<u>Mapping cones</u> [Weibel, Intro to homological algebra, §1.5]

$f: A \to B$ hom. of chain complexes

$\mathrm{Cone}(f)$ is the complex defined by $\mathrm{Cone}(f)_i := B_i \oplus A_{i-1}$ with differential $d_i^{\mathrm{Cone}(f)} : B_i \oplus A_{i-1} \to B_{i-1} \oplus A_{i-2}$ given by $\begin{pmatrix} d_i^B & -f_{i-1} \\ 0 & -d_{i-1}^A \end{pmatrix}$, i.e. $d_i^{\mathrm{Cone}(f)}(b,a) = (d_i^B b - f_{i-1} a, -d_{i-1}^A a)$.

There is a short exact sequence of complexes

$$0 \to B \to \mathrm{Cone}(f) \longrightarrow A[-1] \longrightarrow 0$$

where the left map sends b to $(b,0)$ and the right map sends (b,a) to $-a$. We consider the long exact sequence in homology

$$\begin{tikzcd}
 & {[(b,a)]} \arrow[r, mapsto] & {[-a]} & \\
H_i(B) \arrow[r] & H_i(\mathrm{Cone}(f)) \arrow[r] & H_i(A[-1]) \arrow[r] \arrow[d, equal] & H_{i-1}(B) \\
{[b]} \arrow[r, mapsto] & {[b,a]} & H_{i-1}(A) \arrow[ur] &
\end{tikzcd}$$

connecting hom.

want to show that this is $H_{i-1}(f)$

Let $a \in Z_{i-1}(A)$. Then $(0,-a)$ is a preimage in $\mathrm{Cone}(f)_i$. Let's apply the differential of $\mathrm{Cone}(f)$:

$$d_i^{\mathrm{Cone}(f)}(0,-a) = -f_{i-1}(-a) - d_{i-1}^A(-a) = f_{i-1}(a)$$

$a \in Z_{i-1}(A)$

Therefore the connecting hom. $H_{i-1}(A) \to H_{i-1}(B)$ is $[a] \mapsto [f_{i-1}(a)]$, i.e. it is $H_{i-1}(f)$.

So get long exact sequence

$$\to H_i(A) \xrightarrow{H_i(f)} H_i(B) \longrightarrow H_i(\mathrm{Cone}(f)) \longrightarrow H_{i-1}(A) \to \dots$$

$$[b] \longmapsto [b,0]$$

$$[b,a] \longmapsto [-a]$$

Problem 44a

$0 \to A \xrightarrow{f} B \xrightarrow{p} C \to 0$ s.e.s. of chain complexes

$\Phi : \mathrm{Cone}(f) \longrightarrow C$ defined by

$\Phi_i : B_i \oplus A_{i-1} \to C_i$

$\quad (b, a) \longmapsto p_i b$

Φ is a chain hom. : need to show that

$$\begin{tikzcd} B_i \oplus A_{i-1} \arrow[r, "\Phi_i"] \arrow[d, "d_i^{\mathrm{Cone}(f)}"'] & C_i \arrow[d, "d_i^C"] \\ B_{i-1} \oplus A_{i-2} \arrow[r, "\Phi_{i-1}"'] & C_{i-1} \end{tikzcd}$$

commutes

pick $(b, a) \in B_i \oplus A_{i-1}$.

$d_i^C \, \Phi_i(b,a) = d_i^C(p_i b)$

$\Phi_{i-1} \, d_i^{\mathrm{Cone}(f)}(b,a) = \Phi_{i-1}(d_i^B b - f_{i-1} a, -d_{i-1}^A a) =$

$= p_{i-1}(d_i^B b - f_{i-1} a) = p_{i-1} d_i^B b = d_i^C(p_i b)$

$\uparrow p_{i-1} \circ f_{i-1} = 0$ $\qquad$ $\uparrow$ p is a hom. of complexes

Now we wantto compare the l.e.s. induced by $0 \to A \to B \to C \to 0$ and the long exact sequence induced by $0 \to B \to \mathrm{Cone}(f) \to A[-1] \to 0$:

$$\begin{tikzcd} H_i(A) \arrow[r, "H_i(f)"] \arrow[d, "id"] & H_i(B) \arrow[r] \arrow[d, "id"] \arrow[dr, phantom, "(*)"] & H_i(\mathrm{Cone}(f)) \arrow[r] \arrow[d, "H_i(\Phi)"] \arrow[dr, phantom, "(**)"] & H_{i-1}(A) \arrow[r, "H_{i-1}(f)"] \arrow[d, "id"] & H_{i-1}(B) \arrow[d, "id"] \\ H_i(A) \arrow[r, "H_i(f)"'] & H_i(B) \arrow[r, "H_i(p)"'] & H_i(C) \arrow[r, "-\partial_i"'] & H_{i-1}(A) \arrow[r, "H_{i-1}(f)"'] & H_{i-1}(B) \end{tikzcd}$$

opposite of connecting hom. ∂_i

Need to show that the two squares $(*)$ and $(**)$ commutes:

$(*)$:

$$\begin{tikzcd} & H_i(\mathrm{Cone}(f)) \arrow[d, "H_i(\Phi)"] \\ H_i(B) \arrow[ur] \arrow[r, "H_i(p)"'] & H_i(C) \end{tikzcd} \qquad \begin{tikzcd} & {[b, 0]} \arrow[d, mapsto] \\ {[b]} \arrow[ur, mapsto] \arrow[r, mapsto] & {[p_i b]} \end{tikzcd}$$

ok

(**)
$$\begin{tikzcd} H_i(\mathrm{Cone}(f)) \arrow[d, "H_i(\Phi)"'] \arrow[dr] & \\ H_i(C) \arrow[r, "-\partial_i"'] & H_{i-1}(A) \end{tikzcd}$$

Start from $(b,a) \in Z_i(\mathrm{Cone}(f))$. It is clear that $H_i(\Phi)[b,a] = [\Phi_i(b,a)] = [p_i b]$, whereas the map $H_i(\mathrm{Cone}(f)) \to H_{i-1}(A)$ sends $[b,a]$ to $[-a]$.

So we need to prove that $-\partial_i[p_i b] = [-a]$, i.e. $\partial_i[p_i b] = [a]$.

Since (b,a) is a cycle, we have $d_i^{\mathrm{Cone}(f)}(b,a) = 0$, i.e. $d_i^B b - f_{i-1}a = 0$ and $-d_{i-1}^A a = 0$, i.e. $f_{i-1}a = d_i^B b$ and a is a cycle. From the definition of connecting homomorphism, it is easy to see that $\partial_i[p_i b] = [a]$.

This concludes the proof of the fact that the diagram with the 2 rows with 5 columns is commutative.

By 5-lemma, we have that $H_i(\Phi)$ is an isomorphism for every i. Therefore Φ is a quasi-isomorphism.

Problem 44 b

$0 \to A \xrightarrow{f} B \xrightarrow{p} C \to 0$ s.e.s. of chain complexes

We know that Φ: Cone$(f) \longrightarrow C$, $(b, a) \mapsto pb$ is a quasi-isom. of complexes.

Let's suppose that each layer $0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{p_i} C_i \to 0$ splits, so we have sections $s_i : C_i \longrightarrow B_i$ s.t. $p_i s_i = id_{C_i}$.

Notice that the collection of s_i is not, in general, a morphism of complexes from C to B.

We want to show that in this case Φ is a homotopic equivalence, i.e. there exists a morphism of complexes $\Psi : C \longrightarrow$ Cone(f) such that $\Phi\Psi$ is homotopic to id_C and $\Psi\Phi$ is homotopic to $id_{\text{Cone}(f)}$.

I will suppress the letter f in all formulae

From now on, I will assume that A is a subcomplex of in other words B, i.e. that f is an inclusion.

For every i, define $\Psi_i : C_i \longrightarrow$ Cone$(f)_i$ by $\Psi_i c = (s_i c, (d_i^B s_i - s_{i-1} d_i^C)c)$ $\forall c \in C_i$. This makes sense because $(d_i^B s_i - s_{i-1} d_i^C)c$ is an element of A_{i-1} since $p_{i-1}(d_i^B s_i - s_{i-1} d_i^C)c = (d_i^C p_i s_i - p_{i-1} s_{i-1} d_i^C)c =$
$= (d_i^C - d_i^C)c = 0 \Rightarrow (d_i^B s_i - s_{i-1} d_i^C)c \in \ker p_{i-1} = A_{i-1}$.

Ψ is a morph. of complexes: need to show that the square

$$\begin{tikzcd} C_i \arrow[r, "\Psi_i"] \arrow[d, "d_i^C"'] & \text{Cone}(f)_i \arrow[d, "d_i^{\text{Cone}(f)}"] \\ C_{i-1} \arrow[r, "\Psi_{i-1}"'] & \text{Cone}(f)_{i-1} \end{tikzcd}$$

commutes: for every $c \in C_i$ we have

$$d_i^{Cone(f)} \Psi_i c = d_i^{Cone(f)} \left(s_i c, (d_i^B s_i - s_{i-1} d_i^C) c \right) =$$

$$= \left(d_i^B s_i c - (d_i^B s_i - s_{i-1} d_i^C) c, -d_{i-1}^B (d_i^B s_i - s_{i-1} d_i^C) c \right)$$

$$= (s_{i-1} d_i^C c, d_{i-1}^B s_{i-1} d_i^C c)$$

$$\Psi_{i-1} d_i^C c = (s_{i-1} d_i^C c, (d_{i-1}^B s_{i-1} - s_{i-2} d_{i-1}^C) d_i^C c)$$

$$= (s_{i-1} d_i^C c, d_{i-1}^B s_{i-1} d_i^C c)$$

equal!

$\Phi \Psi = id_C$ because $\forall i, \forall c \in C_i$ $\Phi_i \Psi_i c = \Phi_i(s_i c, ?) =$
$= p_i s_i c = c$.

What is $\Psi \Phi$? $\forall i, \forall (b, a) \in Cone(f)_i$

$$\Psi_i \Phi_i (b, a) = \Psi_i (p_i b) = \left(s_i p_i b, (d_i^B s_i - s_{i-1} d_i^C) p_i b \right) =$$

$$= \left(s_i p_i b, (d_i^B s_i p_i - s_{i-1} p_{i-1} d_i^B) b \right)$$

Construct

$$h_i : Cone(f)_i \longrightarrow Cone(f)_{i+1}$$
$$(b, a) \longmapsto (0, b - s_i p_i b)$$

this is well defined because $b - s_i p_i b \in A_i$
since $p_i(b - s_i p_i b) = p_i b - p_i s_i p_i b = p_i b - p_i b = 0$

$$d_{i+1}^{Cone(f)} h_i (b, a) = d_{i+1}^{Cone(f)} (0, b - s_i p_i b) =$$

$$= \left(-(b - s_i p_i b), -d_i^B (b - s_i p_i b) \right)$$

$$= \left((s_i p_i - id_{B_i}) b, d_i^B (s_i p_i - id_{B_i}) b \right)$$

$$h_{i-1} d_i^{Cone(f)} (b, a) = h_{i-1} (d_i^B b - a, -d_{i-1}^B a) =$$

$$= \left(0, d_i^B b - a - s_{i-1} p_{i-1} (d_i^B b - a) \right)$$

$$= \left(0, d_i^B b - a - s_{i-1} p_{i-1} d_i^B b \right)$$

Then

$$\left(d_{i+1}^{cone(f)} h_i + h_{i-1} d_i^{cone(f)}\right)(b,a) =$$

$$= \left(s_i p_i b - b,\ d_i^B s_i p_i b - a - s_{i-1} p_{i-1} d_i^B b\right)$$

$$= \left(\Psi_i \Phi_i - id_{\,cone(f)_i}\right)(b,a)$$

$\Rightarrow$ the collection $\{h_i\}$ is a homotopy between $\Psi\Phi$ and $id_{cone(f)}$.