Some exercises in commutative algebra

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If A is a ring, then an A-module M is called finite if it is finitely generated (as A-module), or equivalently there exists an integer n and a surjective A-linear map $A^n \to M$.

Let $\varphi\colon A \to B$ be a ring homomorphism. We say that B is finite over A (or that B is a finite A-algebra) if B is a finite A-module.

Let $\varphi\colon A \to B$ be a ring homomorphism. We say that B is an A-algebra of finite type (or that B is a finitely generated algebra or that B is of finite type over A) if there exists a surjective A-algebra homomorphism $A[x_1, \dots, x_n] \to B$. In other words, if there exist finitely many elements $b_1, \dots, b_n \in B$ such that every element in B can be written as a polynomial in the b_i's with coefficients in $\varphi(A)$.

For each of the following statements decide if it is true or false. If it is true, prove it. If it is false, find a counterexample. Let $\varphi\colon A \to B$ be a ring homomorphism and let M be an A-module.

(1) If B is finite over A, then B is of finite type over A.
(2) If B is of finite type over A and A is a noetherian ring, then B is a noetherian ring.
(3) If B is of finite type over A and A is a noetherian ring, then B is a noetherian A-module.
(4) If B is finite over A and A is a noetherian ring, then B is a noetherian ring.
(5) If φ is injective and B is a noetherian ring, then A is a noetherian ring.
(6) If φ is injective, B is a noetherian ring and B is a finite A-module, then A is a noetherian ring. [Very hard!]
(7) If φ is injective and A and B are both noetherian rings, then B is of finite type over A.
(8) Assume that φ is injective and $\psi\colon B \to C$ is an injective ring homomorphism. Assume that A is a noetherian ring, C is of finite type over A, and C is finite over B. Then B is of finite type over A.
(9) Let $K \supseteq k$ be a field extension. If K is a k-algebra of finite type, then K is finite over k.
(10) $\mathbb{Q}$ is not of finite type over $\mathbb{Z}$.
(11) $\mathbb{Q}$ is of finite type over $\mathbb{Z}_{(p)}$, which is the localisation of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$.
(12) If K is a field which is a $\mathbb{Z}$-algebra of finite type, then K is a finite field.
(13) If φ is surjective, then B is finite over A.
(14) If φ is surjective, then B is of finite type over A.
(15) If φ is surjective and B is a noetherian ring, then A is a noetherian ring.
(16) If φ is the localisation with respect to some multiplicative subset of A, then B is finite over A.
(17) If φ is the localisation with respect to some multiplicative subset of A, then B is of finite type over A.
(18) If φ is the localisation with respect to some multiplicative subset of A and A is a notherian ring, then B is a noetherian ring.
(19) If φ is the localisation with respect to some multiplicative subset of A and B is a notherian ring, then A is a noetherian ring.

(20) If φ is injective, B is a field and B is finite over A, then A is a field. [The assumption that B is finite over A can be relaxed to B being integral over A.]
(21) If φ is injective, B is a domain, A is a field and B is finite over A, then B is a field. [The assumption that B is finite over A can be relaxed to B being integral over A.]
(22) Consider the surjective ring homomorphism $\psi\colon A[\![x]\!] \to A$ defined by $f(x) \mapsto f(0)$. Let $\mathfrak{p}$ be a prime ideal in $A[\![x]\!]$ and let $\mathfrak{p}^e \subseteq A$ be the extension of $\mathfrak{p}$ via ψ. Then: $\mathfrak{p}$ is finitely generated if and only $\mathfrak{p}^e$ is finitely generated. [Distinguish the two cases: $x \in \mathfrak{p}$ and $x \notin \mathfrak{p}$.]
(23) If every prime ideal of A is finitely generated, then A is a noetherian ring.
(24) A is a noetherian ring if and only if $A[\![x]\!]$ is a noetherian ring. [Give a proof of $\Rightarrow$ by using the preceding two exercises.]
(25) If M is a finite A-module and A is a noetherian ring, then M is a noetherian A-module.
(26) If M is a non-zero noetherian A-module, then A is a noetherian ring.
(27) If M is a noetherian A-module, then $A/\mathrm{ann}_A(M)$ is a noetherian ring.
(28) If M is a faithful noetherian A-module, then A is a noetherian ring.
(29) If M if a finite A-module, then $M \otimes_A B$ is a finite B-module.
(30) If M is a flat A-module, then $M \otimes_A B$ is a flat B-module.
(31) If $M \otimes_A B$ is a flat B-module, then M is a flat A-module.
(32) If M and N are non-zero A-modules, then $M \otimes_A N$ is non-zero.
(33) If M and N are non-zero A-modules and A is a local ring, then $M \otimes_A N$ is non-zero.
(34) If M and N are non-zero finite A-modules and A is a local ring, then $M \otimes_A N$ is non-zero.
(35) If M and N are finite A-modules, then $M \otimes_A N$ is a finite A-module.
(36) If A is an artinian ring and M is a finite A-module, then M is an A-module of finite length.
(37) If φ is the localisation with respect to some multiplicative subset of A and A is reduced, then B is reduced. [A ring is called reduced if 0 is the unique nilpotent element.]
(38) If A is reduced, then $A_\mathfrak{p}$ is reduced for every $\mathfrak{p} \in \mathrm{Spec}\, A$.
(39) If $A_\mathfrak{p}$ is reduced for every $\mathfrak{p} \in \mathrm{Spec}\, A$, then A is reduced.
(40) If A is a domain, then $A_\mathfrak{p}$ is a domain for every $\mathfrak{p} \in \mathrm{Spec}\, A$.
(41) If $A_\mathfrak{p}$ is a domain for every $\mathfrak{p} \in \mathrm{Spec}\, A$, then A is a domain.
(42) If A is a noetherian ring, then $A_\mathfrak{p}$ is a noetherian ring for every $\mathfrak{p} \in \mathrm{Spec}\, A$.
(43) Assume that A is a boolean ring, i.e. for every element $a \in A$ we have $a^2 = a$. For every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$, the quotient $A/\mathfrak{p}$ and the localisation $A_\mathfrak{p}$ are both isomorphic to $\mathbb{F}_2$.
(44) If $A_\mathfrak{p}$ is a noetherian ring for every $\mathfrak{p} \in \mathrm{Spec}\, A$, then A is a noetherian ring. [Hint: consider the ring $A = \prod_{i \in \mathbb{N}} \mathbb{F}_2$ and use (43).]
(45) Assume that $A_\mathfrak{m}$ is a noetherian ring for every $\mathfrak{m} \in \mathrm{mSpec}\, A$. Assume that for every element $x \in A \setminus \{0\}$ the set $\{\mathfrak{m} \in \mathrm{mSpec}\, A \mid x \in \mathfrak{m}\}$ is finite. Then A is a noetherian ring.
(46) If A is semilocal (i.e. the maximal ideals are finitely many) and $A_\mathfrak{m}$ is a noetherian ring for every $\mathfrak{m} \in \mathrm{mSpec}\, A$, then A is a noetherian ring.
(47) If A is a local ring with $\dim A = 0$, then A is artinian.
(48) If A is a noetherian ring, then $\dim A$ is finite.
(49) If A is a local noetherian ring, then $\dim A$ is finite.
(50) If A is a noetherian ring, then $\mathrm{Spec}\, A$ is a noetherian topological space.
(51) If $\mathrm{Spec}\, A$ is a noetherian topological space, then A is noetherian.

(52) There exists a local ring which is isomorphic to the direct product of two non-zero rings.
(53) If $I \subseteq A$ is a finitely generated ideal such that $(0 : I) = 0$ and $J \subseteq A$ is an arbitrary ideal, then $J \subseteq (IJ : I) \subseteq \sqrt{J}$.
(54) If $I \subseteq A$ is a finitely generated ideal, then the following statements are equivalent:
  (a) the exact sequence $0 \to I \to A \to A/I \to 0$ splits,
  (b) A/I is flat over A,
  (c) $I = I^2$,
  (d) there exists an element $e \in A$ such that $e = e^2$ and $I = Ae$.
(55) If $A \neq 0$ and there exists a surjective A-linear homomorphism $A^n \to A^m$, then $n \geq m$.
(56) If $A \neq 0$ and there exists an isomorphism of A-modules between A^n and A^m, then $n = m$.
(57) If $A \neq 0$ and there exists an injective A-linear homomorphism $A^n \to A^m$, then $n \leq m$.
(58) If M is a finite A-module, then every surjective A-linear endomorphism of M is an isomorphism.
(59) If M is a finite A-module, then every injective A-linear endomorphism of M is an isomorphism.
(60) If M is an artinian A-module, then every injective A-linear endomorphism of M is an isomorphism.
(61) $\operatorname{Ass} M \subseteq \operatorname{Supp} M \subseteq \mathrm{V}(\operatorname{ann}_A(M))$
(62) $\operatorname{Supp} M = \emptyset$ iff $M = 0$.
(63) If M and N are finite A-modules, then $\operatorname{Supp} M \otimes_A N = \operatorname{Supp} M \cap \operatorname{Supp} N$.
(64) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$ and $\operatorname{Ass} M' \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} M' \cup \operatorname{Ass} M''$.
(65) If $M = M_1 \oplus \cdots \oplus M_n$ as A-modules, then $\operatorname{Supp} M = \operatorname{Supp} M_1 \cup \cdots \cup \operatorname{Supp} M_n$ and $\operatorname{Ass} M = \operatorname{Ass} M_1 \cup \cdots \cup \operatorname{Ass} M_n$.
(66) Let G be a finite abelian group (here "finite" means "with finitely many elements") with order n. Let p be a prime number. The ideal $p\mathbb{Z}$ is associated to the $\mathbb{Z}$-module G if and only if $p|n$.
(67) Let G be a finitely generated abelian group. The ideal 0 is associated to the $\mathbb{Z}$-module G iff G is not torsion iff the cardinality of G is infinite.
(68) If M is an A-module with $\operatorname{ann}_A(M) \subseteq I$, then $\operatorname{Ass}_{A/I} M = \operatorname{Ass}_A M$.
(69) If $S \subseteq A$ is multiplicative and N is an $S^{-1}A$-module, then $\operatorname{Supp}_{S^{-1}A} N \subseteq \operatorname{Supp}_A N$ and $\operatorname{Ass}_{S^{-1}A} N = \operatorname{Ass}_A N$.
(70) If $S \subseteq A$ is multiplicative and M is an A-module, then $\operatorname{Supp}_{S^{-1}A} S^{-1}M \supseteq \operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A$ and $\operatorname{Ass}_{S^{-1}A} S^{-1}M \supseteq \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$.
(71) If A is a noetherian ring, $S \subseteq A$ is multiplicative and M is an A-module, then $\operatorname{Ass}_{S^{-1}A} S^{-1}M = \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$.
(72) If A is a noetherian ring, then $\operatorname{Ass} M = \emptyset$ iff $M = 0$.
(73) If M is a finite A-module, then $\operatorname{Supp} M = \mathrm{V}(\operatorname{ann}_A(M))$.
(74) If A is a noetherian ring and M is a finite A-module, then:
  (i) $\operatorname{Ass} M$ is a finite set;
  (ii) $\operatorname{Ass} M$ and $\operatorname{Supp} M$ have the same minimal elements;
  (iii) if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is the set of minimal elements of $\operatorname{Ass} M$, then $\operatorname{Supp} M = \mathrm{V}(\mathfrak{p}_1) \cup \cdots \cup \mathrm{V}(\mathfrak{p}_n)$, $\sqrt{\operatorname{ann}_A(M)} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ and $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is the set of minimal primes of $A/\operatorname{ann}_A(M)$.
(75) Let A be a noetherian ring and M be a finite A-module. Consider a chain $0 = M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n = M$ of submodules where M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for $\mathfrak{p}_i \in \operatorname{Spec} A$. Then $\operatorname{Ass} M \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Supp} M$.

(76) Let A be a noetherian ring and M be a finite A-module. Then M has finite length iff $\operatorname{Ass} M \subseteq \operatorname{mSpec} A$ iff $\operatorname{Supp} M \subseteq \operatorname{mSpec} A$.

(77) Let k be a field and consider the k-algebra $A = k[x,y,z]/(xy - z^2)$ and the ideals $I = (x,z)/(xy - z^2) \subseteq A$ and $\mathfrak{m} = (x,y,z)/(xy - z^2) \subseteq A$. Then $\mathfrak{m}$ is a maximal ideal of A and the ideal $IA_\mathfrak{m} \subseteq A_\mathfrak{m}$ is not principal. [In algebraic geometry, this is the famous example of a Weil divisor which is not Cartier.]

(78) If A is a domain, then $A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_\mathfrak{p} = \bigcap_{\mathfrak{m} \in \operatorname{mSpec} A} A_\mathfrak{m}$, where the intersections take place in the fraction field of A.

(79) Let $\operatorname{Spec} \varphi \colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the continuous map induced by the ring homomorphism $\varphi \colon A \to B$. If $\mathfrak{p} \in \operatorname{Spec} A$ then the fibre $(\operatorname{Spec} \varphi)^{-1}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} B \mid \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\}$ is canonically homeomorphic to $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$.

(80) Consider the ideal $\mathfrak{p} = (2, 1 + \sqrt{-5})$ in the ring $A = \mathbb{Z}[\sqrt{-5}]$. Prove that:
- A is finite over $\mathbb{Z}$;
- $\mathfrak{p}$ is a non-principal maximal ideal of A with residue field $\mathbb{F}_2$;
- the ideal $\mathfrak{p}A_\mathfrak{p} \subseteq A_\mathfrak{p}$ is principal.

(81) Let k be a field and let A be a finite k-algebra. Then the cardinality of $\operatorname{Spec} A$ is not greater than $\dim_k A$. [Hint: use the Chinese remainder theorem.]

(82) If B is generated as an A-module by n elements, then every fibre of $\operatorname{Spec} \varphi$ has cardinality not greater than n. [Use (79) and (81).]

(83) If B is finite over A, then every fibre of $\operatorname{Spec} \varphi$ has finitely many points.

(84) Let A be a local domain with maximal ideal $\mathfrak{m}$. Let K be the fraction field of A and let $k = A/\mathfrak{m}$. Let M be a finite A-module. Prove the following statements:
(a) $\dim_k M \otimes_A k \geq \dim_K M \otimes_A K$;
(b) M is flat A-module iff M is a free A-module iff $\dim_k M \otimes_A k = \dim_K M \otimes_A K$.

(85) Let k be a field and consider the inclusion of rings $k[x^2, x^3] \subseteq k[x]$. Is $k[x]$ finite/of finite type/flat over $k[x^2, x^3]$?

(86) Let k be a field and let A be a finite k-algebra. Show that A is an artinian ring. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the prime ideals of A. For each $i = 1, \ldots, n$, let e_i be the length of $A_{\mathfrak{m}_i}$ and let f_i be the dimension of $\kappa(\mathfrak{m}_i)$ as a k-vector space. Then

$$\dim_k A = \sum_{i=1}^{n} e_i f_i.$$

[Hint: via the structure theorem for artinian rings, reduce to the case when A is local]

(87) Let A be a ring and M be an A-module. M is flat if and only for every finitely generated ideal $I \subseteq A$ the natural map $I \otimes_A M \to M$ is injective.

(88) Let A be a PID and let M be an A-module. M is flat if and only if M is torsion free.

(89) Let A be a subring of $\mathbb{C}$ which is finite over $\mathbb{Z}$. Let $\iota \colon \mathbb{Z} \to A$ be the unique ring homomorphism from $\mathbb{Z}$ to A. We say that a prime ideal $\mathfrak{q} \in \operatorname{Spec} A$ *lies over* (or *is lying over*) $p\mathbb{Z}$, for some prime number p, if it is in the fibre $(\operatorname{Spec} \iota)^{-1}(p\mathbb{Z})$, i.e. if $\mathfrak{q} \cap \mathbb{Z} = p\mathbb{Z}$. If $\mathfrak{q} \in \operatorname{Spec} A$ lies over $p\mathbb{Z}$, then there is a natural local ring homomorphism $\mathbb{Z}_{(p)} \to A_\mathfrak{q}$, where $\mathbb{Z}_{(p)}$ denotes the localisation of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$. If $\mathfrak{q} \in \operatorname{Spec} A$ lies over $p\mathbb{Z}$, then the *ramification index* is defined to be

$$e(\mathfrak{q}/p) := \text{length of } A_\mathfrak{q}/pA_\mathfrak{q} \text{ as } A_\mathfrak{q}\text{-module}$$

and the *inertia degree* is defined to be

$$f(\mathfrak{q}/p) := \dim_{\mathbb{F}_p} \kappa(\mathfrak{q}),$$

where $\kappa(\mathfrak{q})$ denotes the residue field of $\mathfrak{q}$. Let K be the fraction field of A. Fix a prime $p \in \mathbb{Z}$ and prove the following statements.

(a) A is flat over $\mathbb{Z}$. [Hint: use (88).]

(b) A is a free $\mathbb{Z}$-module of finite rank. [Hint: use the structure theorem of finitely generated abelian groups.]

(c) The ring of fractions $(\mathbb{Z} \setminus \{0\})^{-1}A$ coincides with K. [Hint: By (21) we have that $(\mathbb{Z} \setminus \{0\})^{-1}A$ is a field, because it is a domain and is finite over $\mathbb{Q}$.]

(d) A is a free $\mathbb{Z}$-module of rank equal to $\dim_{\mathbb{Q}} K$. [Hint: use the natural isomorphism $(\mathbb{Z} \setminus \{0\})^{-1}A \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$.]

(e) If $\mathfrak{q} \in \operatorname{Spec} A$ lies over $p\mathbb{Z}$, then $\mathfrak{q}$ is a maximal ideal of A and $A_{\mathfrak{q}}/pA_{\mathfrak{q}}$ is a local artinian ring.

(f) A/pA is an artinian ring and $\dim_{\mathbb{F}_p} A/pA = \dim_{\mathbb{Q}} K$.

(g) We have an isomorphism of A-algebras

$$A/pA \simeq \prod_{\mathfrak{q} \text{ lying over } p\mathbb{Z}} A_{\mathfrak{q}}/pA_{\mathfrak{q}}.$$

(h) There is the equality:

$$\dim_{\mathbb{Q}} K = \sum_{\mathfrak{q} \text{ lying over } p\mathbb{Z}} e(\mathfrak{q}/p) \cdot f(\mathfrak{q}/p).$$

[This is a fundamental formula in algebraic number theory. Hint: consider $\dim_{\mathbb{F}_p} A/pA$ and use (86).]

[From (82) we knew that the number of primes of A lying over a fixed prime ideal of $\mathbb{Z}$ is not greater that $\dim_{\mathbb{Q}} K$. In the formula in (h) we are giving a quantitative interpretation for the difference between $\dim_{\mathbb{Q}} K$ and the number of primes of A lying over a fixed prime ideal of $\mathbb{Z}$. Also, the formula in (h) has a geometric meaning for the morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ which is similar, but slightly more complicated, to the theory of non-constant holomorphic maps between Riemann surfaces.]

(90) Consider the ring $A = \mathbb{Z}[i]$ of Gaussian integers. Prove that A is finite over $\mathbb{Z}$. For every prime $p \in \mathbb{Z}$, study the primes of A which lie over $p\mathbb{Z}$, their ramification indexes and their inertia degrees.

(91) Consider the ring $A = \mathbb{Z}[\sqrt[3]{2}]$. Prove that A is finite over $\mathbb{Z}$. Show the following statements.

(a) There is a unique prime of A lying over $2\mathbb{Z}$; it has ramification index 3 and inertia degree 1.

(b) There is a unique prime of A lying over $3\mathbb{Z}$; it has ramification index 3 and inertia degree 1.

(c) There are two primes of A lying over $5\mathbb{Z}$ and their ramification indexes and inertia degrees are $(e, f) = (1, 1)$ and $(e, f) = (1, 2)$.

(d) There is a unique prime of A lying over $7\mathbb{Z}$; it has ramification index 1 and inertia degree 3.