Topics in Global Analysis

The course will be concerned with aspects of the theory of elliptic differential operators in which global phenomena, as opposed to local, are a dominant feature. Before listing the specific topics we illustrate the point with some examples.

First, local solvability versus global solvability for elliptic operators. The equation $du/dx = f$ has a solution for any given f near a point x_0 (say with f continuous), but viewed as a problem on the circle, in which x is the angular variable, does not always have a solution: the vanishing of the integral of f over the circle is a necessary condition for the existence of u. This example is easily extended to one for a partial differential equation on a torus.

Boundary value problems provide another example of the contrast of local versus global settings. With Δ being the Laplacian (in 2 dimensions, for example), the equation $\Delta u - \lambda u = f$ in a bounded region with smooth boundary, with $u = 0$ on the boundary, may not have a solution. If for a given λ existence fails for some reasonable right hand side, then there is a necessary (and sufficient) condition (depending on that λ) similar to the one in the previous paragraph the ensures existence. The condition singles out a space of finite codimension in the space of all data (say in L^2) for which a solution exists.

Concerning boundary value problems, consider the following problem. Given a bounded region Ω with smooth boundary and a differential operator P which is elliptic in a neighborhood of the closure of Ω. Is there a boundary condition B for which the problem $Pu = f$ in Ω, $Bu = g$ on $\partial\Omega$, has a finite dimensional kernel while also a solution for all (f, g) in a space of finite codimension in the Hilbert space from which f and g are taken (typically L^2 and Sobolev spaces)? The operator B, should act on boundary values of u and normal derivatives as a matrix of differential (or pseudodifferential) operators of certain orders. The answer is, no: there is a topological obstruction to the existence of such B.

Other places where the global aspect introduces interesting phenomena are cohomology of elliptic complexes on closed manifolds (i.e. compact manifolds without boundary), where we have finite cohomology for the global problem, but sometimes trivial cohomology for the local problem, and no general theorem known for the local problem) and eigenvalue problems.

The aim is to go into each of the following topics in sufficient detail that the beauty of each and of the mathematics involved becomes apparent.

1. Elliptic boundary value problems following Chapter 5 of [4], to the extent that we reach the point where we can describe the Atiyah-Bott obstruction [1] to the existence of Lopatinskii-Schapiro conditions for a given elliptic differential operator on a given compact manifold (or open set in $\mathbb{R}^n$) with boundary.

2. Hodge theory (which concerns elliptic complexes on closed manifolds), originally in [5] but we will not follow this reference. This will require some basic knowledge of pseudodifferential operators, for which the plan is to give only enough information as is necessary to develop this and other upcoming topics.

3. The Atiyah-Bott formula for the Lefschetz number [2]. The Lefschetz number [6] of a continuous map from a closed manifold to itself discriminates between the case where the map is homotopic to one without fixed points and one which must have fixed points. The Atiyah-Bott formula (which restricts some of Lefschetz's

version but extends it in other directions) gives the same number in case the map is smooth with "nondegenerate" fixed points.

4. Eigenvalues and the ζ function of an elliptic operator. This again concerns an elliptic differential operator on a closed manifold, acting as an unbounded operator on a suitable L^2 space. The operator is assumed to be positive. The definition of its zeta function is inspired by the Riemann zeta function, initially defined for complex numbers with sufficiently large real part, has a meromorphic extension to all of $\mathbb{C}$ without pole at 0; we will follow [8] in this exposition. That 0 is a regular point allows for the definition of a determinant for such operators (the ζ-regularized determinant). The zeta function of laplacians was introduced by Minakshisundaram and Pleijel [7] in the 1940's. Several other problems are generated by the existence of this determinant, including the next topic.

5. The gluing formula for determinants of elliptic differential operators. Suppose that in a closed manifold we have a suitable elliptic operator and a separating hypersurface (the complement has two connected components). The elliptic operator has its own zeta regularized determinant, and one can define similarly determinants for the operator on either component of the complement of the hypersurface associated to boundary value problems on each side, with the appropriate transmission condition. The question of the relation between these three determinants (is the first the product of the two others? no but there is a correction term) was resolved by Burghelea, Friedlander and Kappeler [3].

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