \mathbb{C}^l)$, then $\forall \phi \in C_0^\infty(\mathbb{R}^m)$ ϕu belongs to the Besov space $B_{1,\infty}^k(\mathbb{R}^m; \mathbb{C}^l)$, which contains $W^{k,1}(\mathbb{R}^m; \mathbb{C}^l)$ and is contained in $W^{k-1,1}(\mathbb{R}^m; \mathbb{C}^{l_2})$. As, if P_s is elliptic, $P_{s\Phi}$ is elliptic for every system (Φ, U, L_E, L_{F_s}), we deduce, again from Lemma 1, that, from $u \in \Gamma_{L_\mu^1}(X,E)$ and $P_s u \in \Gamma_{L_\mu^1}(X,E)$, it follows $u \in \Gamma_{W_{loc}^{k-1,1}}(X,E)$, so that Theorem 1 is applicable. $\square$

Remark 1. It is quite clear that it cannot be expected that Theorem 1 holds for $p = \infty$. Nevertheless, one can replace $\Gamma_{L^\infty}(X,E)$ with $\Gamma_{C \cap L^\infty}(X,E)$, the Banach space of bounded

continuous sections. The crucial condition becomes

$$\{u \in \Gamma_{C \cap L^\infty}(X, E) : P_j u \in \Gamma_{C \cap L^\infty}(X, F_j) \forall j \in \{1, \ldots, s\}\} \subseteq \Gamma_{C^{k-1}}(X, E).$$

Proposition 1, in a weak form, analogous to the one valid in case $p = 1$, can be extended to this case, because one can show that, if P_s is elliptic of order k, from $u \in \Gamma_{C \cap L^\infty}(X, E)$ and $P_s u \in \Gamma_{C \cap L^\infty}(X, E)$, it follows $u \in \Gamma_{C^{k-1}}(X, E)$

Before considering our last example (Sobolev spaces in not necessarily compact Riemannian manifolds), we want to discuss the optimality of Theorem 1.

First of all, it is clear that the condition $W_\mu^{\mathcal{P},p}(X, E) \subseteq \Gamma_{W_{loc}^{k-1,p}}(X, E)$ is not necessary to get the conclusion: for example, let $X = \mathbb{R}^m$, μ the standard Lebesgue measure, $E = \mathbb{R}^m \times \mathbb{C}$ the trivial bundle and assume that the operators $P_1, \ldots, P_s$ have constant coefficients and (for simplicity) are defined in $C^\infty(\mathbb{R}^m)$ with values in $C^\infty(\mathbb{R}^m)$. Then $\Gamma_{C^\infty}(X, E) \cap W_\mu^{\mathcal{P},p}(X, E)$ is dense in $W_\mu^{\mathcal{P},p}(X, E)$. To show this, it suffices to consider the usual regularisation procedure: fix $\omega \in C_0^\infty(\mathbb{R}^m)$, with $\int_{\mathbb{R}^m} \omega(x) dx = 1$ and, set, for $\epsilon > 0$, $\omega_\epsilon(x) := \epsilon^{-m} \omega(\frac{x}{\epsilon})$. Suppose, for simplicity, that $u \in W_\mu^{\mathcal{P},p}(X, \mathbb{C})$. Then the convolutions $(\omega_\epsilon * u)_{\epsilon > 0}$ converge to u in $L^p(\mathbb{R}^m; \mathbb{C})$ (as $\epsilon \to 0$) and, for each $j \in \{1, \ldots, s\}$, $(P_j(\omega_\epsilon * u))_{\epsilon > 0} = (\omega_\epsilon * P_j u)_{\epsilon > 0}$ converge to $P_j u$ in $L^p(\mathbb{R}^m; \mathbb{C})$, regardless of the condition $W_\mu^{\mathcal{P},p}(X, \mathbb{C}) \subseteq \Gamma_{W_{loc}^{k-1,p}}(X, \mathbb{C})$.

Neverthless, this condition is, in some sense, optimal, because it may happen that $W_\mu^{\mathcal{P},p}(X, \mathbb{C}) \subseteq \Gamma_{W_{loc}^{k-2,p}}(X, \mathbb{C})$ and $\Gamma_{C^\infty}(X, E) \cap W_\mu^{\mathcal{P},p}(X, E)$ is not dense in $W_\mu^{\mathcal{P},p}(X, E)$. In fact let us consider the following example.

Example 1. We consider the following operator P in $\mathbb{R}$, equipped with the Lebesgue measure:

$$Pu(x) = -x u^{(3)}(x) + (x - 1) u^{(2)}(x).$$

We set $\mathcal{P} := \{P\}$. We take $p \in (1, \infty)$, write $W^{\mathcal{P},p}$ instead of $W^{\mathcal{P},p}(\mathbb{R}, \mathbb{C})$, and start by observing that

$$W^{\mathcal{P},p} = \{u \in L^p(\mathbb{R}) : x u'' \in W^{1,p}(\mathbb{R})\}.$$

In fact, $P = (1 - \partial) \circ x\partial^2$. Let $u \in W^{\mathcal{P},p}$ and set $v = xu''$. Then v is a tempered distribution and $v - v' = Pu \in L^p(\mathbb{R})$. Employing the Fourier transform $\mathcal{F}$, we see that $v = \mathcal{F}^{-1}((1 - i\xi)^{-1}\mathcal{F}Pu)$, which belongs to $W^{1,p}(\mathbb{R})$ because $Pu \in L^p(\mathbb{R})$.

We set $f := Pu$. We have that

$$u''(x) = x^{-1}v(x) = x^{-1}v(0) + x^{-1}\int_0^x v'(y)dy = x^{-1}v(0) + h(x), \quad x \in \mathbb{R} \setminus \{0\},$$

with $h(x) = x^{-1}\int_0^x v'(y)dy$. By Hardy's inequality, $h \in L^p(\mathbb{R})$. It follows that

(9)
$$u''(x) = v(0)p.v.(\frac{1}{x}) + h(x) + k(x),$$

with k distribution with support in $\{0\}$. From $xu'' = v$ it follows $xk(x) = 0$, which implies $k(x) = c\delta(x)$. So from (9) we deduce

$$u'(x) = v(0)\ln(|x|) + \int_0^x h(y)dy + cH(x) + const,$$

where we have denoted by $H(x)$ the Heaviside function. We infer that $u' \in L^p_{loc}(\mathbb{R})$, so that $W^{\mathcal{P},p} \subseteq W^{1,p}_{loc}(\mathbb{R})$. Nevertheless, $W^{\mathcal{P},p}$ is not contained in $W^{2,p}_{loc}(\mathbb{R})$. In fact, set

$$u(x) := \phi(x)x\ln(|x|),$$

with $\phi \in C_0^\infty(\mathbb{R}), \phi(x) = 1$ in some neighbourhood of 0. It is easily seen that $u \in W^{\mathcal{P},p}$, as

(10)
$$xu''(x) = 1,$$

in some neighbourhood of 0. u does not belong to $W^{2,p}_{loc}(\mathbb{R})$, because

$$u''(x) = p.v.(\frac{1}{x})$$

in some neighbourhood of 0. Now we check that $C^\infty(\mathbb{R}) \cap W^{\mathcal{P},p}$ is not dense in $W^{\mathcal{P},p}$. We argue by contradiction: we suppose that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C^\infty(\mathbb{R}) \cap W^{\mathcal{P},p}$, such that

$$\|u_k - u\|_{L^p(\mathbb{R})} + \|Pu_k - Pu\|_{L^p(\mathbb{R})} \to 0 \quad (k \to \infty).$$

We set $v := xu''$, $v_k := xu_k''$. Then, as $v_k = \mathcal{F}^{-1}((1 - i\xi)^{-1}\mathcal{F}Pu_k)$, $(v_k)_{k\in\mathbb{N}}$ converges to v in $W^{1,p}(\mathbb{R})$, implying that $(v_k(0))_{k\in\mathbb{N}}$ converges to $v(0)$. However, evidently, $v_k(0) = 0$, while, by (10), $v(0) = 1$. We have reached a contradiction.

We conclude by extending the theorem of Meyers-Serrin to (not necessarily compact) Riemannian manifolds. We begin by defining the Sobolev space $W^{m,p}(X)$, with X Riemannian manifold. We take the measure μ induced by the metric in X and denote by $T(X)$ the tangent bundle. We recall that the Riemannian (or Levi-Civita) connection is the unique connection in X, such that, $\forall X, Y, Z \in \Gamma_{C^\infty}(X, T(X))$,

$$D_X(Y) - D_Y(X) = [X, Y],$$

with $[X, Y]$ commutator of X and Y, and

$$X(Y \cdot Z) = D_X(Y) \cdot Z + Y \cdot D_X(Z),$$

with $(X \cdot Y)(x) := X(x) \cdot_x Y(x)$, $\forall x \in X$ and $\cdot_x$ is, of course, the inner product in $T_x(X)$. In local coordinates, if $X = \sum_{j=1}^m X_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^m Y_j \frac{\partial}{\partial x_j}$, we have

$$D_X(Y) = \sum_{i,j=1}^m X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k=1}^m \Gamma_{ij}^k X_i Y_j \frac{\partial}{\partial x_k}.$$

Here

$$D_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where the functions Γ_{ij}^k are the so called Christoffel symbols.

Given $X \in \Gamma_{C^\infty}(X, T(X))$, we can define $D_X(\alpha)$ $\forall \alpha \in \Gamma_{C^\infty}(X, T^{*\otimes j}(X))$ in a unique way so that the following conditions are satisfied:

(I) $D_X(\alpha) \in \Gamma_{C^\infty}(X, T^{*\otimes j}(X))$ $\forall \alpha \in \Gamma_{C^\infty}(X, T^{*\otimes j}(X))$;

(II) if $f \in C^\infty(X)$, $D_X(f) = X(f)$;

(III) if $\alpha \in \Gamma_{C^\infty}(X, T^*(X))$ and $Y \in \Gamma_{C^\infty}(X, T(X))$,

$$(D_X(\alpha), Y) = X((\alpha, Y)) - (\alpha, D_X(Y)),$$

with $(\alpha, Y)(x) := (\alpha(x), Y(x)) \in C^\infty(X)$;

(IV) if $\alpha \in \Gamma_{C^\infty}(X, T^{*\otimes j}(X))$ and $\beta \in \Gamma_{C^\infty}(X, T^{*\otimes k}(X))$,

$$D_X(\alpha \otimes \beta) = D_X(\alpha) \otimes \beta + \alpha \otimes D_X(\beta).$$

So, we can easily deduce that, if $X \in \Gamma_{C^\infty}(X, T(X))$, $\alpha \in \Gamma_{C^\infty}(X, T^*(X))$ and, in local coordinates, $X = \sum_{i=1}^m X_i \frac{\partial}{\partial x_i}$, $\alpha = \sum_{j=1}^m \alpha_j dx_j$, we have, again in local coordinates,

$$(11) \qquad D_X(\alpha) = \sum_{i,j=1}^m X_i \frac{\partial \alpha_j}{\partial x_i} dx_j - \sum_{i,j,k=1}^m \Gamma_{ij}^k X_i \alpha_k dx_j.$$

Now we introduce the operators ∇^j ($j \in \mathbb{N}$), each of which is a differential operator from $C^\infty(X)$ to $\Gamma_{C^\infty}(X, T^{*\otimes j(X)})$. If $f \in C^\infty(X)$, we define $\nabla^j f$ recursively: given $\nabla^j f \in \Gamma_{C^\infty}(X, T^{*\otimes j(X)})$, we define $\nabla^{j+1} f \in \Gamma_{C^\infty}(X, T^{*\otimes(j+1)(X)})$ setting, $\forall X^1, X^2, \ldots X^{j+1} \in \Gamma_{C^\infty}(X, T(X))$,

$$(12) \qquad \nabla^{j+1} f(X^1, X^2, \ldots, X^{j+1}) := D_{X^1}(\nabla^j f)(X^2, \ldots, X^{j+1}).$$

So, for example, we have $\nabla^1 f = df = \sum_{j=1}^m \frac{\partial f}{\partial x_j} dx_j$ (in local coordinates) and, from (11),

$$\nabla^2 f = \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i x_j} dx_i \otimes dx_j - \sum_{i,j,k=1}^m \Gamma_{ij}^k \frac{\partial f}{\partial x_k} dx_i \otimes dx_j.$$

In general one can easily see that, in local coordinates,

$$(13) \qquad \nabla^j f = \sum_{i_1,\ldots,i_j=1}^m \left(\frac{\partial^j f}{\partial x_{i_1} \ldots dx_{i_j}} + P_{i_1,\ldots,i_j}(f) \right) dx_{i_1} \otimes \cdots \otimes dx_{i_j},$$

with $P_{i_1,\ldots,i_j}(f)$ operator of order strict less than j. An outline of this theory can be found in [3].

Now we equip each bundle $T^{*\otimes j}(X)$ with a Hermitian structure in a natural way: if $\alpha, \beta \in T_x^*(X)$, we set

$$(\alpha, \beta)_x := (V(\alpha), V(\beta))_x,$$

with $V(\alpha) \in T_x(X)$ such that $(\alpha, w) = (V(\alpha), w)_x$, $\forall w \in T_x(X)$. Next, we define $(\alpha, \beta)_x$, with $\alpha, \beta \in T_x^{*\otimes j}(X)$, employing the fact that, given linear spaces X and Y, equipped with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, there is a unique inner product $(\cdot, \cdot)$ in $X \otimes Y$ such that, $\forall \alpha_1, \alpha_2 \in X$, $\forall \beta_1, \beta_2 \in Y$

$$((\alpha_1 \otimes \beta_1), (\alpha_2 \otimes \beta_2)) = (\alpha_1, \alpha_2)_X (\beta_1, \beta_2)_Y.$$

See for this [1], Chapter 1.3.

Now we are able to define the Sobolev spaces $W^{s,p}(X)$:

Definition 2. *Let X be a Riemannian manifold, equipped with the measure μ induced by the Riemannian structure. For each $j \in \mathbb{N}$, we equip each space $T^*(X)^{\otimes j}$ with the Hermitian structures previously described and consider the differential operators ∇^j from $C^\infty(X)$ to $\Gamma_{C^\infty}(X, T^*(X)^{\otimes j})$. Let $p \in [1, \infty)$ and $s \in \mathbb{N}$. We set*

$$W^{s,p}(X) := \{f \in L^p_\mu(X) : \forall j \in \{1, \ldots, s\} \ \ \nabla^j f \in \Gamma_{L^p_\mu}(X, T^*(X)^{\otimes j})\}.$$

$W^{s,p}$ is a Banach space with the norm

(14) $$\|f\|_{W^{s,p}(X)} := (\|f\|^p_{L^p_\mu(X)} + \sum_{j=1}^s \|\nabla^j f\|^p_{\Gamma_{L^p_\mu}(X, T^*(X)^{\otimes j})})^{1/p}.$$

From Theorem 1, we easily deduce the following generalisation of Meyers-Serrin theorem:

Theorem 2. *Let $W^{s,p}(X)$ be the spaces described in Definition 2. Then $C^\infty(X) \cap W^{s,p}(X)$ is dense in $W^{s,p}(X)$.*

Proof. Let (U, Φ) be a local chart. If $x \in U$, $j \in \mathbb{N}$ and $F = T^*(X)^{\otimes j}$, we can take

$$L_F(x)(\sum_{i_1,\ldots,i_j=1}^m \alpha_{i_1,\ldots,i_j}(x) dx_{i_1} \otimes \ldots dx_{i_j}) = (\alpha_{i_1,\ldots,i_j}(x))_{1 \le i_1,\ldots,i_j \le m}.$$

It follows that, if $y \in \Phi(U)$, employing the usual notations (2)-3), we have

$$\nabla^j_\Phi(\phi)(y) \ = (\frac{\partial^j f_\phi}{\partial x_{i_1}\ldots dx_{i_j}}(\Phi^{-1}(y)) + P_{i_1,\ldots,i_j}(f_\phi)(\Phi^{-1}(y)))_{1 \le i_1,\ldots,i_j \le m}$$

$$= (\frac{\partial^j \phi}{\partial r_{i_1}\ldots dr_{i_j}}(y) + Q_{i_1,\ldots,i_j}(\phi)(y)))_{1 \le i_1,\ldots,i_j \le m},$$

with each operator $Q_{i_1,\ldots,i_j}$ of degree less than j. We deduce from Lemma 1 that, if $f_\phi \in W^{j-1,p}_{loc}(X)$, $\phi \in W^{j,p}_{loc}(\Phi(U); \mathbb{C}^{m^j})$. We conclude that each space $W^{s,p}(X)$ is contained in $W^{s,p}_{loc}(X)$. So we can apply Theorem 1. $\square$

Remark 2. The conclusion of Theorem 2 continue to hold if we replace, in the previous construction of Sobolev spaces $W^{s,p}(X)$, the Levi-Civita connection with any other smooth connection: of course, we obtain a different Sobolev space, for which we have a corresponding Meyers-Serrin type theorem.

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