An Inverse Problem in Potential Theory for Picone's Elliptic-Parabolic PDEs

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The ancestor of our problem

- [Gauss-Köebe]

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be open.

Then

$$u \in \mathcal{H}(\Omega) := \{\Delta u = 0\} \quad \Leftrightarrow \quad u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\, dx \quad \forall \overline{B_r(x_0)} \subseteq \Omega$$

(IP) [Harmonic characterization of the Euclidean balls]

Let $D \subseteq \mathbb{R}^N$ $(N \geq 3)$ be open and bounded.

Suppose, for a suitable $x_0 \in D$,

$$u(x_0) = \frac{1}{|D|} \int_D u(y)\, dy \qquad \forall u \in \mathcal{H}(D), u \geq 0.$$

Then: $D = B_r(x_0)$, $r = \left(\frac{|D|}{\omega_N}\right)^{1/N}$.

Epstein (1961), Epstein-Schiffer (1965), Kuran (1972)

Some weighted Mean Value Theorems for harmonic functions and the related inverse problem

Let $f :]0, \infty[\to]0, \infty[$ be a continuous function, s.t.

$$F(r) := \int_0^r \rho^{N-1} f(\rho)\, d\rho < \infty \qquad \forall r > 0.$$

Let Ω be an open subset of $\mathbb{R}^N$ ($N \geq 3$).

Then

$$u \in \mathcal{H}(\Omega) \quad \Leftrightarrow \quad u(x_0) = \frac{1}{N\omega_N F(r)} \int_{B_r(x_0)} u(x) f(|x - x_0|)\, dx \quad \forall \overline{B_r(x_0)} \subseteq \Omega.$$

Note: $N\omega_N F(r) = \int_{B_r(x_0)} f(|x - x_0|)\, dx.$

(IP)

Let $D \subseteq \mathbb{R}^N$ be open and bounded.
Suppose, for a suitable $x_0 \in D$ and $c > 0$,

$$u(x_0) = \frac{1}{c} \int_D u(x) f(|x - x_0|)\, dx \qquad \forall u \in \mathcal{H}(D), u \geq 0.$$

Then: $c = \int_D f(|x - x_0|)\, dx$ and

$$D = B_r(x_0), \quad r = F^{-1}\left(\frac{c}{N\omega_N}\right)$$

Densities with the mean value property for harmonic functions

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, be an open set and let

$$w : \Omega \to [0, \infty[$$

be a l.s.c. function s.t.

$$\text{int}\{w = 0\} = \emptyset.$$

We say that

w is a density with the mean value property for harmonic functions in Ω w.r.t. $x_0 \in \Omega$

if

(i) $w(\Omega) := \int_\Omega w(x)\, dx < \infty$,

(ii) $u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(x) w(x)\, dx \quad \forall u \in \mathcal{H}(\Omega),\ u \geq 0.$

Remark:

Every positive continuous radial function

$$x \mapsto f(|x - x_0|)$$

is a density with the mean value property for harmonic functions in $B_r(x_0)$, $r > 0$, w.r.t. x_0.

General examples

- Hansen-Netuka (1995); Aikawa (1997)

Every bounded sufficiently regular open set $\Omega \subseteq \mathbb{R}^N$ supports densities with the mean value property for harmonic functions with respect to any fixed $x_0 \in \Omega$.

Precisely:

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, be bounded with $\partial\Omega \in \text{Lip}$.

Then, for every $x_0 \in \Omega$,

$$w(x) := \varphi(G(x_0, x))|\nabla G(x_0, x)|^2,$$

where G is the Green function of Ω and φ is a positive measurable function s.t. $\displaystyle\int_0^\infty \varphi = 1$,

is a density with the mean value property for the harmonic functions in Ω, w.r.t. x_0.

Δ-triples and related inverse problem

Notation:

(Ω, x_0, w) is a Δ-triple if

- $\Omega \subseteq \mathbb{R}^N$ is open
- $x_0 \in \Omega$
- w is a density with the mean value property for the harmonic functions in Ω, w.r.t. x_0; i.e.

$$u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(x) w(x)\, dx \qquad \forall u \in \mathcal{H}(\Omega),\ u \geq 0.$$

A natural extension of the classical harmonic characterization of the Euclidean balls (Epstein, Schiffer, Kuran) is the following inverse problem:

(IP) If (Ω, x_0, w) and (D, x_0, w') are Δ-triples, such that

$$\frac{w}{w(\Omega)} = \frac{w'}{w'(D)} \qquad \text{in } \Omega \cap D,$$

is it true that $\Omega = D$?

Positive answer to (IP)

With G. Cupini (very recently: 2016) we proved that basically

(IP) has a **positive answer** if (Ω, x_0, w) is a strong Δ-triple.

DEFINITION:

(Ω, x_0, w) is a strong Δ-triple if

(i) (Ω, x_0, w) is a Δ-triple; i.e.

$$u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(x)w(x)\,dx \qquad \forall u \in \mathcal{H}(\Omega),\ u \geq 0$$

(ii) $u(x_0) > \dfrac{1}{w(\Omega)} \displaystyle\int_\Omega u(x)w(x)\,dx \quad \forall u \in \overline{\mathcal{H}}(\Omega),\ \Delta u \not\equiv 0 \text{ in } \Omega,\ u \geq 0.$

Remark:

- $(B_r(x_0), x_0, w)$ is a strong Δ-triple for **every radial** w; i.e. $w(x) := f(|x - x_0|)$,
- (Ω, x_0, w) is a strong Δ-triple if

$$w(x) := \varphi(G(x_0, x))|\nabla G(x_0, x)|^2.$$

Our answer to (IP) for Δ

THEOREM [Cupini-Lanc. 2016]

Let (Ω, x_0, w) and (D, x_0, w') be Δ-triples such that

(i) (Ω, x_0, w) is strong

(ii) $\dfrac{w}{w(\Omega)} = \dfrac{w'}{w'(D)}$ in $\Omega \cap D$,

(iii) Ω is a solid set, i.e., $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\Omega = \operatorname{int}\overline{\Omega}$.

Then $D = \Omega$.

Remark:

This theorem contains Epstein-Schiffer-Kuran Theorem.
It corresponds to the case

$$\Omega = B_r(x_0) \qquad \text{and} \qquad w = w' = 1.$$

Caloric Mean Value Theorem

Notation:

$$z = (x,t) \in \mathbb{R}^{N+1}, \; x \in \mathbb{R}^N, \; t \in \mathbb{R}, \; \Delta = \Delta_x.$$

- Heat operator in $\mathbb{R}^{N+1}$

$$H := \Delta - \partial_t$$

- Gauss-Weierstrass kernel:

$$\Gamma(z) := \Gamma(x,t) = \begin{cases} 0 & \text{if } t \le 0, \\ \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t > 0. \end{cases}$$

Γ is the **fundamental solution** of H.

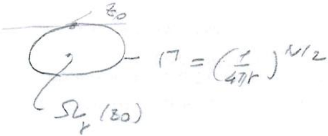
- Heat balls:

$$\Omega_r(z_0) := \left\{\zeta \in \mathbb{R}^{N+1} \; : \; \Gamma(z_0 - \zeta) > \left(\frac{1}{4\pi r}\right)^{N/2}\right\}, \quad z_0 \in \mathbb{R}^{N+1}, \; r > 0.$$

- Watson kernel:

$$W(z) := W(x,t) = \frac{1}{4}\frac{|x|^2}{t^2}.$$

Caloric Mean Value Theorem - II



Note: The *center* of $\Omega_r(z_0)$ is on the boundary of $\Omega_r(z_0)$: $z_0 \in \partial\Omega_r(z_0)$.

Gauss-Köebe type Th. for caloric functions

THEOREM [Pini (1951), Watson (1971)]

Let $O \subseteq \mathbb{R}^{N+1}$ be open.

Let $u : O \to \mathbb{R}$ be a continuous function.

Then:

$u \in \mathcal{C}(O) := \{u \in C^\infty(O) \,:\, Hu = 0\}$ if and only if

$$u(z_0) = \left(\frac{1}{4\pi r}\right)^{N/2} \int_{\Omega_r(z_0)} u(\zeta) W(z_0 - \zeta)\, d\zeta \qquad \forall\, \overline{\Omega_r(z_0)} \subseteq O.$$

Caloric characterization of the heat balls

Epstein-Schiffer-Kuran type Th. for H

THEOREM [Suzuki-Watson (2001)]

Let $D \subseteq \mathbb{R}^{N+1}$ be open and bounded.

Let $z_0 \in \mathbb{R}^{N+1}$ be such that:

(i) $u(z_0) = \frac{1}{c} \int_D u(\zeta) W(z_0 - \zeta)\, d\zeta \qquad \forall u \in \mathcal{C}(D \cup \{z_0\}),\ u \geq 0,$

for a suitable $c > 0$,

(ii) $\exists\, p > \frac{N}{2} + 1$ such that

$$\zeta \mapsto \left(\chi_D - \chi_{\Omega_r(z_0)}\right)(\zeta) W(z_0 - \zeta) \text{ is in } L^p(\mathbb{R}^{N+1}).$$

Then $D = \Omega_r(z_0)$, with $(4\pi r)^{N/2} = c = \int_D W(z_0 - \zeta)\, d\zeta.$

Note: (ii) $\Rightarrow z_0 \in \partial D$.

Well behaved Mean Value formulas for caloric functions

- The Watson kernel is not bounded
- With N. Garofalo (in 1989) we obtained a weighted Mean Value Theorem for caloric functions with **good kernels**, regular as one wants.

Idea:

If $(\Delta_x - \partial_t)u(x,t) = 0$, then

$$(\Delta_x + \Delta_y - \partial_t)u(x,t) = 0, \quad y \in \mathbb{R}^m.$$

Writing Pini-Watson Mean Value formula for

$$(x,y,t) \mapsto v(x,y,t) := u(x,t)$$

as a solution to the heat equation in $\mathbb{R}^{N+m+1}$, one obtains:

$$u(z_0) = \int_{\Omega_r^{(m)}(z_0)} u(\zeta) W_r^{(m)}(z_0 - \zeta)\, d\zeta,$$

where

$$\Omega_r^{(m)}(z_0) := \left\{ \zeta = (\xi,\tau) \,:\, (4\pi(t_0 - \tau))^{-m/2}\,\Gamma(z_0 - \zeta) > (4\pi r)^{-(N+m)/2} \right\}$$

and $W_r^{(m)}$ is ≥ 0, bounded and C^k for every fixed k, if m is large enough.

General problem

The harmonic and the caloric (IP)'s previously showed are particular cases of an inverse problem for a wide class of

Picone's Elliptic-Parabolic PDEs

of the following kind:

$$\mathcal{L} := \operatorname{div}(A(x)D) + \langle\, b(x)\,,\, D\,\rangle, \quad x \in X$$

where

- X is an open set in $\mathbb{R}^n$
- $A(x) = (a_{ij})_{i,j=1,\cdots,n}, \quad a_{ij} \in C^\infty(X)$

  $A(x)$ symmetric and $A(x) \geq 0 \ \forall x \in X$,
- $b(x) = (b_1(x), \cdots, b_n(x)), \quad b_i \in C^\infty(X)$
- $\mathcal{L}$ hypoelliptic.

$\mathcal{L}$-fundamental solution - I

$\mathcal{L}$ is endowed with a fundamental solution

$$\Gamma : X \times X \to [0, \infty[$$

such that

- $(x, y) \mapsto \Gamma(x, y)$ is l.s.c. and smooth in $\{x \neq y\}$
- $\Gamma(\cdot, y) \in L^1_{\mathrm{loc}}(X) \ \ \forall y \in X$
- $\Gamma(x, \cdot) \in L^1_{\mathrm{loc}}(X) \ \ \forall x \in X$
- $\forall K \subset\subset X$,

$$\sup_{y \in K} \Gamma(x, y) \to 0 \quad \text{as } x \to \infty,$$

$$\sup_{x \in K} \Gamma(x, y) \to 0 \quad \text{as } y \to \infty.$$

$\mathcal{L}$-fundamental solution - II

- $\forall \varphi \in C_0^\infty(X)$ the functions

$$x \mapsto \int_X \Gamma(x,y)\varphi(y)\,dy, \quad y \mapsto \int_X \Gamma(x,y)\varphi(x)\,dx,$$

are smooth and satisfy:

1) $\mathcal{L} \int_X \Gamma(x,y)\varphi(y)\,dy = -\varphi(x),$

2) $\int_X \Gamma(x,y)\mathcal{L}\varphi(y)\,dy = -\varphi(x),$

3) $\mathcal{L}^* \int_X \Gamma(x,y)\varphi(x)\,dx = -\varphi(y),$

4) $\int_X \Gamma(x,y)\mathcal{L}^*\varphi(x)\,dx = -\varphi(y).$

Densities with the mean value property for $\mathcal{L}$-harmonic functions

DEFINITION:

Let $\Omega \subseteq \mathbb{R}^n$ be open and let

$$w : \Omega \to [0, \infty[$$

be a l.s.c. function s.t.

$$\mathrm{int}\{w = 0\} = \emptyset.$$

We say that

w is a density with the mean value property for $\mathcal{L}$-harmonic functions in Ω w.r.t. $x_0 \in \Omega$

if

(i) $w(\Omega) := \int_\Omega w(x)\, dx < \infty,$

(ii) $u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(x) w(x)\, dx \quad \forall u \in \mathcal{L}(\Omega) := \{\mathcal{L}u = 0 \text{ in } \Omega\},\ u \geq 0.$

Basic examples

Define, for $x_0 \in \Omega$ and $r > 0$,

$$\Omega_r(x_0) := \left\{ x \in \Omega \,:\, \Gamma(x_0, x) > \frac{1}{r} \right\}$$

and

$$w := \frac{\langle AD\Gamma, D\Gamma \rangle}{\Gamma^2}.$$

Then, for several classes of operator $\mathcal{L}$, one has

$$u(x_0) = \frac{1}{w(\Omega_r(x_0))} \int_{\Omega_r(x_0)} u(x) w(x)\, dx \qquad \forall u \in \mathcal{L}(\Omega),\ \forall\, \overline{\Omega_r(x_0)} \subseteq \Omega.$$

Hence

w is a **density with the mean value property** for $\mathcal{L}$-harmonic functions in $\Omega_r(x_0)$ w.r.t. x_0.

$\mathcal{L}$-triples

DEFINITION:

(Ω, x_0, w) is a $\mathcal{L}$-triple if

- $\Omega \subseteq \mathbb{R}^n$ is open
- $x_0 \in \overline{\Omega}$
- w is a density with the mean value property for $\mathcal{L}$-harmonic functions in Ω w.r.t. x_0.

(IP) If (Ω, x_0, w) and (D, x_0, w') are $\mathcal{L}$-triples, such that

$$\frac{w}{w(\Omega)} = \frac{w'}{w'(D)} \qquad \text{in } \Omega \cap D,$$

is it true that $\Omega = D$?

Γ-triples - I

$$(\Omega, x_0, w) \text{ is a } \mathcal{L}\text{-triple}$$

$$\Downarrow$$

$$\Gamma(x_0, y) = \frac{1}{w(\Omega)} \int_\Omega \Gamma(x, y)\, dx \qquad \forall\, y \notin \Omega \cup \{x_0\}. \tag{*}$$

Note:

(*) can be rewritten as

$$\Gamma(x_0, y) = \int_X \Gamma(x, y)\, d\mu(x) =: \Gamma_\mu(y) \qquad \forall\, y \notin \Omega \cup \{x_0\}, \tag{**}$$

where

$$d\mu(x) = \chi_\Omega(x) \frac{w(x)}{w(\Omega)}\, dx.$$

Hence:

$$\mu \in \mathcal{M}(\Omega) := \{\text{non-negative Radon measure } \mu \text{ s.t. } \mu(\Omega^c) = 0\}$$

Γ-triples - II

DEFINITION:

(Ω, x_0, μ) is a Γ-triple if

(i) $\Omega \subseteq \mathbb{R}^n$ is open and $x_0 \in \overline{\Omega}$

(ii) $\mu \in \mathcal{M}(\Omega)$ and

$$\Gamma_\mu(y) = \Gamma(x_0, y) \quad \forall\, y \notin \Omega \cup \{x_0\}.$$

As we observed above:

$$(\Omega, x_0, w) \ \mathcal{L}\text{-triple}$$

$$\Downarrow$$

$$(\Omega, x_0, \mu) \ \Gamma\text{-triple}, \quad \mu := \frac{w}{w(\Omega)} m \llcorner \Omega.$$

(IP) for Γ-triples

(IP) If (Ω, x_0, μ) and (D, x_0, ν) are Γ-triples, such that

$$\mu \llcorner \Omega \cap D = \nu \llcorner \Omega \cap D,$$

is it true that $\Omega = D$?

Note:

Positive answers to (IP) for Γ-triples

$$\Downarrow$$

Positive answers to (IP) for $\mathcal{L}$-triples.

Strong Γ-triples

DEFINITION:

(Ω, x_0, μ) is a strong Γ-triple if

(i) (Ω, x_0, μ) is a Γ-triple; i.e.

$$\Gamma_\mu(y) = \Gamma(x_0, y) \qquad \forall y \notin \Omega \cup \{x_0\}$$

(ii) $\Gamma_\mu(y) < \Gamma(x_0, y) \quad \forall\, y \in \Omega.$

This is our **key notion** to answer (IP) for Γ-triples.

Main Theorem

THEOREM [Cupini-Lanc.]

Let Ω and D be open subsets of X s.t. $(\overline{\Omega} \cup \overline{D})^c \neq \emptyset$.

Assume that

(i) (Ω, x_0, μ) is a strong Γ-triple

(ii) (D, x_0, ν) is a Γ-triple

(iii) $\Gamma_\mu - \Gamma_\nu$ is a continuous function

(iv) $\mu \llcorner \Omega \cap D = \nu \llcorner \Omega \cap D$

(v) $\partial D \subseteq \operatorname{supp} \nu$

(vi) Ω is a solid set

(vii) $P((\overline{\Omega} \cup \overline{D})^c, \overline{\Omega}^c) = \overline{\Omega}^c$.

Then $D = \Omega$.

A corollary for $\mathcal{L}$-triples

Corollary:

Let Ω and D be open subsets of X s.t. $(\overline{\Omega} \cup \overline{D})^c \neq \emptyset$.

Assume that

(i) (Ω, x_0, w) is a strong $\mathcal{L}$-triple

(ii) (D, x_0, w') is a $\mathcal{L}$-triple

(iii) $\Gamma_\mu - \Gamma_\nu$ is a continuous function with $\mu := \frac{w}{w(\Omega)}\chi_\Omega$ and $\nu := \frac{w'}{w'(D)}\chi_D$

(iv) $\frac{w}{w(\Omega)} = \frac{w'}{w'(D)}$ in $\Omega \cap D$

(v) Ω is a solid set

(vi) $P((\overline{\Omega} \cup \overline{D})^c, \overline{\Omega}^c) = \overline{\Omega}^c$.

Then $D = \Omega$.

$\mathcal{L}$-subharmonic functions

DEFINITION:

$\Omega \subseteq X$ open, $u : \Omega \to [-\infty, \infty[$.

$$u \in \underline{\mathcal{L}}(\Omega) := \{\mathcal{L}\text{-subharmonic functions in } \Omega\}$$

if

(i) u is u.s.c. and $\overline{\{u > -\infty\}} = \Omega$

(ii) $V \subseteq \overline{V} \subseteq \Omega$, V open and bounded,

$$h \in \mathcal{L}(V) \cap C(\overline{V}) \;:\; u \leq h \text{ on } \partial V$$

$$\Downarrow$$

$$u \leq h \text{ in } V$$

THEOREM [Maximum Principle for $u \in \underline{\mathcal{L}}$]:

$u \in \underline{\mathcal{L}}(\Omega)$, $\Omega \subseteq X$ open and bounded,

$$\limsup_{x \to y} u(x) \leq 0 \quad \forall y \in \partial\Omega$$

$$\Downarrow$$

$$u \leq 0 \text{ in } \Omega$$

Γ-potential and $\mathcal{L}^*$

Let μ be a non-negative Radon measure s.t. $\mu(X) < \infty$.

Define

$$\Gamma_\mu(x) := \int_X \Gamma(y, x)\, d\mu(y), \qquad x \in X\,.$$

Then

- $\Gamma_\mu \in L^1_{\mathrm{loc}}(X)$
- $\mathcal{L}^* \Gamma_\mu = -\mu \quad \text{in } \mathcal{D}'(X)$

In particular:

- Γ_μ is smooth in $\mathbb{R}^n \setminus \operatorname{supp} \mu$
- $\mathcal{L}^* \Gamma_\mu = 0 \quad \text{in } \mathbb{R}^n \setminus \operatorname{supp} \mu$.

Propagation of Maxima - I

Note:

$\mathcal{L}^* := \mathrm{div}(A(x)D) - \langle\, b(x)\,,\ D\,\rangle$

Let $X_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$ be the j-th column of A, $j = 1, \cdots, n$

and $Y = -\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

DEFINITION ($\mathcal{L}^*$-trajectory):

A curve γ is a $\mathcal{L}^*$-trajectory if

$$\gamma = \gamma^{(1)} + \cdots + \gamma^{(n)} + \gamma^{(0)}$$

where:

$$\gamma^{(j)} = \text{integral curve of } \pm X_j \quad (j = 1, \cdots, n)$$
$$\gamma^{(0)} = \text{integral curve of } Y.$$

Propagation of Maxima - II

DEFINITION ($\mathcal{L}^*$-Propagation set):

Let $\Omega \subseteq \mathbb{R}^n$ be open and let $x \in \Omega$.

Define

$$P(x,\Omega) = \mathcal{L}^*\text{-propagation set of } x \in \Omega$$
$$:= \{y \mid \exists\, \mathcal{L}^*\text{-trajectory } \gamma : [0,1] \to \Omega, \text{ s.t. } \gamma(0) = x,\ \gamma(1) = y\}\,.$$

Notation:

If $A \subseteq \Omega$ we let

$$P(A,\Omega) := \bigcup_{x \in A} P(x,\Omega).$$

Propagation of Maxima - III

THEOREM:

Let $\Omega \subseteq \mathbb{R}^n$ be open and let

$$u \in \underline{\mathcal{L}}^*(\Omega), \quad u \leq 0.$$

Assume there exists $x_0 \in \Omega$ s.t.

$$u(x_0) = 0.$$

Then $u \equiv u(x_0)$ in $P(x_0, \Omega)$.

Note 1 $\mathcal{L}$ elliptic:

$$\Omega \text{ connected}, \; x_0 \in \Omega$$

$$\Downarrow$$

$$P(x_0, \Omega) = \Omega$$

Note 2 $\mathcal{L} = H$ heat operator, $\Omega = O \times]a, b[$,

$$O \subseteq \mathbb{R}^n \text{ open}, \; z_0 = (x_0, t_0)$$

$$\Downarrow$$

$$P(z_0, \Omega) = O \times]t_0, b[$$

Previous results

Our Main Theorem and its corollary **improve and extend**

- Epstein-Schiffer-Kuran characterization of the Euclidean balls via harmonic functions (1962-1965-1972)
- Aharonov-Schiffer-Zalcman inverse problem for Newtonian potential (1981)
- Lanconelli characterization of the gauge balls in stratified Lie group $\mathbb{G}$ via $\Delta_{\mathbb{G}}$-harmonic functions (2013)
- Abbondanza-Bonfiglioli characterization of the level sets of the fundamental solution of suitable hypoelliptic degenerate elliptic PDEs (2007)
- Cupini-Lanconelli inverse problem for $\Delta_{\mathbb{G}}$-triples (2016)
- Suzuki-Watson characterization of the **heat balls** via caloric functions (2001)
- Kogoj-Lanconelli-Tralli characterization of the level sets of the fundamental solutions for suitable evolution equation in homogeneous groups (2014).