SOME ASPECTS OF HARDY TYPE INEQUALITIES

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ABSTRACT. We give a general answer to the following fundamental problem posed by Shmuel Agmon 30 years ago:

> Given a (symmetric) linear elliptic operator P of second-order in $\mathbb{R}^n$, find a continuous, nonnegative weight function W which is "as large as possible" such that for some neighborhood of infinity Ω_R the following inequality holds

$$(P\phi, \phi) \geq \int_{\Omega_R} W(x)|\phi|^2\, dx \qquad \forall \phi \in C_0^\infty(\Omega_R).$$

We construct, for a *general* subcritical second-order elliptic operator P in a domain $\Omega \subset \mathbb{R}^n$ (or a noncompact manifold), a Hardy-weight W which is *optimal* in the following natural sense sense. The operator $P - \lambda W$ is subcritical in Ω for all $\lambda < 1$, null-critical in Ω for $\lambda = 1$, and supercritical near any neighborhood of infinity in Ω for any $\lambda > 1$. Moreover, in the symmetric case, if $W > 0$, then the spectrum and the essential spectrum of $W^{-1}P$ are equal to $[1, \infty)$.

Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators on a general domain Ω or on a noncompact manifold. Moreover, the results can be generalized to certain p-Laplacian type operators. The constructed weight W is given by an explicit simple formula involving two positive solutions of the equation $Pu = 0$.

References

[1] B. Devyver, M. Fraas, Y. Pinchover, *Optimal Hardy Weight for Second-Order Elliptic Operator: an answer to a problem of Agmon*, 61 pp., to appear in J. Functional Anal., arXiv: 1208.2342v2.

[2] B. Devyver, Y. Pinchover, *Optimal L^p Hardy-type inequalities*, 32 pp., arXiv: 1312.6235.