Wiener-type criteria for evolution equations

Ermanno Lanconelli

Joint work with

Giulio Tralli and Francesco Uguzzoni

The evolution PDO

$$\mathcal{H} = \sum_{i,j=1}^{N} q_{i,j}(z)\,\partial_{x_i,x_j} + \sum_{k=1}^{N} q_k(z)\,\partial_k - \partial_t.$$

The coefficients $q_{i,j} = q_{j,i}, q_k$ are of class C^∞ in the strip

$$S = \{z = (x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\ T_1 < t < T_2\},$$

(with $-\infty \le T_1 < T_2 \le \infty$). Moreover

$$q_{\mathcal{H}}(z,\xi) := \sum_{i,j=1}^{N} q_{i,j}(z)\,\xi_i\xi_j \ \ge 0, \quad q_{\mathcal{H}}(z,\cdot) \not\equiv 0 \quad \text{for every} \quad z \in S$$

Qualitative assumptions

- $\mathcal{H}$ e $\mathcal{H}^*$ are hypoelliptic
- $\mathcal{H}$ has a global fundamental solution

$$(z, \zeta) \mapsto \Gamma(z, \zeta)$$

of class C^∞ in $\{z \neq \zeta\}$

Γ is a fundamental solution in the following sense:

(i) $\Gamma(\cdot, \zeta) \in L^1_{\text{loc}}(S)$ e $\mathcal{H}(\Gamma(\cdot, \zeta)) = -\delta_\zeta$, for every $\zeta \in S$;

(ii) for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for every $x_0 \in \mathbb{R}^N$, and $\tau \in]T_1, T_2[$, one has

$$\int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau)\, \varphi(\xi)\, d\xi \to \varphi(x_0), \qquad \text{as} \quad (x, t) \to (x_0, \tau), t > \tau.$$

Crucial assumptions

Γ satisfies a two sided Gaussian bound w.r.t a distance in $\mathbb{R}^N$.

Precisely: given a metric $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, we call d-Gaussian of exponent $a > 0$ the function

$$G_a^{(d)}(z,\zeta) = 0 \quad \text{if} \quad z = (x,t),\ \zeta = (\xi,\tau) \ \text{and } t \le \tau,$$

$$G_a^{(d)}(z,\zeta) = \frac{1}{|B_d(x,\sqrt{t-\tau})|} \exp\left(-a\frac{d(x,\xi)^2}{t-\tau}\right) \ \text{if } t > \tau.$$

We assume the existence of a metric d in $\mathbb{R}^N$ such that

(H) $\frac{1}{\Lambda}\, G_{b_0}^{(d)}(z,\zeta) \le \Gamma(z,\zeta) \le \Lambda\, G_{a_0}^{(d)}(z,\zeta), \quad \forall z,\zeta \in S$

and suitable positive constants a_0, b_0 and Λ

Hypotheses on d

(D1) The d-topology is the Euclidean one, $(\mathbb{R}^N, d)$ is complete and, for $x \in \mathbb{R}^N$, $d(x, \xi) \to \infty$ if and only if $\xi \to \infty$ w.r.t. the usual Euclidean norm

(D2) $(\mathbb{R}^N, d)$ is *doubling* w.r.t the Lebesgue measure, i.e.,

$$|B(x, 2r)| \leq c_d |B(x, r)|, \quad \forall x \in \mathbb{R}^N, \forall r > 0.$$

(D3) $(\mathbb{R}^N, d)$ has the *segment property*, i.e., for every $x, y \in \mathbb{R}^N$ there exists a continuous path $\gamma : [0, 1] \to \mathbb{R}^N$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(x, y) = d(x, \gamma(t)) + d(\gamma(t), y) \quad \forall t \in [0, 1].$$

Example : 1

$$\mathcal{H} = \text{div}(A(x)D) \; - \; \partial_t, \quad x \in \mathbb{R}^N$$

- A matrix with C^∞ entries, symmetric, ≥ 0 and such that tr$A > 0$.
- $\mathcal{H}$ hypoelliptic
- $(\mathbb{R}^N, d_A)$ metric space satisfying (D1)-(D3)

(Grigoryan and Saloff-Coste; classical parabolic operators: D.G. Aronson)
Some particular cases:

- Classical parabolic operators
- Heat operators on complete, unbounded Riemannian Manifolds with Ric ≥ 0
- Heat operators on stratified Lie groups

Examples: 2

$\mathcal{H} = \sum_{i.j=1}^{m} a_{i,j}(z)\, X_i X_j$ in an open subset of $\mathbb{R}^{N+1}$

- $(a_i, j)_{i,j=1\dots m}$ strictly positive matrix with C^∞ entries
- $\mathcal{X} = (X_1, \dots, X_m)$ Hormander system in $\mathbb{R}^N$

$\mathcal{H}$ satisfies our general assumptions w. r. t. the control distance related to the system $\mathcal{X}$.

$\mathcal{H}$ has a fundamental solution satisfying a two sided Gaussian bound w.r.t d, with constants only depending on Holder norms of the $a_{i,j}$

(Bramanti -Brandolini- Lanconelli- Uguzzoni (2010))

Potential Analysis for $\mathcal{H}$

Methods and results from Abstract Potential Theory apply to the operator $\mathcal{H}$. Indeed, letting

$$\text{H(S)} := \{\text{sheaf of the solutions to } \mathcal{H}u = 0 \text{ in some open sbset of S}\},$$

then

$$(\text{S}, \text{H(S)}) \quad \text{is a} \quad \beta - \text{Harmonic Space}$$

(Lanconelli-Uguzzoni (2010)). As a consequence: the Dirichlet problem

$$\mathcal{H}\,u = 0 \text{ in } \Omega, \quad u = \varphi \text{ su } \partial\Omega,$$

has a generalized solution (in the sense of Perron)

$$u = u_\varphi^\Omega,$$

for every open set $\Omega \subset\subset$ S and for every $\varphi \in C(\partial\Omega)$.

The Perron solution

The Perron solution of the previous Dirichlet problem is

$$u_\varphi^\Omega := \inf \{ v \in \overline{\mathcal{H}}(\Omega) \ / \ \liminf_{\partial\Omega} v \geq \varphi \}$$

By definition: $v \in \overline{\mathcal{H}}(\Omega)$ if it is continuous and , for every open set $V \subset\subset \Omega$ and every solution h of $\mathcal{H}\, h = 0$ in V one has

$$v \geq h \ \text{ in } \ V \quad \text{if} \quad \liminf_{\partial\Omega} (v - h) \geq 0 \ \text{ in } \ V.$$

NOTE. If $v \in C^2(\Omega)$ then $v \in \overline{\mathcal{H}}(\Omega)$ if and only if $\mathcal{H}v \leq 0$ in Ω

We always have

$$u_\varphi^\Omega \in C^\infty(\Omega) \quad \text{e} \quad \mathcal{H}\, u_\varphi^\Omega = 0 \ \text{ in } \ \Omega$$

$\mathcal{H}$-regular points of $\partial\Omega$

In general: u_φ^Ω does not assume the datum φ on $\partial\Omega$.

The points $z_0 \in \partial\Omega$ such that

$$\lim_{z \to z_0} u_\varphi^\Omega(z) = \varphi(z_0), \quad \text{for every} \quad \varphi \in C(\partial\Omega)$$

are called $\mathcal{H}$-regular for Ω.

Crucial question in studying Dirichlet problem with Perron method :

which points of $\partial\Omega$ are regular?

Potential Theory gives some general criteria, working for every operator $\mathcal{H}$.

Bouligand 's criterion

$$z_0 \in \partial\Omega \quad \text{è} \quad \mathcal{H} - \text{regular for } \Omega$$

$$\Updownarrow$$

$$\text{there exists } \mathcal{H} - \text{barrier for } \Omega \text{ in } z_0$$

DEFINITION. v is a $\mathcal{H}$ – barrier for Ω in z_0 if there exists a neighborhood V of z_0 such that

$$v \in \overline{\mathcal{H}}(V \cap \Omega), \quad v > 0 \text{ in } V \cap \Omega, \quad \lim_{z \to z_0} v(z) = 0,$$

Balayage criterion. Preliminaries

Let $K \subset \mathsf{S}$ be compact. We define

$$V_K(z) := \liminf_{\zeta \to z} W_K(\zeta), \quad z \in \mathsf{S},$$

where

$$W_K := \inf\{v \in \overline{\mathcal{H}}(\mathsf{S}) \; / \; v \geq 0 \text{ in } \mathsf{S}, \; v \geq 1 \text{ on } K\}.$$

V_K is called the $\mathcal{H}$-balayage of K

NOTE. V_K is also called the $\mathcal{H}$-equilibrium potential of K

- $V_K \in \overline{\mathcal{H}}(\mathsf{S}), \quad \mathcal{H}\, V_K = 0$ in $S \setminus \partial K$
- $0 \leq V_K \leq 1, \quad V_K = 1$ in $\mathrm{int}(K)$

Balayage criterion

Let $z_0 \in \partial\Omega$. Define

$$\Omega_\rho^c := \{z \in \mathsf{S} \setminus \Omega \ : \ |z| \le \rho\}.$$

Then

$$z_0 \in \partial\Omega \quad \text{è} \ \mathcal{H} - \text{regular for } \Omega$$

$$\Updownarrow$$

$$V_{\Omega_\rho^c}(z_0) = 1, \quad \forall \rho > 0$$

NOTE. If $z_0 \in \partial\Omega$ is not $\mathcal{H}$ – regular for Ω, then

$$\lim_{\rho \to 0} V_{\Omega_\rho^c}(z_0) = 0$$

Geometrical Criteria

Bouligand's and Balayage criteria are not really geometrical criteria.

Several geometrical regularity criteria are present in literature, mainly for classical Elliptic and Parabolic operators. Their progenitor is the Wiener criterion for Δ, the Laplace operator.

WIENER TEST (1924). Let $x_0 \in \partial\Omega$. Fix $\lambda \in]0,1[$. For every $k \in \mathbb{N}$ define

$$\Omega_k^c(x_0) := \{x \in \mathbb{R}^N \setminus \Omega \ : \ \lambda^{k+1} \leq |x - x_0| \leq \lambda^k\}$$

Then: x_0 is Δ-regular for Ω if and only if

$$\sum_{k=1}^{\infty} V_{\Omega_k^c(x_0)}(x_0) \ = \ \infty$$

Equivalent formulation of Wiener test

- There exists a unique nonnegative Radon measure ν in Ω_k^c such that

$$V_{\Omega_k^c}(x_0) = \Gamma \star \nu, \quad \text{where} \quad \Gamma = \text{fundamental solution of } \Delta$$

- $\text{cap}(\Omega_k^c(x_0)) := \nu(\Omega_k^c(x_0))$
  cap is called the Newtonian capapcity, or the Laplacian capacity
- $V_{\Omega_k^c}(x_0) \cong \frac{\text{cap}(\Omega_k^c(x_0))}{\lambda^{k(N-2)}}$

Then(Wiener): x_0 is Δ-regular for Ω if and only if

$$\sum_{k=1}^{\infty} \frac{\text{cap}\,(\Omega_k^c(x_0))}{\lambda^{k(N-2)}} = \infty$$

Equivalent formulation of Wiener test, continuation

Letting $\mu = \lambda^{N-2}$ we can write

- $\Omega_k^c(x_0) \;=\; \left\{ x \in \mathbb{R}^N \setminus \Omega \;:\; (\frac{1}{\mu})^k \le \Gamma(x_0 - x) \le (\frac{1}{\mu})^{k+1} \right\}$
- $\frac{\text{cap}(\Omega_k^c(x_0))}{\lambda^{k(N-2)}} = \frac{\text{cap}(\Omega_k^c(x_0))}{\mu^k}$

Hence (Wiener) : x_0 is Δ-regular for Ω if and only if

$$\sum_{k=1}^{\infty} \frac{\text{cap}\,(\Omega_k^c(x_0))}{\mu^k} \;=\; \infty$$

NOTE. If Ω satisfies the exterior cone condition at x_0 then

$$\frac{\text{cap}\,(\Omega_k^c(x_0))}{\mu^k} \;\cong\; 1$$

so that the Wiener's series is divergent, and

. $\quad x_0 \quad \text{is} \quad \Delta - \text{regular} \quad \text{for} \quad \Omega$

(Zaremba cone criterion (1909))

Wiener test for the Heat equation

Let $\mathcal{H} := \Delta - \partial_t$ be the Heat operator in $\mathbb{R}^{N+1}$.

Let $z_0 \in \partial\Omega$ $(\Omega \subset \mathbb{R}^{N+1})$. Then : z_0 is $\mathcal{H}$-regular for Ω if and only if

$$\sum_{k=1}^{\infty} \frac{\text{cap}\,(\Omega_k^c(z_0))}{\mu^k} \;=\; \infty$$

where

$$\Omega_k^c(z_0) \;=\; \left\{ x \in \mathbb{R}^{N+1} \setminus \Omega \;:\; (\frac{1}{\mu})^k \leq \Gamma(z_0 - z) \leq (\frac{1}{\mu})^{k+1} \right\}$$

Here : Γ denotes the fundamental solution of the Heat equation, and cap denotes the caloric capacity

(Pini (1954), Lanconelli (1973), Evans-Gariepy (1985))

Exterior parabolic cone condition

NOTE. If Ω satisfies the exterior parabolic cone condition at z_0 then

$$\frac{\text{cap}\,(\Omega_k^c(z_0))}{\mu^k} \cong 1$$

so that the Wiener's series is divergent, and

$$\cdot \qquad z_0 \quad \text{is} \quad \mathcal{H}-\text{regular} \quad \text{for} \quad \Omega$$

(Effros-Kazdan cone criterion (1970))

Variable coefficients. Elliptic equations

Let $L := \text{div}(A(x)D)$ be a uniformly Elliptic operator in $\mathbb{R}^N$.

Let $x_0 \in \partial\Omega$ $(\Omega \subset \mathbb{R}^N)$. Then:

x_0 is L-regular for Ω $\Leftrightarrow$ x_0 is Δ-regular for Ω

(Littman-Stampacchia-Weinberger (1964))

The parabolic case

Let

$$\mathcal{H}_a := a\,\Delta - \partial_t \quad \text{and} \quad \mathcal{H}_b := b\,\Delta - \partial_t$$

If $0 < a < b$ then

$$\mathcal{H}_a \ \text{regularity} \quad \Rightarrow \quad \mathcal{H}_b \ \text{regularity}$$

and does not hold the reverse implication (Petrowski (1934))

More generally, let

$$\mathcal{H}_A = \text{div}(A(z)D) \ - \ \partial_t, \quad \text{and} \quad \mathcal{H}_B = \text{div}(B(z)D) \ - \ \partial_t,$$

two uniformly parabolic operators in S. Then :

$\mathcal{H}_A$ and $\mathcal{H}_B$ have the same regularity points if and only if $A \equiv B$

(Lanconelli (1977))

A more precise results

Let

$$\mathcal{H} = \mathrm{div}(A(z)D) \; - \; \partial_t,$$

be a uniformly parabolic operators in S with smooth coefficients.

Let $z_0 \in \Omega \subset\subset \mathrm{S}$ and let

$$\mathcal{H}_{z_0} = \mathrm{div}(A(z_0)D) \; - \; \partial_t,$$

the constant coefficients frozen operator at z_0. Then

$$z_0 \text{ is } \mathcal{H}_{z_0} - \text{regular for } \Omega \quad \Leftrightarrow \quad z_0 \text{ is } \mathcal{H} - \text{regular for } \Omega$$

(Garofalo and Lanconelli (1988))

Some bibliographical comments

In the stationary case, Wiener-type tests for second order degenerate-elliptic equations with underlying sub-Riemannian structures are well settled in literature: papers by Hueber (1985), Hansen and Hueber (1987), Negrini and Scornazzani (1987)); Tralli and Uguzzoni (2014).
We also mention a paper by Ferrari and Franchi (2003) related to the doubling property for the Harmonic measure.

On the contrary, as far as we know, only a few papers have been devoted to the Wiener test for evolution equations in sub-Riemannan settings: we mention a paper by Scornazzani (1981), where a Wiener test for a Kolmogorov equation is proved, and the work of Garofalo and Segala (1990) in which the Wiener test for the Heat equation on the Heisenberg group is established.

Some bibliographical comments, continuation

In sub-Riemannian settings , more literature is available relating to the boundary behavior, in sufficiently regular domains, of *nonnegative* solutions to evolution equations: see e.g. the papers by:

C. Cinti, K. Nyström, S. Polidoro (2012)
M. Frentz, N. Garofalo, E. Götmark, I. Munive, K. Nyström (2012)
I.Munive (2012)

An alternative definition of capacity. $G_a^{(d)}-$capacity

Let Γ be the fundamental solution of Δ in $\mathbb{R}^N$ (of $\Delta - \partial_t$ in $\mathbb{R}^{N+1}$).

For every compact set $K \subset \mathbb{R}^N$ ($K \subset \mathbb{R}^{N+1}$) we have

cap$(K) = \sup\{\mu(K) : \mu$ Radon measure supported in K $s.t.$ $\Gamma\star\mu \leq 1\}$

Then, analogously, for every compact set $K \subset$ S we define

$$\mathcal{C}_a(K) := \sup\{\mu(K) : \mu \in \mathcal{M}_a(K)\}$$

where

$\mathcal{M}_a(K) := \{$Radon measure μ supported in K : $G_a^{(d)} \star \mu \leq 1$ in S$\}$

and

$$(G_a^{(d)} \star \mu)(z) \;:=\; \int_K G_a^{(d)}(z,\zeta)\; d\mu(\zeta).$$

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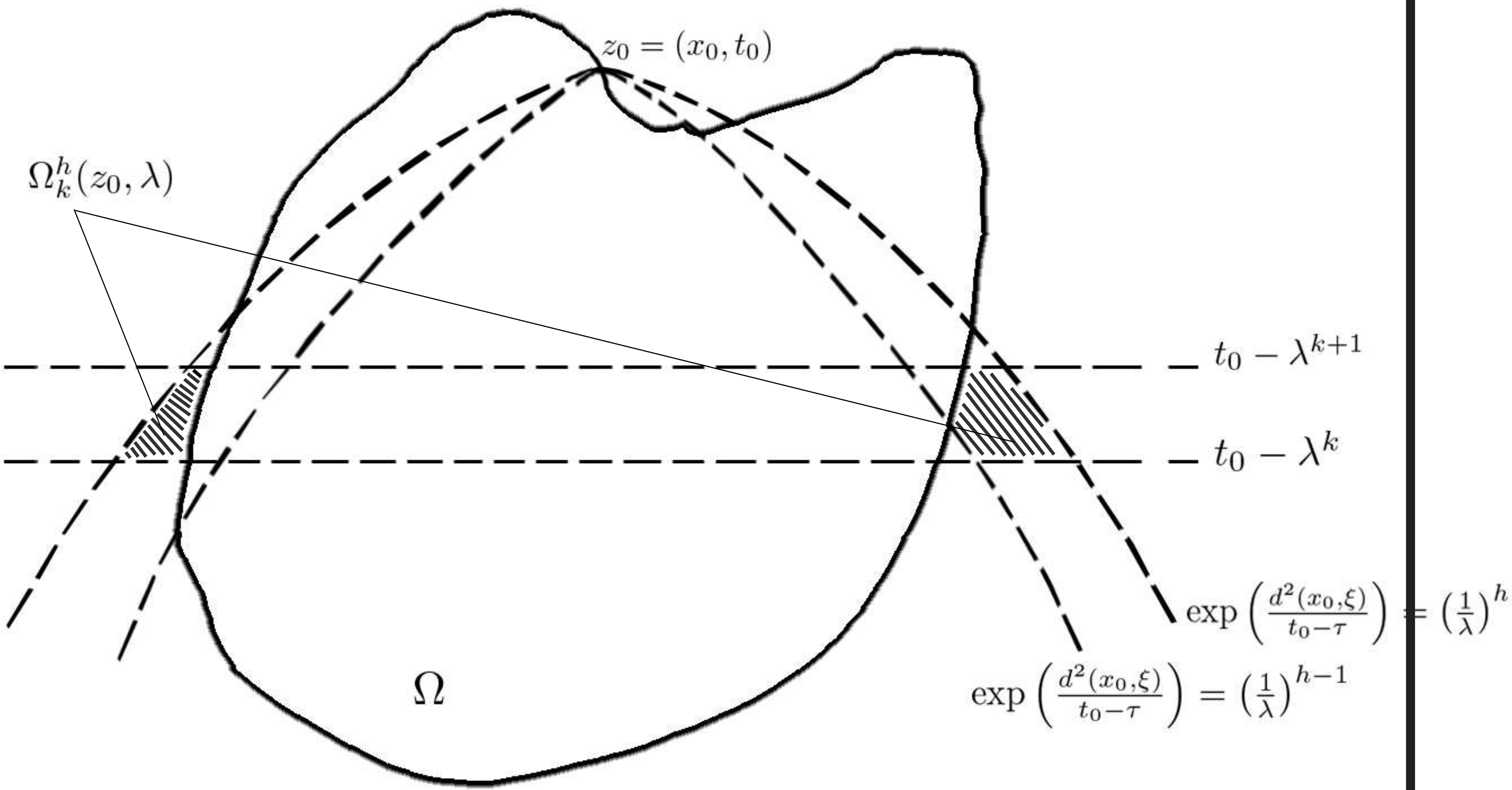
Our Winer-type criteria for $\mathcal{H}$. Preliminary definition

Let $\Omega \subset\subset$ S and let $z_0 = (x_0, t_0) \in \partial\Omega$. For every $k, h \in \mathbb{N}$ define

$$\Omega^c_{k,h}(z_0) = \Bigg\{ \zeta = (\xi, \tau) \in S \setminus \Omega \,:\, \lambda^{k+1} \le t_0 - \tau \le \lambda^k, \\ \left(\frac{1}{\lambda}\right)^{h-1} \le \exp\left(\frac{d^2(x_0, \xi)}{t_0 - \tau}\right) \le \left(\frac{1}{\lambda}\right)^h \Bigg\}.$$

for a fixed $\lambda \in]0, 1[$.

$\Omega^c_{k,h}$ picture



Our Winer-type criteria for $\mathcal{H}$

THEOREM . Let $z_0 = (x_0, t_0) \in \partial\Omega,\ \Omega \subset\subset$ S, and let $\lambda \in]0,1[$.

(i) If there exists $b > b_0$ such that

$$\sum_{h,k=1}^{\infty} \frac{\mathcal{C}_a\left(\Omega^c_{k,h}(z_0)\right)}{\left|B\left(x_0, \sqrt{\lambda^k}\right)\right|}\lambda^{bh} = \infty$$

then the point z_0 is $\mathcal{H}$-regular.

(ii) If the point z_0 is $\mathcal{H}$-regular, then

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_b\left(\Omega^c_{k,h}(z_0)\right)}{\left|B\left(x_0, \sqrt{\lambda^k}\right)\right|}\lambda^{ah} = \infty$$

for every $0 < a \leq a_0$.

Exterior $d-$ cone condition

Let $\Omega \subset\subset$ S and let $z_0 = (x_0, t_0) \in \partial\Omega$

We say that Ω satisfies the exterior d-cone condition at z_0 if:

there exist $M, R, \theta > 0$ such that

$$|\{x \in \overline{B_d(x_0, M\,r)}\ :\ (x, t_0 - r^2) \notin \Omega\}|\ \geq\ \theta\,|B_d(x_0, M\,r)|$$

for every $0 < r \leq R$.

COROLLARY (Zaremba-Effros-Kazdan type criterion). If Ω satisfies the exterior d-cone condition at z_0 then z_0 is $\mathcal{H}-$ regular for Ω

Further results

We also proved estimates of the continuity modulus of u_φ^Ω at the $\mathcal{H}-$ regular points of $\partial\Omega$, in terms of the Wiener series.

In particular, if Ω satisfies the exterior d-cone condition at z_0, and φ is Holder continuous at z_0, then u_φ^Ω is Holder continuous at z_0

We stress that all our estimates depend on $\mathcal{H}$ only in terms of the constants in the Gaussian bounds of Γ