SOME ASPECTS OF HARDY TYPE INEQUALITIES

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Abstract. We give a general answer to the following fundamental problem posed by Shmuel Agmon 30 years ago:

Given a (symmetric) linear elliptic operator $P$ of second-order in $\mathbb{R}^n$, find a continuous, nonnegative weight function $W$ which is “as large as possible” such that for some neighborhood of infinity $\Omega_R$ the following inequality holds

$$(P\phi, \phi) \geq \int_{\Omega_R} W(x)|\phi|^2 \, dx \quad \forall \phi \in C_0^\infty(\Omega_R).$$

We construct, for a general subcritical second-order elliptic operator $P$ in a domain $\Omega \subset \mathbb{R}^n$ (or a noncompact manifold), a Hardy-weight $W$ which is optimal in the following natural sense sense. The operator $P - \lambda W$ is subcritical in $\Omega$ for all $\lambda < 1$, null-critical in $\Omega$ for $\lambda = 1$, and supercritical near any neighborhood of infinity in $\Omega$ for any $\lambda > 1$. Moreover, in the symmetric case, if $W > 0$, then the spectrum and the essential spectrum of $W^{-1}P$ are equal to $[1, \infty)$.

Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators on a general domain $\Omega$ or on a noncompact manifold. Moreover, the results can be generalized to certain $p$-Laplacian type operators. The constructed weight $W$ is given by an explicit simple formula involving two positive solutions of the equation $Pu = 0$.

References
