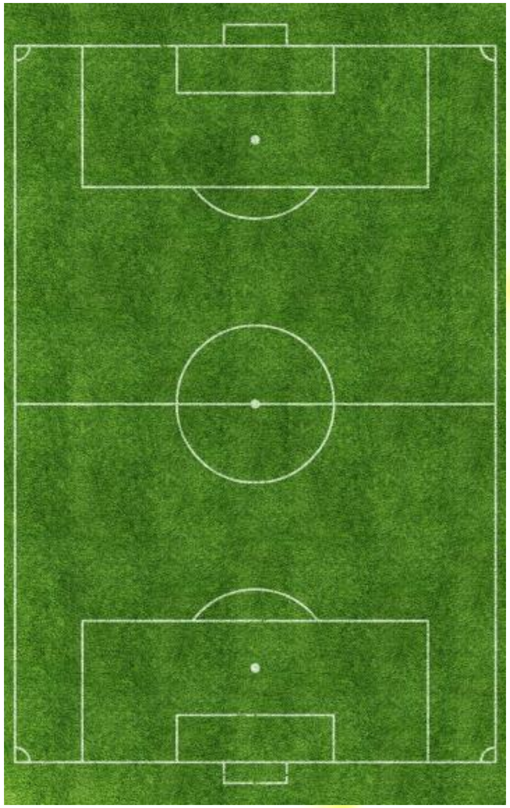
Persistence for shape analysis

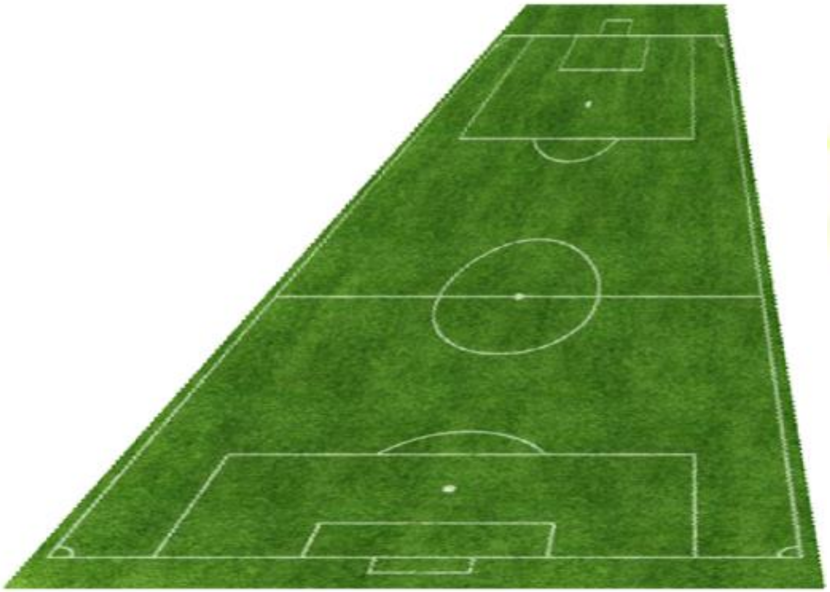
Massimo Ferri

Persistence for shape analysis

- **Shape: a topological problem?**
- Persistent Betti Numbers and Persistence Diagrams
- Shape analysis
- Not only images
- But...

Shape: a topological problem?



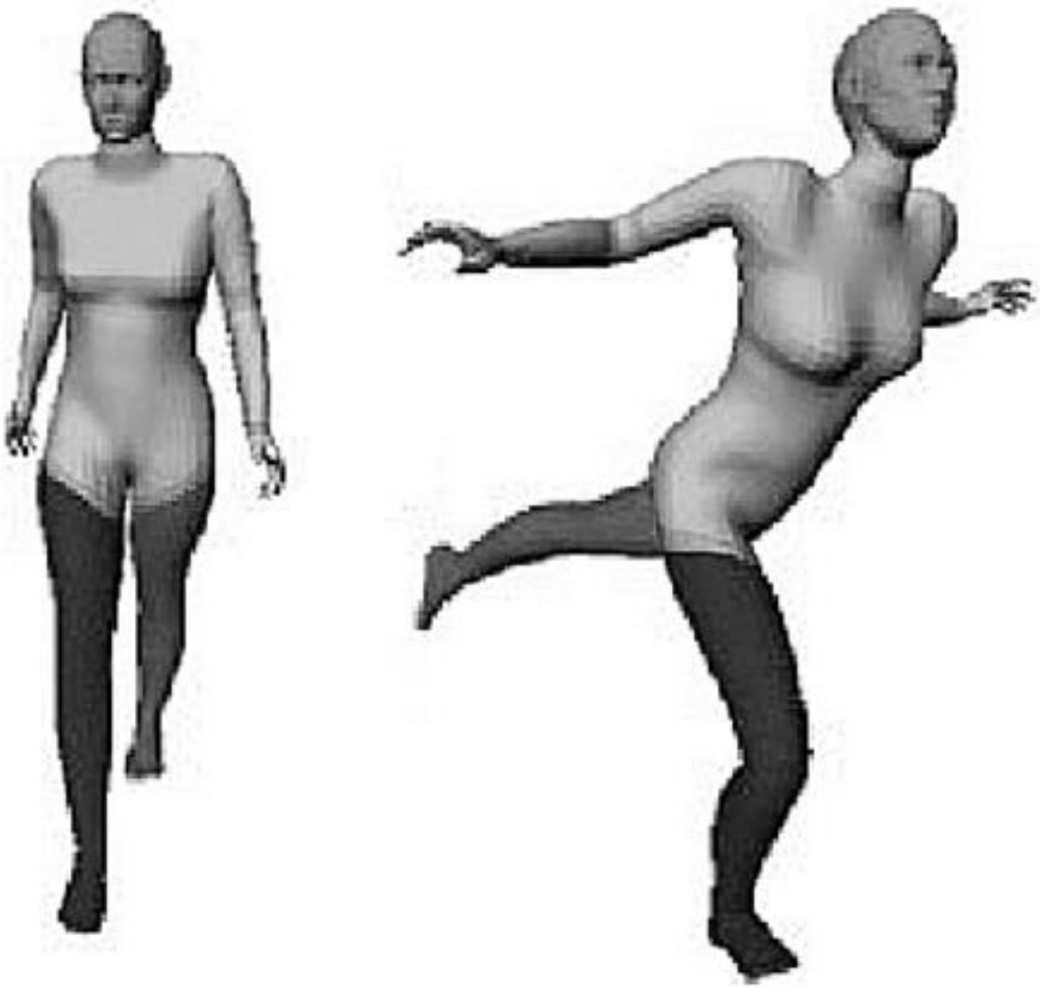


$$\begin{cases} x' = ax + by + et \\ y' = cx + dy + ft \\ t' = gx + hy + kt \end{cases}$$

$$adk + bfg + ech - edg - afh - bck \neq 0$$

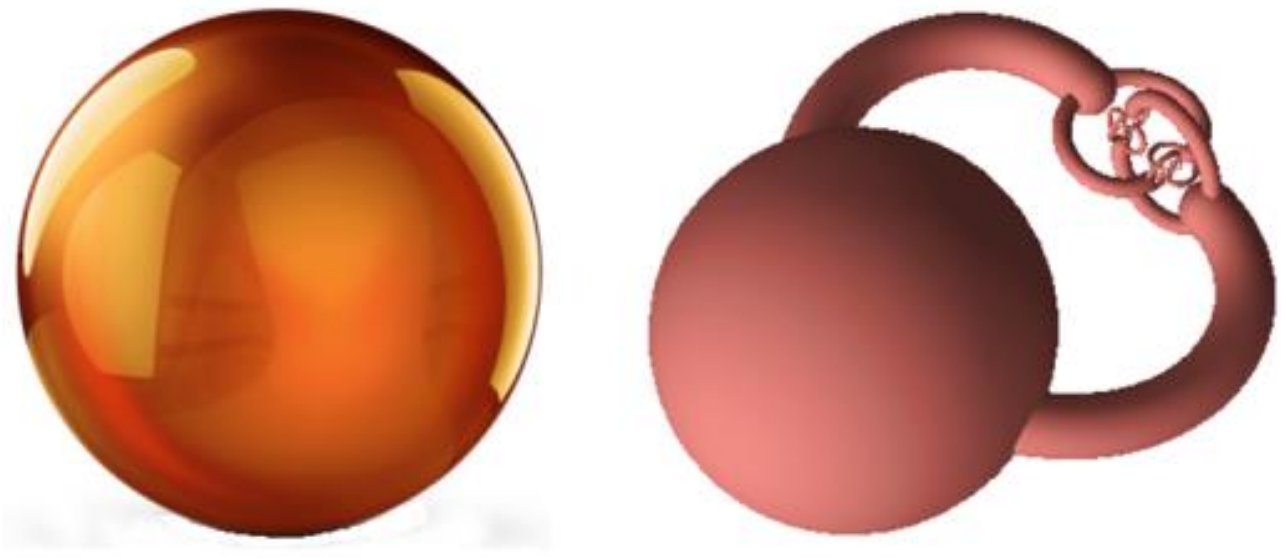
In image recognition, linear similarity is performing well

Shape: a topological problem?



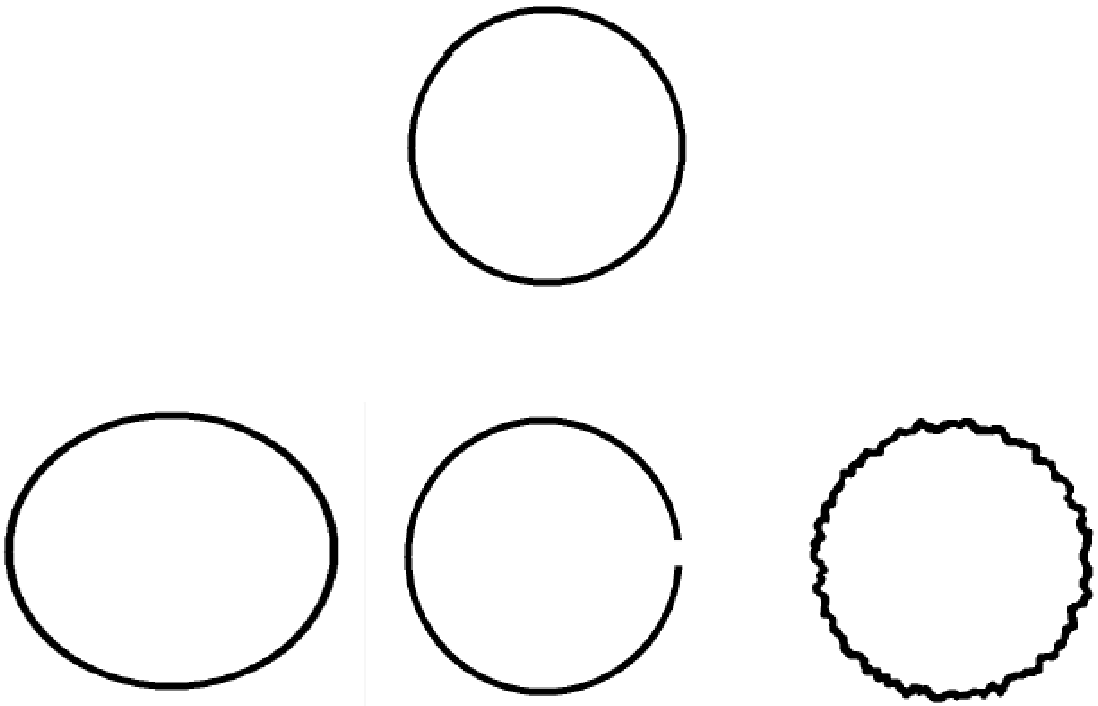
still sometimes you need homeomorphism

Shape: a topological problem?



but here we have a homeomorphism too!

Shape: a topological problem?



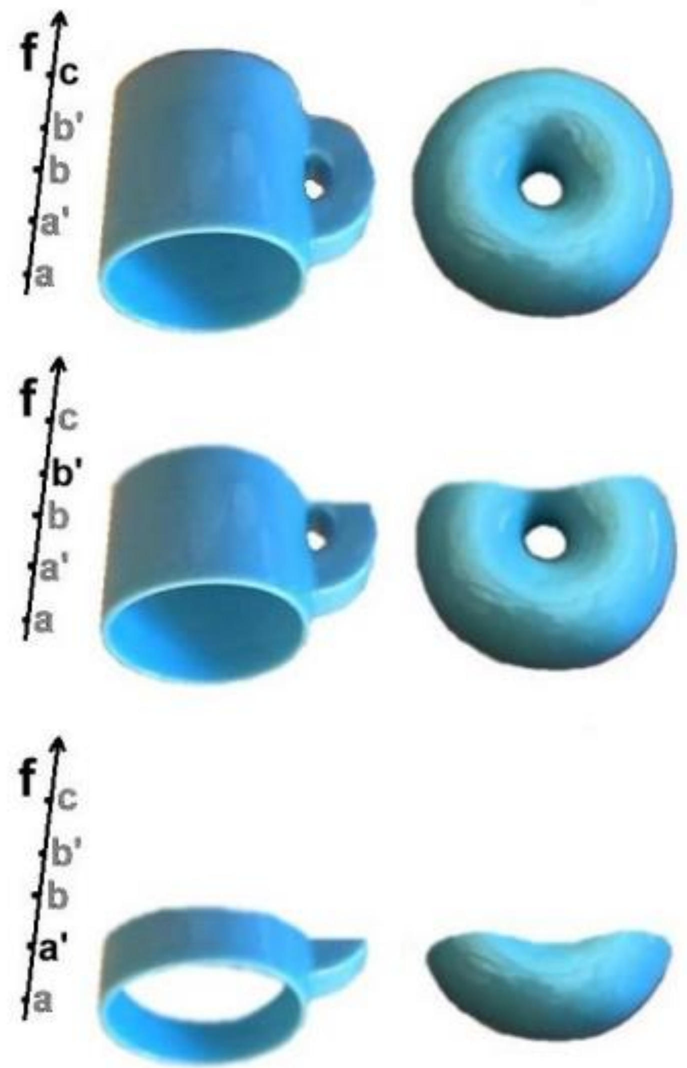
Which object has "the same shape" as the upper circle?

In our opinion, this depends on the observer (his/her viewpoint, interest, tasks…). This can be formalized by a *filtering function* defined on the object of interest.

Shape: a topological problem?

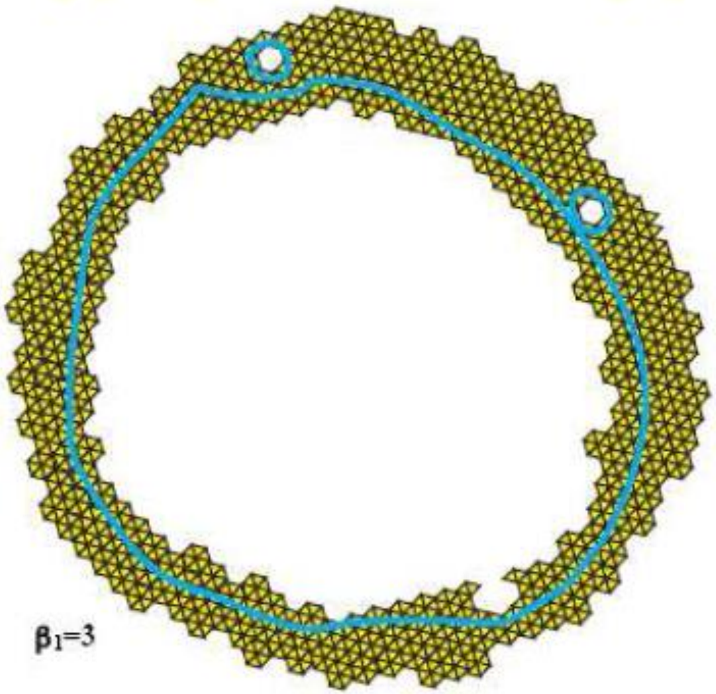
The key idea of persistence is to consider not just a space *X*, but a filtration of it, e.g. through sublevel spaces of a *filtering map* $f: X \to \mathbf{R}^n$. (Mostly $n=1$). The object becomes the *size pair (X,f)*.

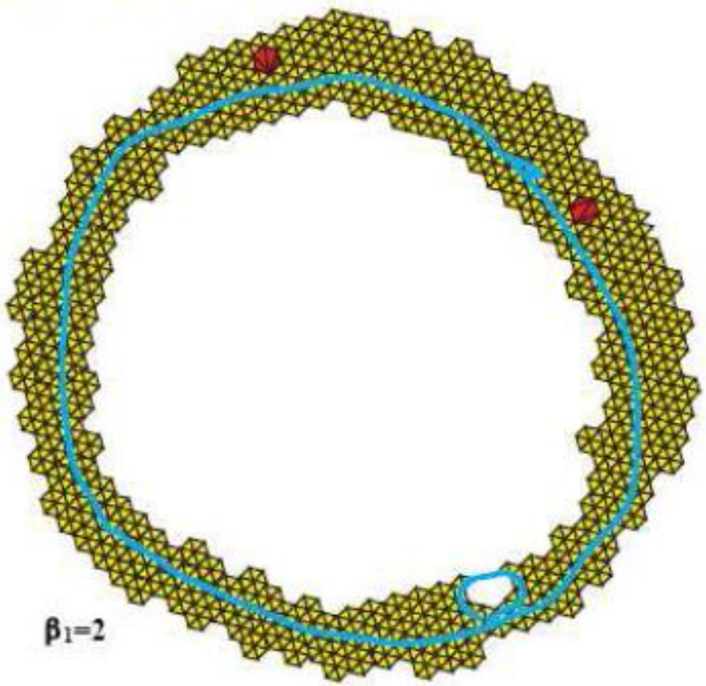
With f = - ordinate we can then distinguish mug and doughnut, while recognizing different mugs and different doughnuts as similar.



Shape: a topological problem?

Persistence was (re)invented in Stanford for guessing the topology of a sampled space:

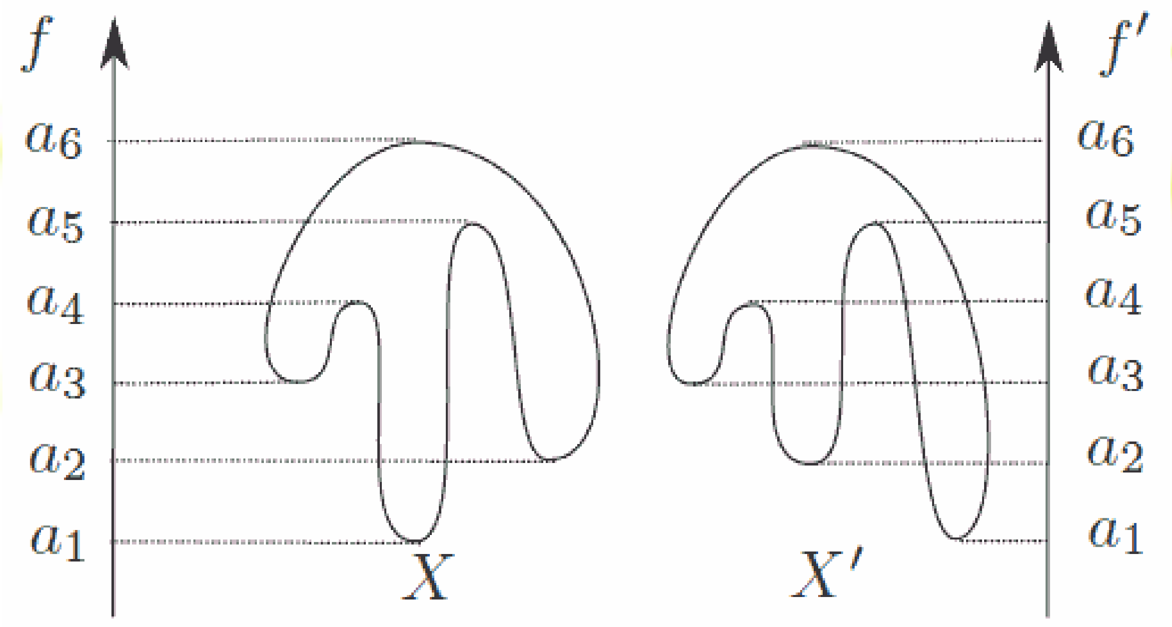




G. Carlsson, A. Collins, L. Guibas and A. Zomorodian. *Persistence barcodes for shapes*. In Proc. 2nd Sympos. Geometry Process., 2004, 127–138.

Shape: a topological problem?

A classification problem:



How much do these two curves differ w.r.t. ordinate as a filtering function?

Our proposal: the minimum cost, in terms of *f* and *f'*, of transforming one in the other.

Shape: a topological problem?

Definition

The natural pseudo-distance between the size pairs $(M, \boldsymbol{\varphi})$ and $(N, \boldsymbol{\psi})$ is

$$d\Big((M,\varphi),(N,\psi)\Big) = \begin{cases} \inf_{h \in H(M,N)} \max_{P \in M} \|\varphi(P) - \psi(h(P))\|_\infty, \\ +\infty \qquad \text{if } H(M,N) = \emptyset, \end{cases}$$

$H(M, N)$ being the set of all the homeomorphisms between M and N.

P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bull. of the Belgian Math. Soc. - Simon Stevin, 6 (1999), 455-464.

Shape: a topological problem?

But the natural pseudodistance is difficult (or even impossible) to compute.

Therefore we need a computable lower bound for it.

Luckily, we have it: a distance between *Persistent Betti Number* functions (*PBN*'s) of the size pairs.

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PBN's and PD's

For each $i \in \mathbb{Z}$, the i-th *Persistent Betti Number (PBN) function* of $(X, \vec{f})$ is $\rho_{(X,\vec{f},i)} : \Delta^+ \to \mathbb{N}$ defined as
$\rho_{(X,\vec{f},i)}(\vec{u}, \vec{v}) = \dim(Imf_{\vec{u}}^{\vec{v}}),\ \vec{u} \prec \vec{v}$ with

$$f_{\vec{u}}^{\vec{v}} : H_i(X\langle \vec{f} \preceq \vec{u} \rangle) \to H_i(X\langle \vec{f} \preceq \vec{v} \rangle),$$

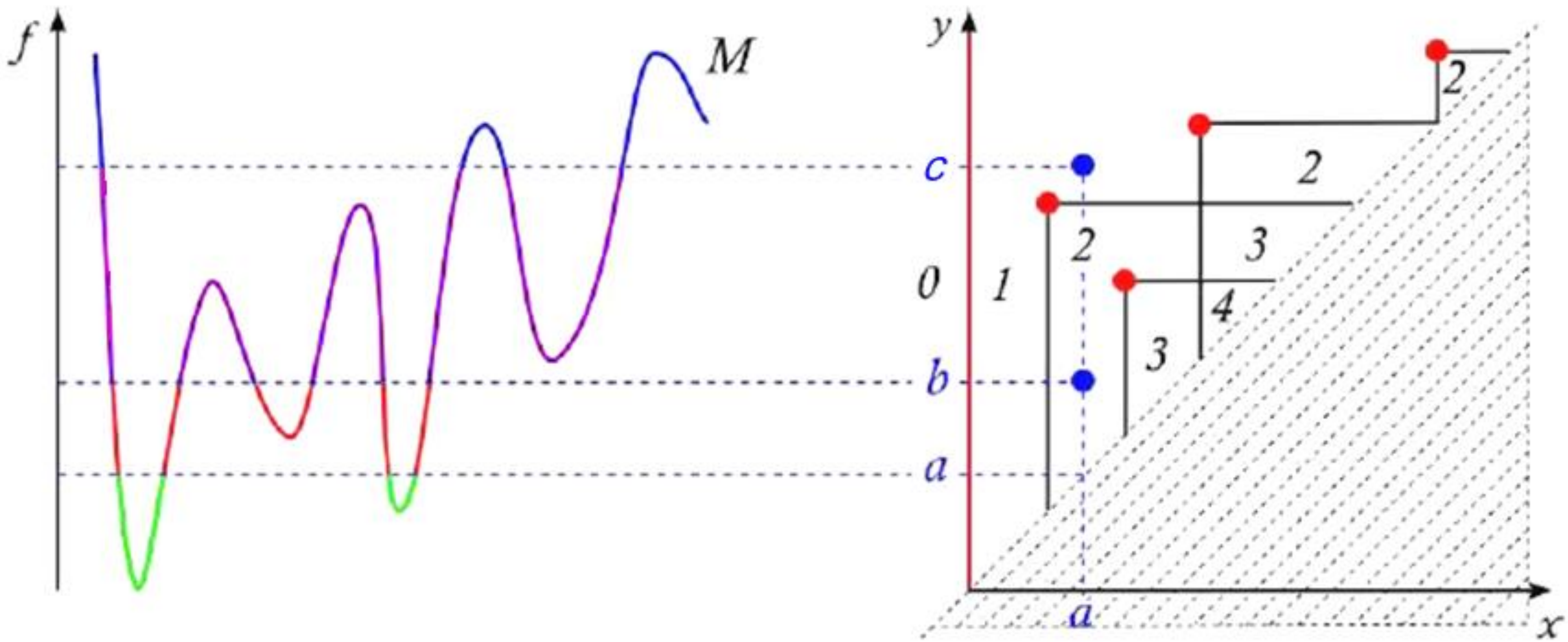
where $f_{\vec{u}}^{\vec{v}}$ is the homomorphism induced by the inclusion map of lower level sets $X\langle \vec{f} \preceq \vec{u} \rangle \subseteq X\langle \vec{f} \preceq \vec{v} \rangle$

H. Edelsbrunner, D. Letscher and A. Zomorodian, *Topological Persistence and Simplification*, Proc. 41st Ann. IEEE Sympos. Found Comput. Sci. (2000), 454–463.

G. Carlsson and A. Zomorodian, *The Theory of Multidimensional Persistence*, Symposium on Computational Geometry, June 6–8, 2007, Gyeongiu, South Korea (2007) 184–193.

PBN's and PD's

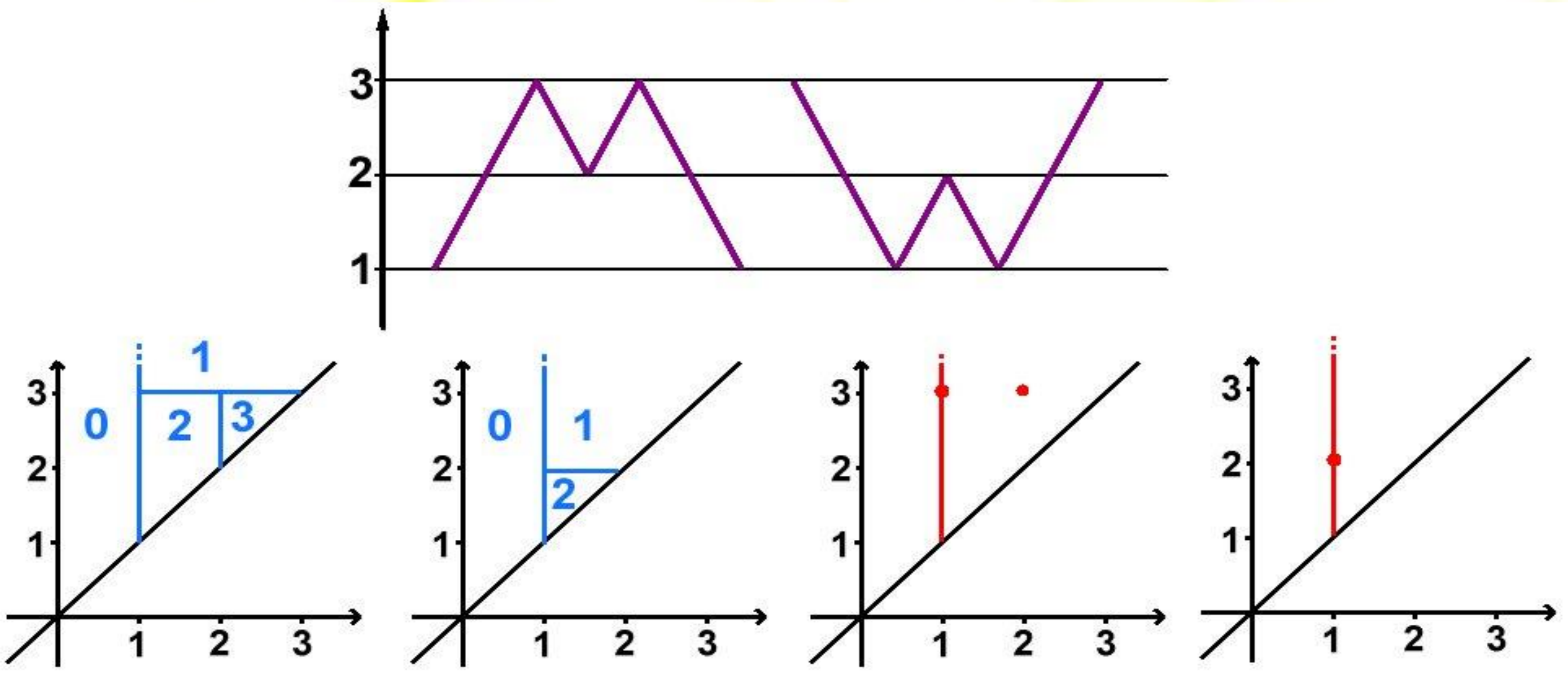
An easy example of 0-degree Persistent Betti Number function (also called *size function*) with $f : M \to \mathbb{R}$:



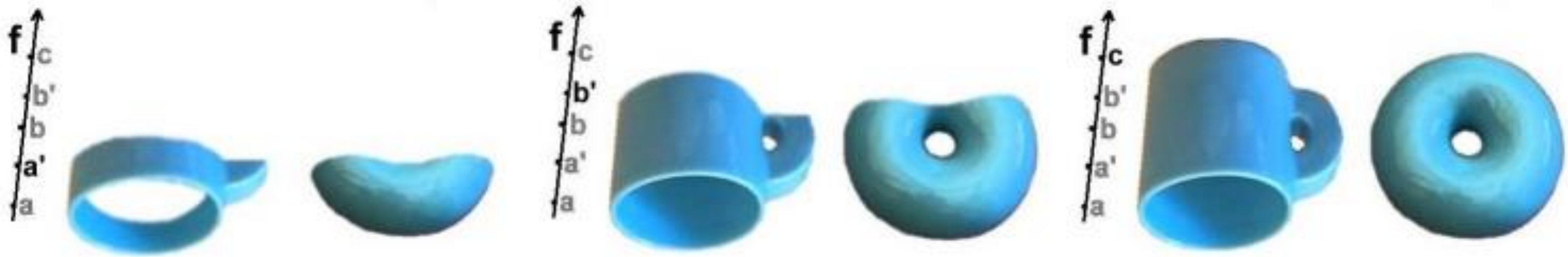
P. Frosini *Measuring shapes by size functions*, Proc. of SPIE, Intelligent Robots and Computer Vision X: Algorithms and Techniques, Boston, MA 1607 (1991).

PBN's and PD's

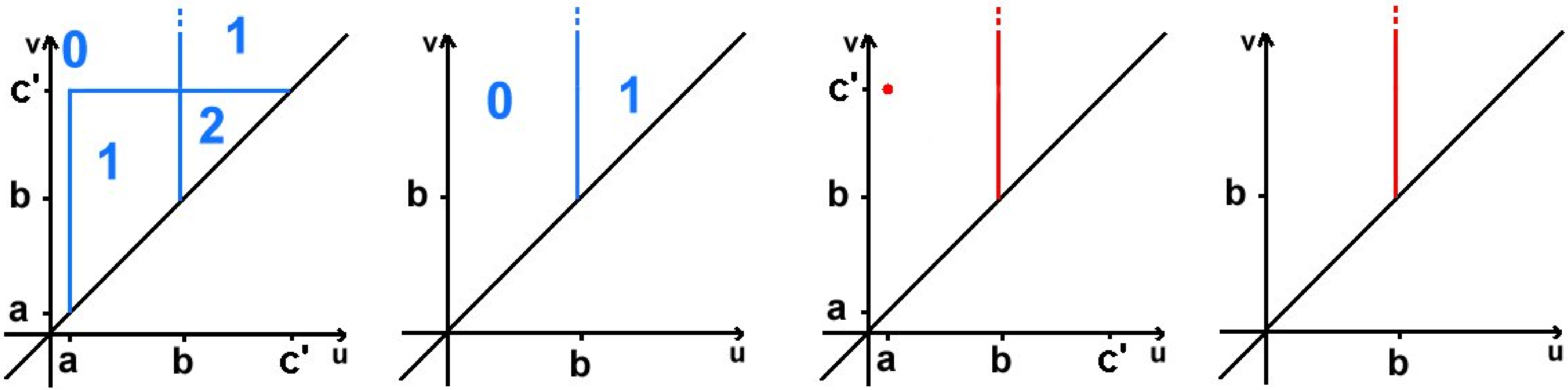
All information carried by a PBN function can be condensed in its *cornerpoints* and *cornerlines* (or *cornerpoints at infinity*). They built what is called a *Persistence Diagram (PD).*



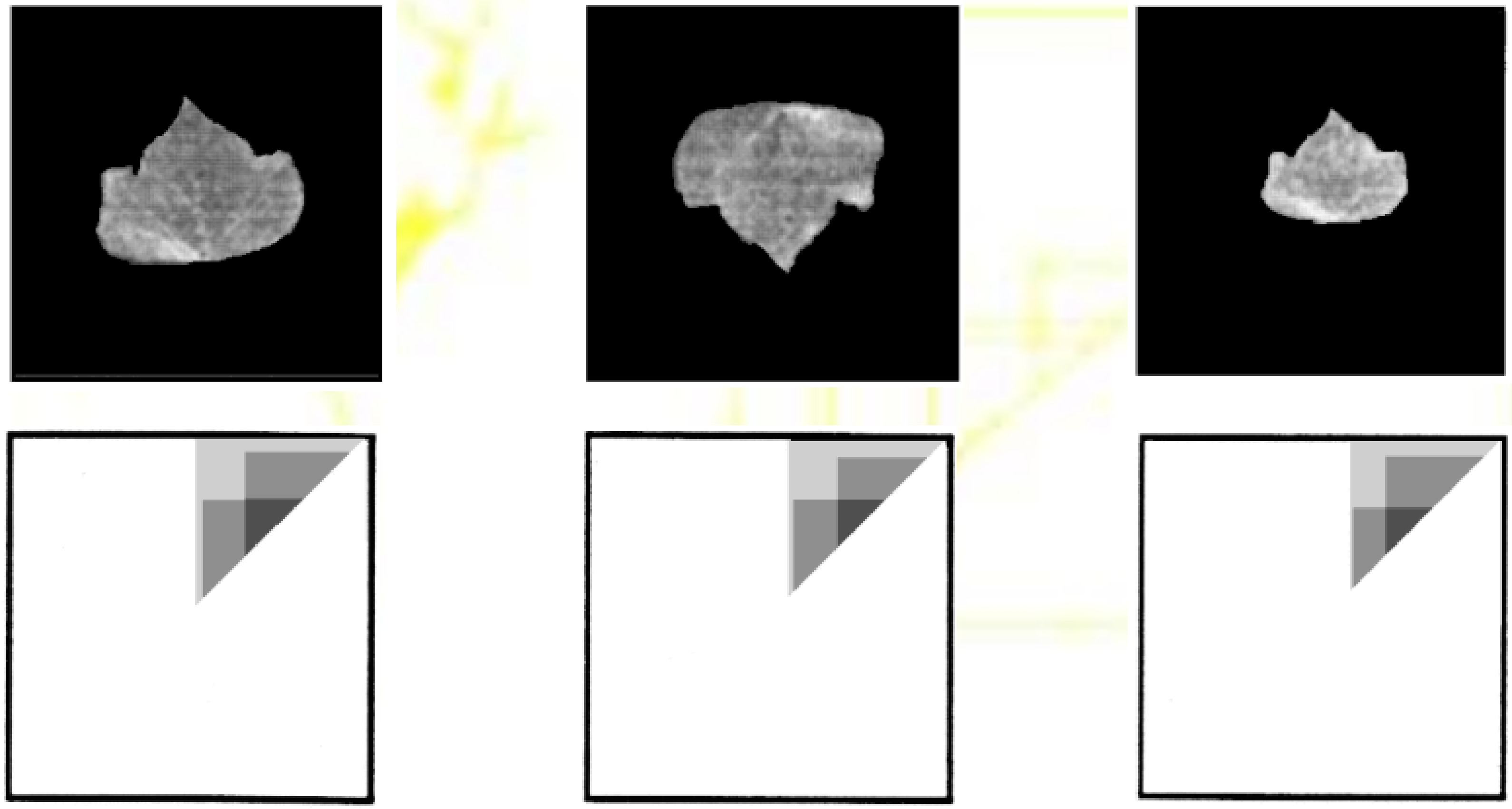
PBN's and PD's



The PBN functions of degree 1 of mug and doughnut and the corresponding Persistence Diagrams.

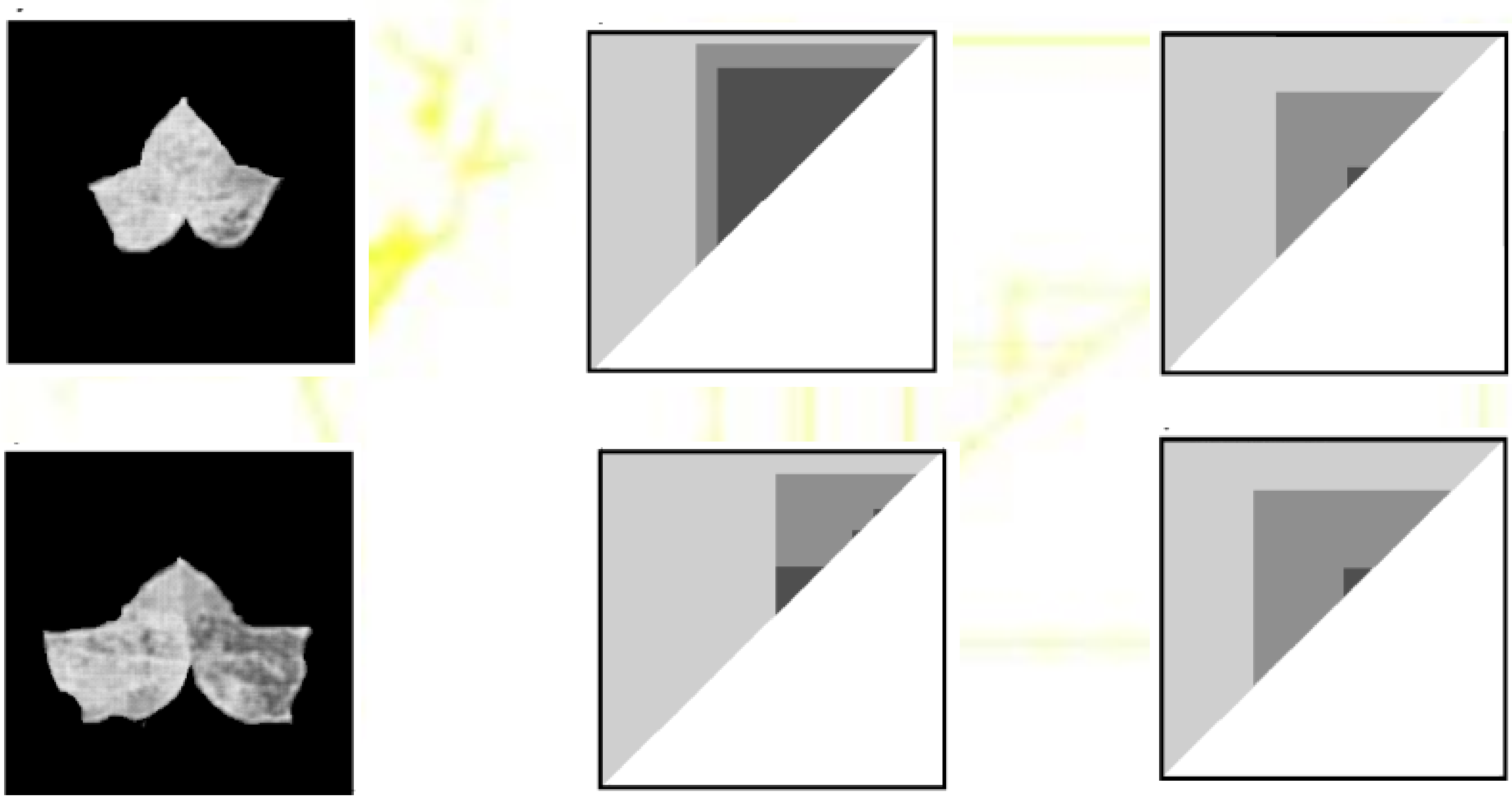


PBN's and PD's



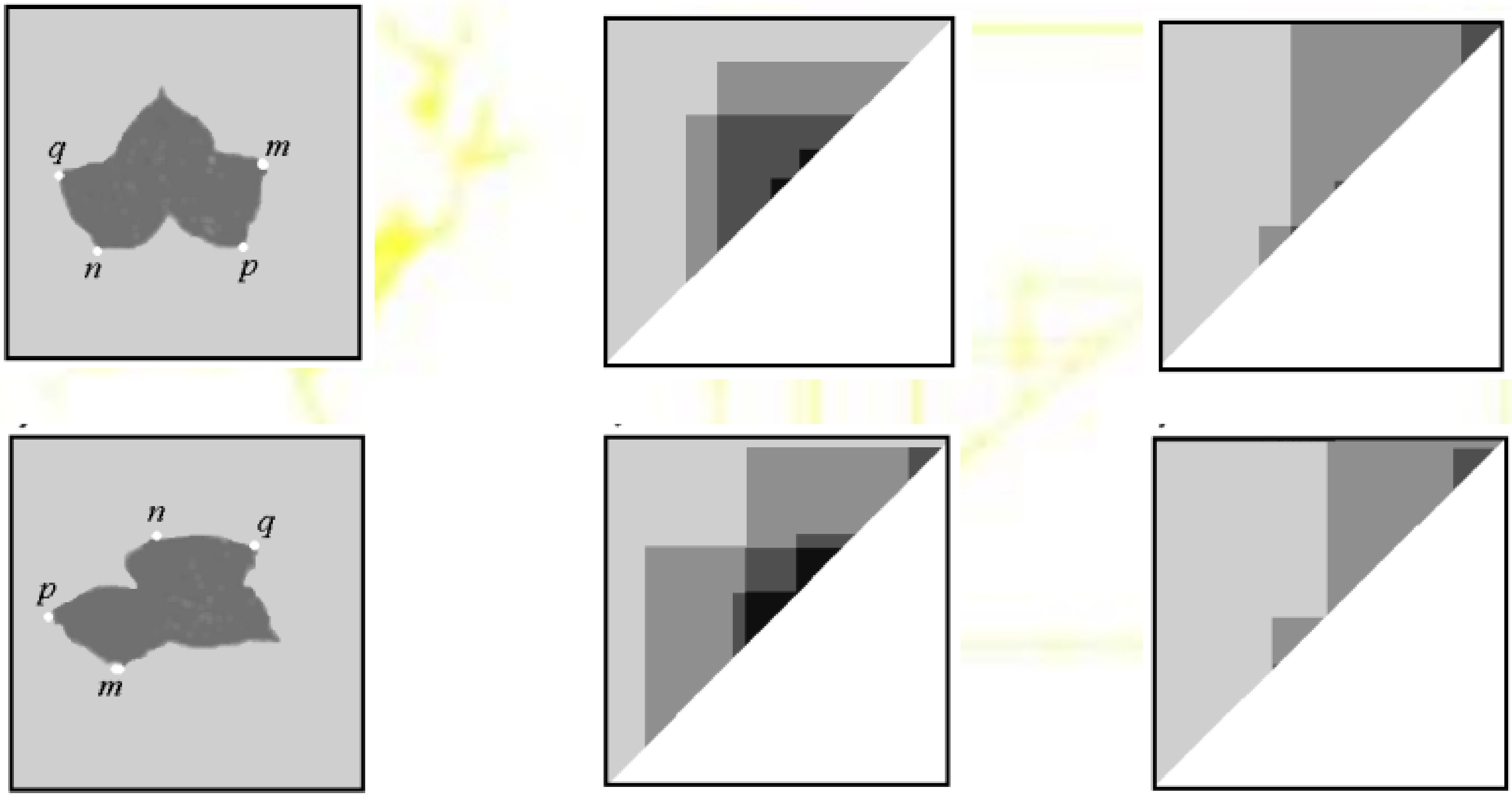
similitudes

PBN’s and PD’s



affinities

PBN's and PD's



homographies

PBN's and PD's

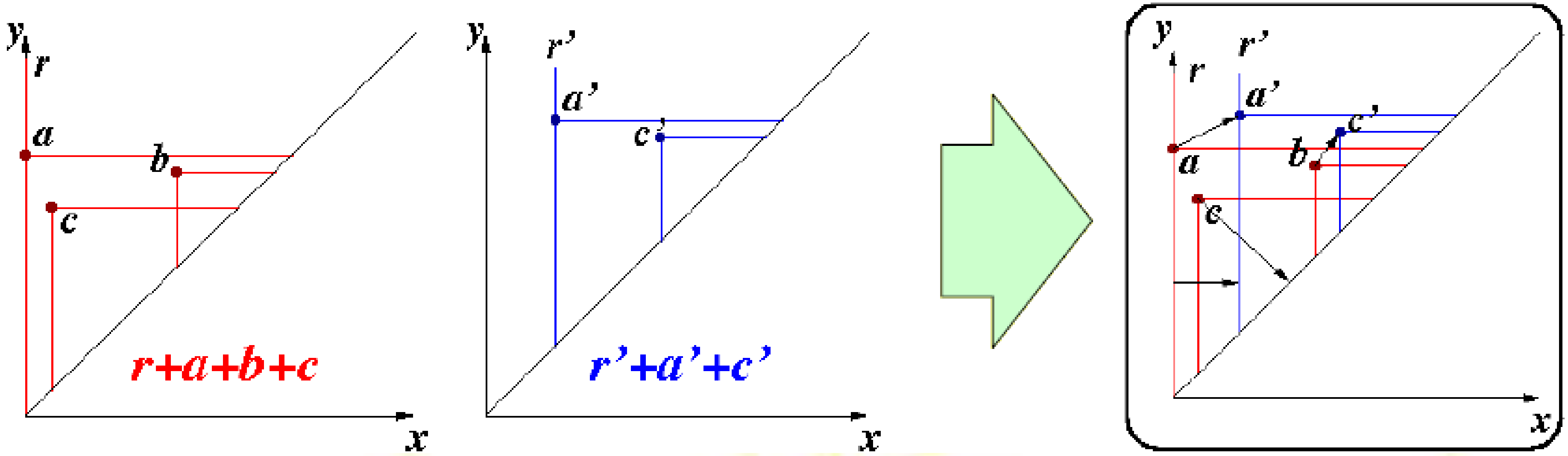
Given PD's *D* and *D'*, add to *D* the projections of the cornerpoints of *D'* on the diagonal and conversely.

Let σ be a bijection between the so enriched PD's *D* and *D'*. Let δ be the 1-norm distance between corresponding points. The matching (or bottleneck) distance between *D* and *D'* is then defined as:

$$d_{match}(D, D') = \inf_{\sigma} \sup_{P \in D} \delta(P, \sigma(P))$$

This distance is stable w.r.t. perturbations of the filtering functions.

PBN's and PD's



The matching distance

P. Frosini, C. Landi, *Size functions and formal series*, Appl. Algebra Engrg. Comm. Comput. 12 (2001), 327-349.

PBN's and PD's

It turns out that:

$$d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}) \leq d((\mathcal{M},\varphi),(\mathcal{N},\psi))$$

i.e. the matching distance between size functions yields a lower bound to the natural pseudodistance.
It is actually the best possible one which we can get from PD's.

S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, C. Landi *Multidimensional size functions for shape comparison* Journal of Mathematical Imaging and Vision 32 (2008), 161–179.

A. Cerri, B. Di Fabio, M. Ferri, P. Frosini, C. Landi *Betti numbers in multidimensional persistent homology are stable functions* Math. Meth. Appl. Sci. DOI: 10.1002/mma.2704 (2012).

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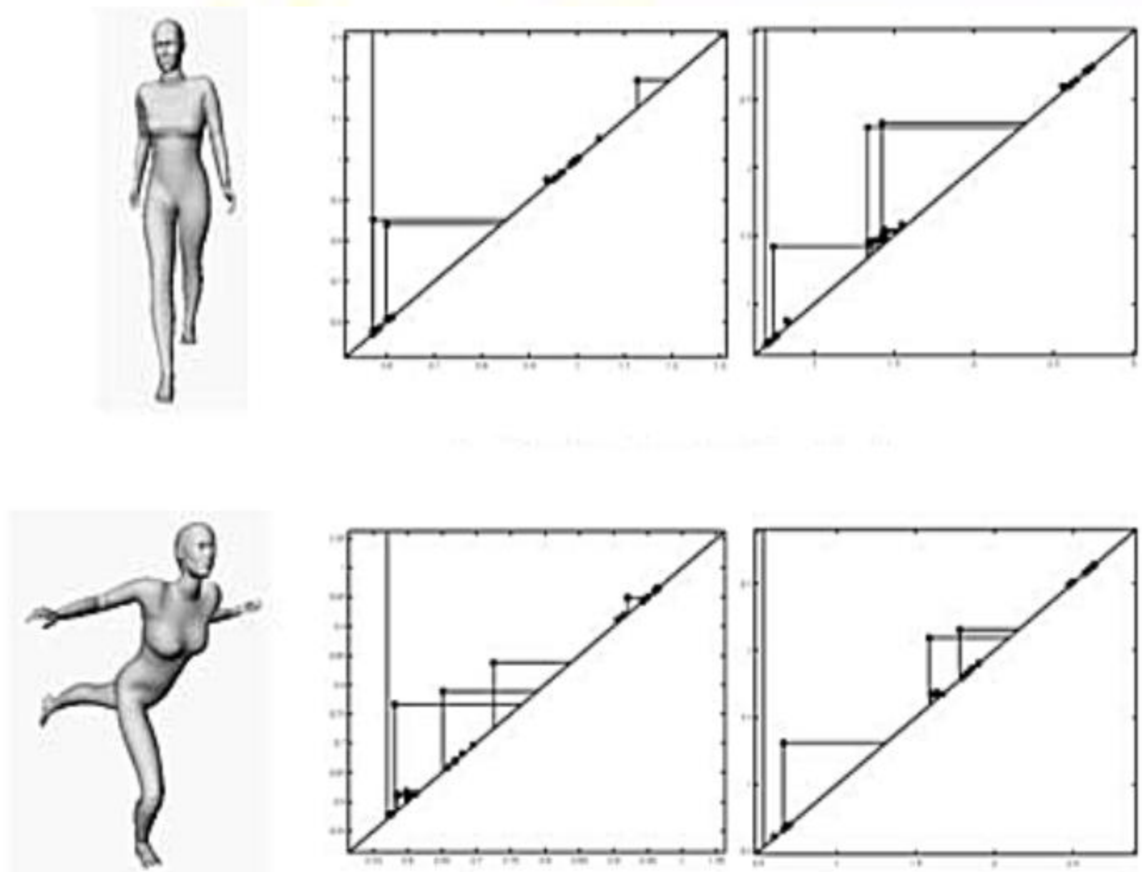
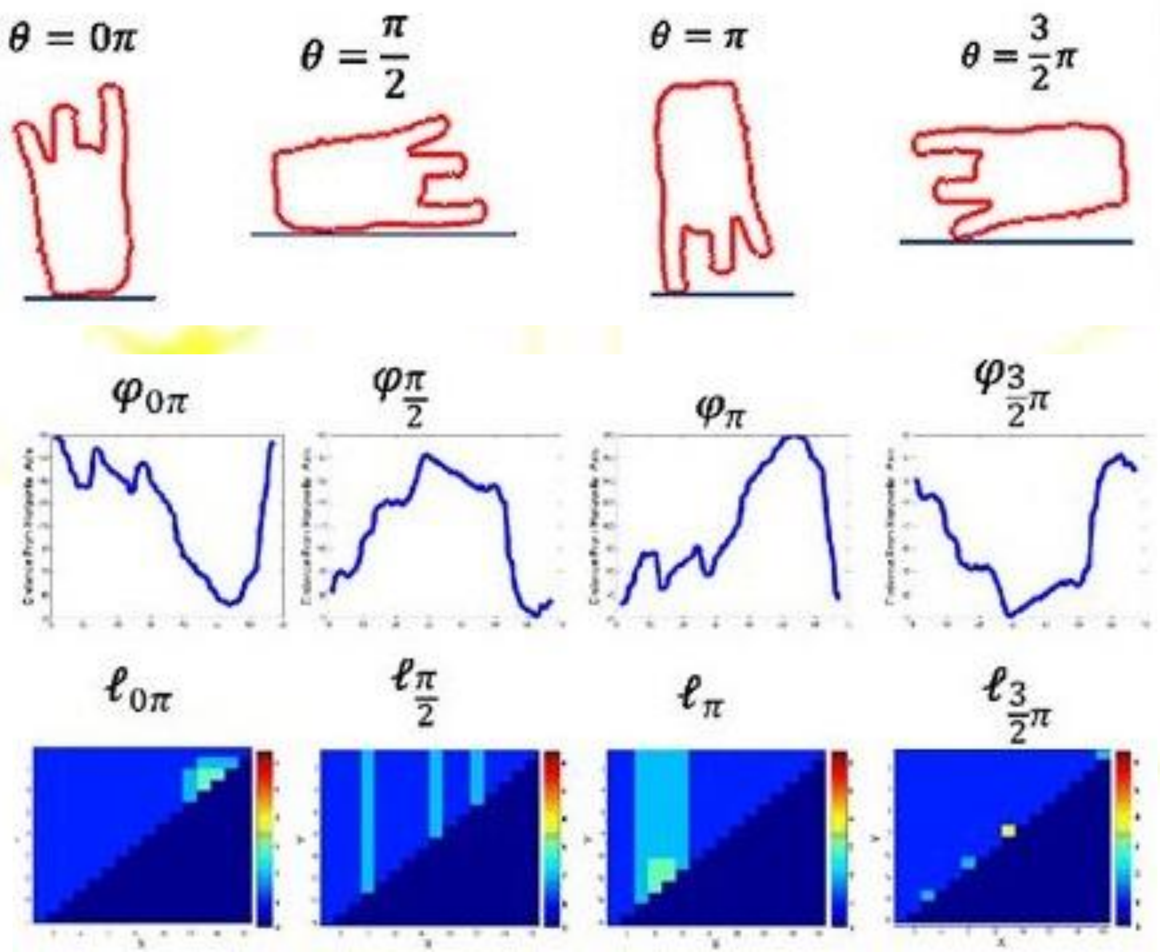
Shape analysis

Different filtering functions for different tasks and viewpoints



S. Biasotti, D. Giorgi, M. Spagnuolo and B. Falcidieno. *Size functions for comparing 3D models*, Pattern Recognition 41 (2008), 2855-2873.

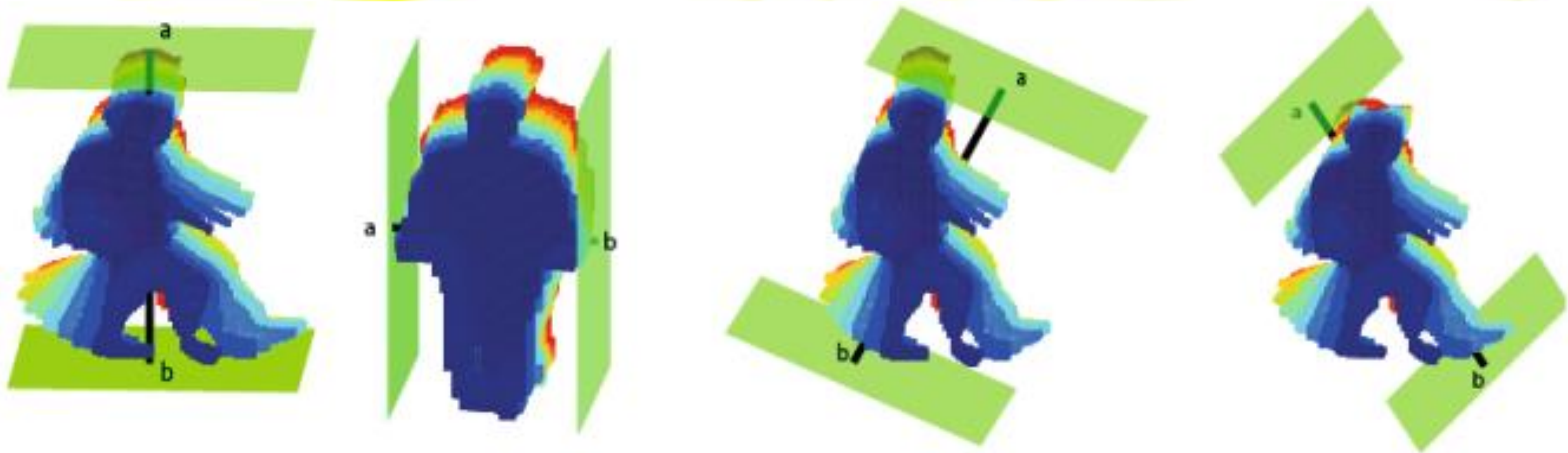
Shape analysis



Kelly, D., McDonald, J., Lysaght, T., Markham, C.: Analysis of sign language gestures using size functions and principal component analysis. In: Machine Vision and Image Processing Conference, IMVIP 2008. International, pp. 31–36. IEEE (2008)

Shape analysis

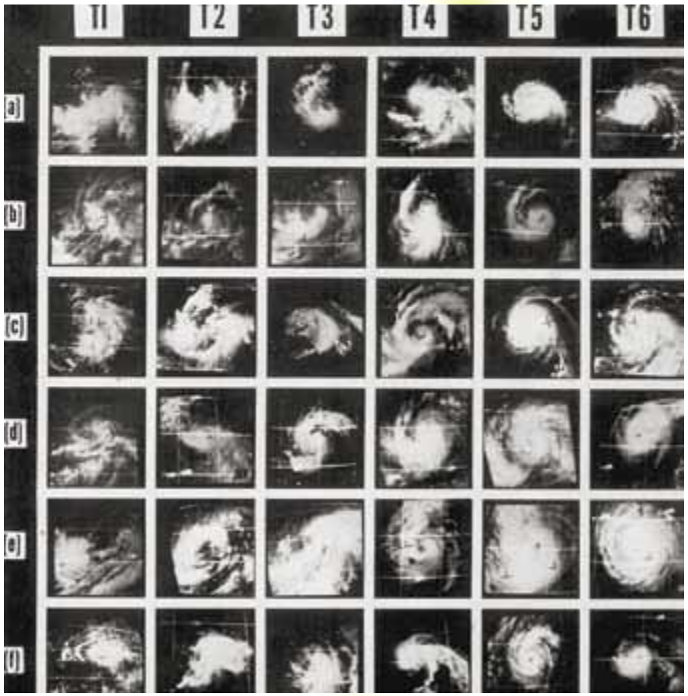
Considering a stack of silhouettes as a 3D object, and using four different filtering functions, makes 0- and 1-degree persistent homology a tool for identifying people through their gait.

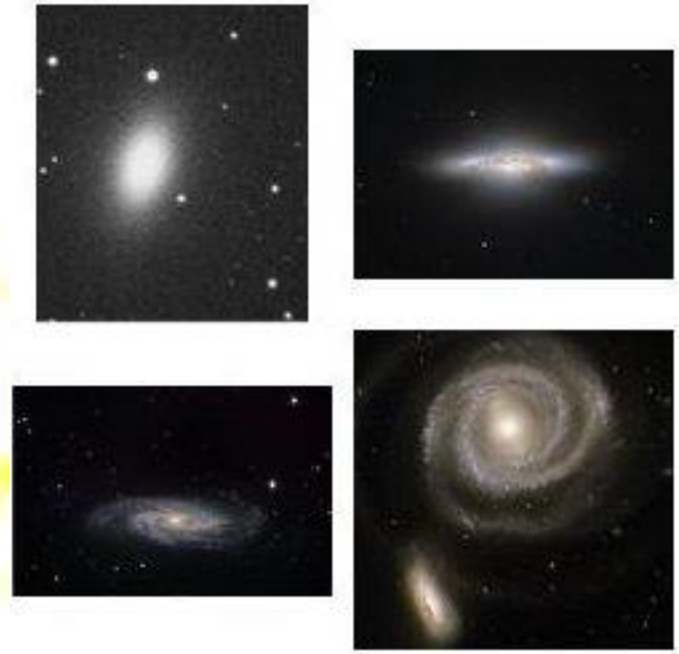


Lamar-León, J., García-Reyes, E.B., and Gonzalez-Diaz, R. *Human gait identification using persistent homology*. Progress in Pattern Recognition, Image Analysis, Computer Vision, and Applications. Springer Berlin Heidelberg (2012), 244-251.

Shape analysis

Two different types of spirals: hurricanes and galaxies; both are analyzed through persistence.



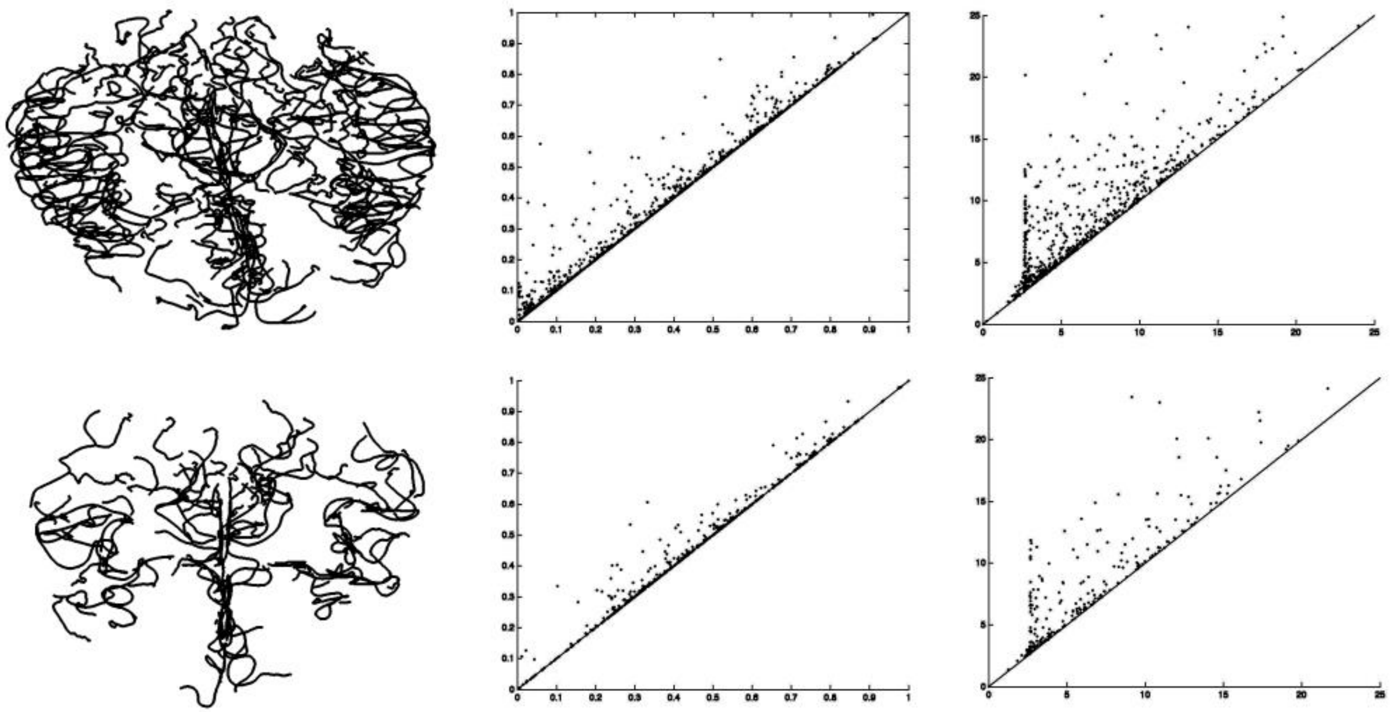


Banerjee, S. *Size Functions in the Study of the Evolution of Cyclones*. International Journal of Meteorology 36.358 (2011), 39-46.

Banerjee, S. *Size Functions In Galaxy Morphology Classification*. International Journal of Computer Applications 100.3 (2014), 1-4.

Shape analysis

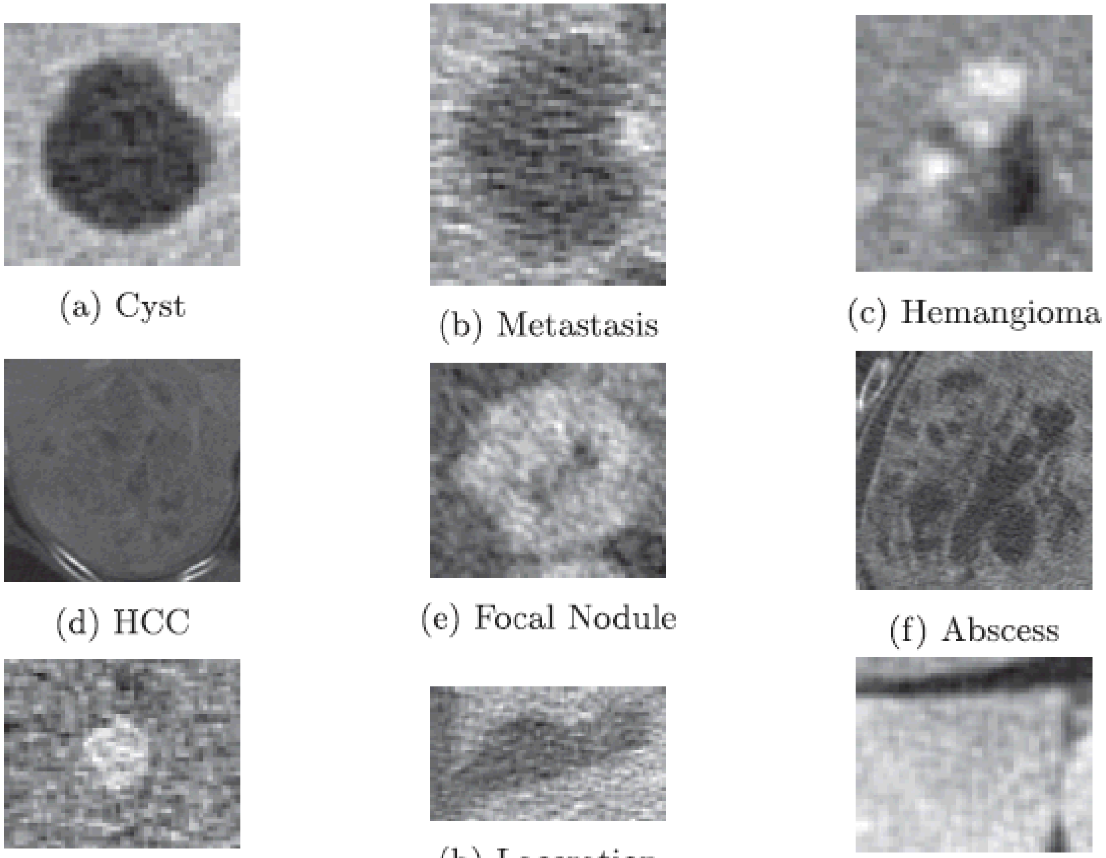
Besides density and thickness, bending of brain arteries is a sign of age. Its assessment is best done through 0- and 1-degree persistent homology.



Bendich, P., Marron, J. S., Miller, E., Pieloch, A. and Skwerer, S. *Persistent homology analysis of brain artery trees*. The Annals of Applied Statistics 10 (2014), 198-218.

Shape analysis

Experimental evidence (on hepatic lesions) confirms that filtering functions with an nD range carry more information than their single components.

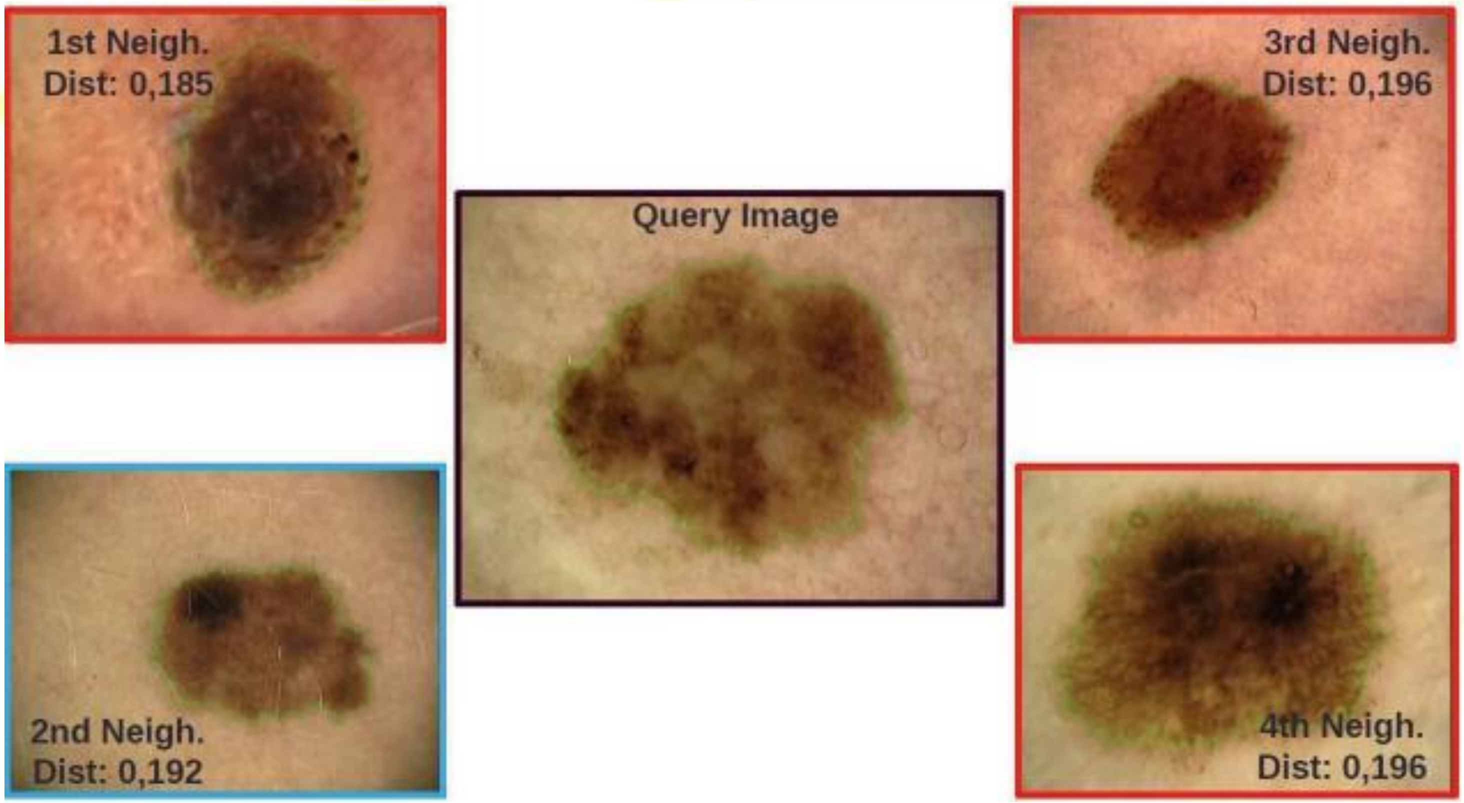


(a) Cyst (b) Metastasis (c) Hemangioma
(d) HCC (e) Focal Nodule (f) Abscess
(g) NeN (h) Laceration (i) Fat Deposit

Adcock, A., Rubin, D. and Carlsson, G. *Classification of hepatic lesions using the matching metric*. Computer vision and image understanding 121 (2014), 36-42.

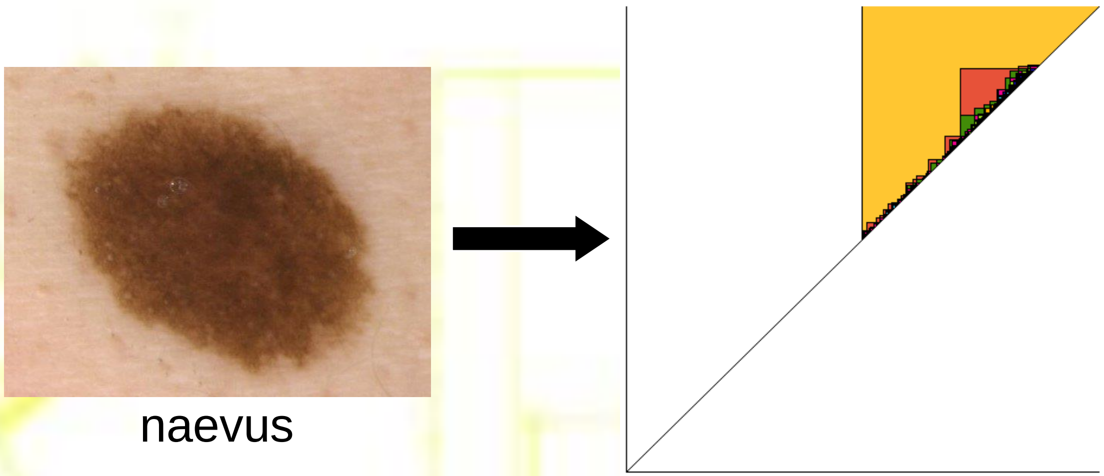
Shape analysis

Several colour-based filtering functions are the key to image retrieval for dermatologists.

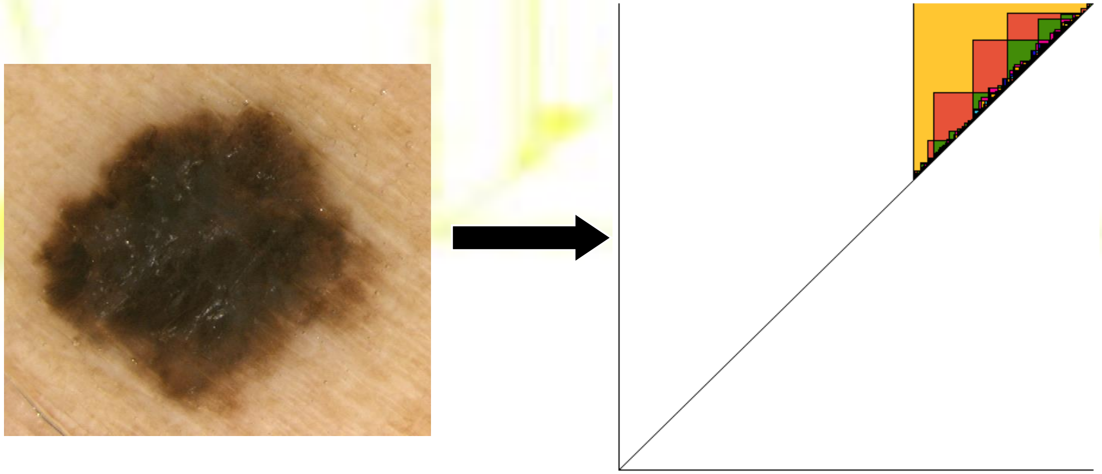


Ferri, M., Tomba, I., Visotti, A. and Stanganelli, I. *A feasibility study for a persistent homology based k-Nearest Neighbor search algorithm in melanoma detection*. J. Math. Imaging Vis. 57 (2017), 324-339.

Shape analysis



naevus



melanoma

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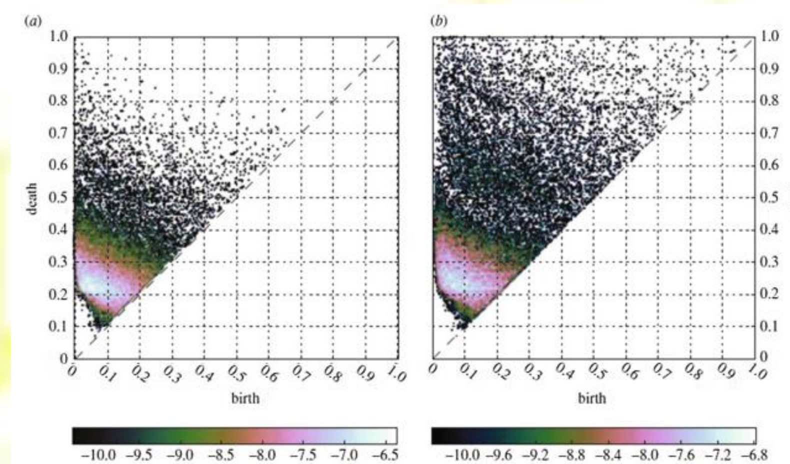
Not only images

Cycle persistence is at the base of two applications in the biological domain:

- Through Vietoris-Rips complexes for isolating genetic pathways to Coronary Artery Disease
- Through clique complexes for analyzing the effect of psilocybin on brain networks.



Platt, D. E., Basu, S., Zalloua, P. A. and Parida, L. *Characterizing redescriptions using persistent homology to isolate genetic pathways contributing to pathogenesis*. BMC Systems Biology 10 (2016), 107-119.

Petri, G., Expert, P., Turkheimer, F., Carhart-Harris, R., Nutt, D., Hellyer, P. J. and Vaccarino, F. *Homological scaffolds of brain functional networks*. Journal of The Royal Society Interface 11.101 (2014): 20140873.

Not only images

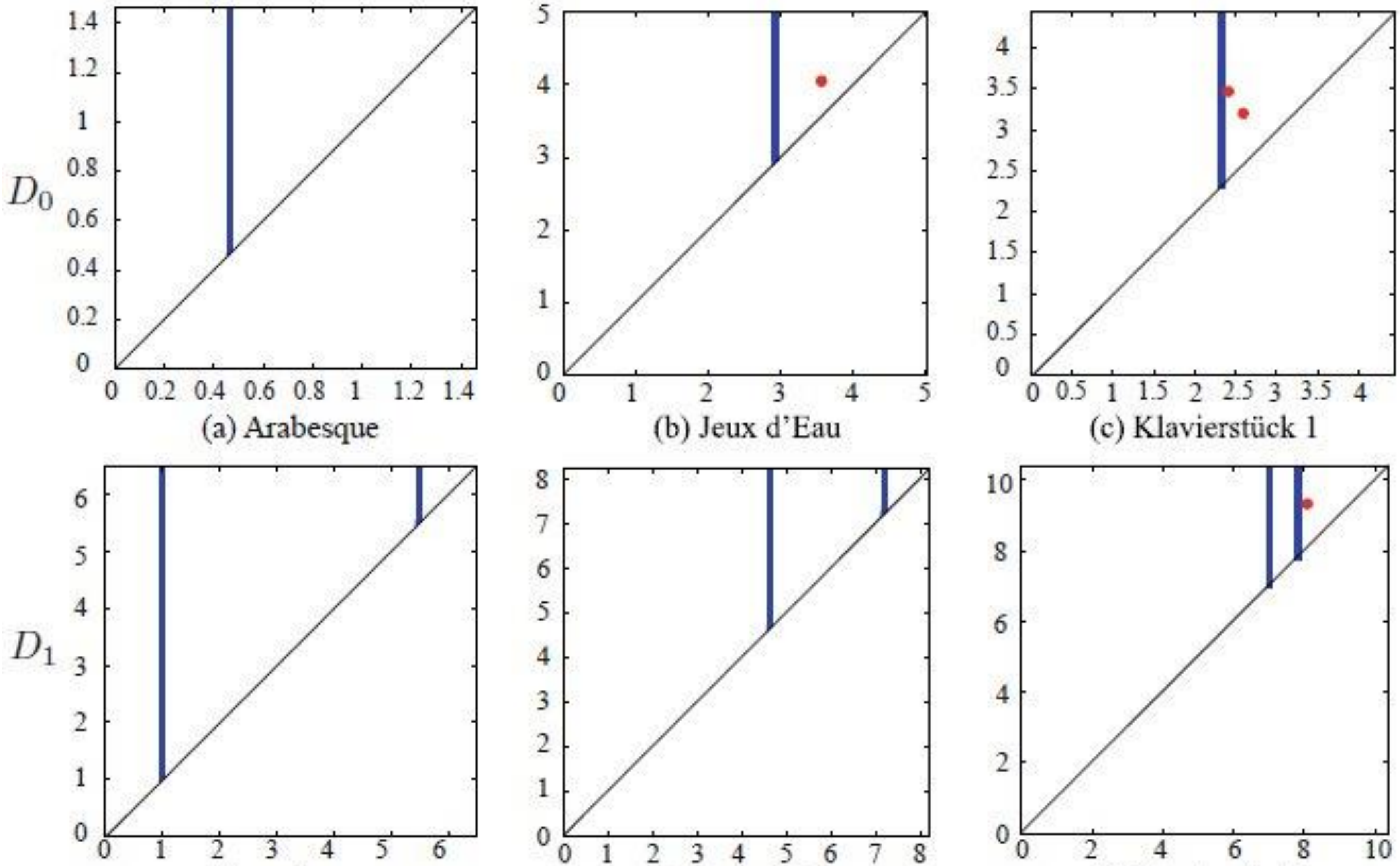
More applications:

- Detection of linguistic subfamilies (via syntactic proximity and Vietoris-Rips complexes) through 0-degree persistence; speculation on the meaning of 1-degree persistence
- Recognition of tonal, modal and atonal music is performed through alignment of time series and time evolution of persistence diagrams (work in progress).

Port, A., Gheorghita, I., Guth, D., Clark, J. M., Liang, C., Dasu, S. and Marcolli, M. *Persistent topology of syntax*. arXiv preprint arXiv:1507.05134 (2015).

Bergomi, M.G. *Dynamical and topological tools for (modern) music analysis*. Theses, Université Pierre et Marie Curie - Paris VI, December 2015. https://tel.archives-ouvertes.fr/tel-01293602

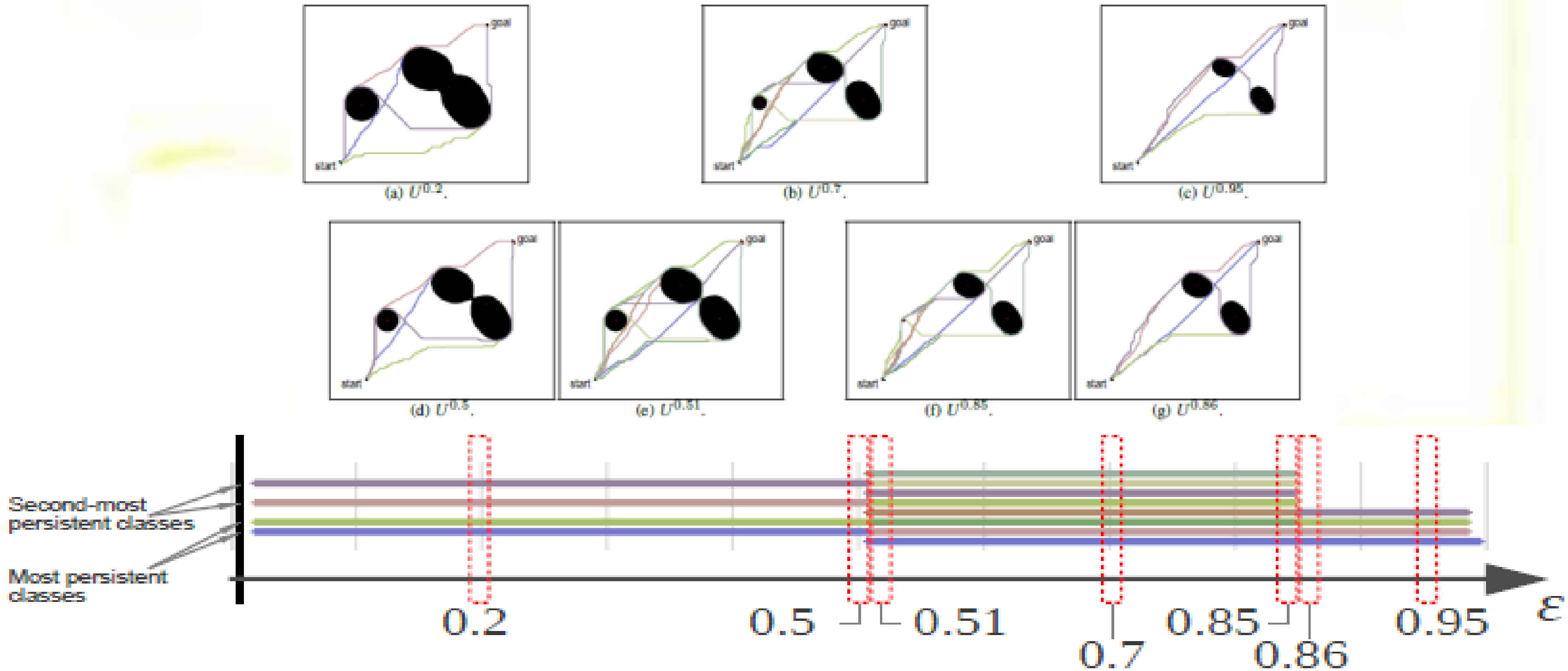
Not only images



(a) Arabesque
(b) Jeux d'Eau
(c) Klavierstück 1
(d) Arabesque
(e) Jeux d'Eau
(f) Klavierstück 1

Not only images

Robot navigation needs a map of obstacles; different thresholds of occupancy probability yield different maps. Preferred routes are the ones homologically persistent when varying the threshold.



Bhattacharya, S., Ghrist, R., & Kumar, V. *Persistent homology for path planning in uncertain environments*. IEEE Transactions on Robotics 31.3 (2015), 578-590.

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But...

Persistent Homology proves to be an extremely powerful tool for analysis and classification in a great variety of contexts.

BUT...

Do we really need topological spaces (or simplicial complexes), continuous functions, homology?

But...

What we actually need of all the Persistence machinery is:

- filtering
- a natural pseudodistance
- a function with a behaviour similar to that of PBN
- in particular a function determined by "cornerpoints"
- i.e. a representation by a persistence diagram.

This can be done in a much wider context!

Persistence for shape analysis

THANKS FOR YOUR ATTENTION!

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