SUMMARY OF EXERCISE CLASSES OF ALGEBRA I

FREIE UNIVERSITÄT BERLIN, WINTER SEMESTER 2018/19

Andrea Petracci
andrea.petracci@fu-berlin.de
https://userpage.fu-berlin.de/petracci/201819Algebra1

- Mi 17. Okt. (Prime) ideals, operations of ideals, invertible elements, nilpotent elements, zero-divisors in the rings $\mathbb{Z}$ and $\mathbb{Z}/60\mathbb{Z}$. In a ring with finitely many elements, each element is either invertible or a zero-divisor. Recap of inductively ordered sets and Zorn lemma. An ideal which is maximal among the ideals which contain a given ideal and are disjoint from a multiplicative set is prime. An ideal which is maximal among non finitely generated ideals is prime (left as exercise). In a ring, all ideals are finitely generated if and only if all prime ideals are finitely generated. Every prime ideal contains a minimal prime ideal (left as exercise; hint: consider the poset of prime ideals with the reversed inclusion).
- Mo 22. Okt. Corrections of Problems 1, 3, 4. Extensions and contractions of prime/maximal ideals. Spec is a functor. If A is a UFD, then $A[x]$ is a UFD; Gauss' lemma (statement). The polynomial $y^2 - x^3$ is irreducible in $k[x, y]$, when k is a field. Recap of category theory; initial objects and examples. If F is a functor from a category $\mathcal{A}$ to the category of sets and $\mathcal{C}$ is the category cofibred over $\mathcal{A}$ associated to F, then $\mathcal{C}$ has an initial object if and only if F is corepresentable.
- Mi 24. Okt. Corrections of Problems 1, 2, 3, 4. Spec is a functor. The functor $\mathrm{h}^X = \mathrm{Hom}(X, \cdot)$. Definition of corepresentable functor and universal pair. If F is a functor from a category $\mathcal{A}$ to the category of sets and $\mathcal{C}$ is the category cofibred over $\mathcal{A}$ associated to F, then $\mathcal{C}$ has an initial object if and only if F is corepresentable.
- Mo 29. Okt. Corrections of Problems 5 and 8. Definition of full, faithful and fully faithful functors. Definition of functor represented by an object: $\mathrm{h}_X = \mathrm{Hom}(\cdot, X)$. Definition of natural transformation and of category of functors. Yoneda lemma. If a is an element in the ring A, then $A[x]/(x-a)$ is isomorphic to A. How to decompose explicit rings in products of fields: $\mathbb{Z}[i]/(3)$, $\mathbb{Z}[i]/(5)$ and $\mathbb{Z}[\sqrt[3]{2}]/(5)$.
- Mi 31. Okt. Corrections of Problems 5 and 8. Definition of opposite category. Definition of full, faithful and fully faithful functors. Definition of natural transformation and of category of functors. Definition of functor represented by an object: $\mathrm{h}_X = \mathrm{Hom}(\cdot, X)$. Yoneda lemma. Nilpotents, units and zero-divisors in $A[x]$. Nilpotents and units in $A[\![x]\!]$. The ring $\mathbb{Z}[i]$ is isomorphic to $\mathbb{Z}[x]/(x^2 + 1)$. If f is the minimal polinomial of a over the field K, then $K[a]$ is isomorphic to $K[x]/(f)$. How to decompose explicit rings in products of fields: $\mathbb{Z}[i]/(3)$, $\mathbb{Z}[i]/(5)$ and $\mathbb{Z}[\sqrt[3]{2}]/(5)$.
- Mo 5. Nov. Definition of adjoint functors. The tensor product corepresents the functor of bilinear forms; the tensor product functor is adjoint to the Hom functor. Correction of Problems 10 and 12a.

- Mi 7. Nov. Corrections of Problems 9, 10, 12a, 12b. If I is an ideal in a ring A, then A/I has only one prime ideal if and only if the radical of I is maximal. Definition of graded ring w.r.t. a commutative monoid, definition of homogeneous ideal; an ideal I is homogeneous if and only if all homogeneous parts of every element of I lie in I.
- Mo 12. Nov. (Exercise session led by Karin Schaller) Correction of Problem 13. Several proofs of the fact that $M \otimes_A A/I$ is isomorphic to M/IM. Comparison between $I \otimes_A M$ and IM.
- Mi 14. Nov. Definition of adjoint functors. The tensor product corepresents the functor of bilinear forms; the tensor product functor is adjoint to the Hom functor. Extension and restriction of scalars with respect to a ring homomorphism are adjoint functors. Correction of Problem 14. Comparison between IM and $I \otimes_A M$. The element $2 \otimes_{\mathbb{Z}} (1 + 2\mathbb{Z})$ is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, but is not zero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Several proofs of the fact that $M \otimes_A A/I$ is isomorphic to M/IM. Statement of snake lemma. Correction of Problem 13. The natural map from $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ to $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R}$ is surjective, but not injective; $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathbb{R}$ as a $\mathbb{Q}$-vector space; the two natural $\mathbb{R}$-vector space structures on $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ are different from each other and they are not isomorphic to the $\mathbb{R}$-vector space $\mathbb{R}$.
- Mo 19. Nov. Relationship between tensor product and direct product/sum of modules. Correction of Problem 17. If $A \to B$ a ring homomorphism, then the natural map $B \otimes_A A[\![x]\!] \to B[\![x]\!]$ needn't be injective nor surjective; the subrings $A[\![x]\!][y]$ and $A[y][\![x]\!]$ of $A[\![x, y]\!]$ are different; if I is a non finitely generated ideal of the ring A, then the extended ideal of I to $A[\![x]\!]$ can be strictly smaller than $I[\![x]\!]$. If B and C are A-algebras, then the tensor product $B \otimes_A C$ has a natural A-algebra structure. Base change of an algebra of finite type: if $A \to B$ is a ring homomorphism and I is an ideal in $A[x_1, ..., x_n]$, then $B \otimes_A A[x_1, ..., x_n]/I$ is isomorphic to $B[x_1, ..., x_n]/I^e$, where I^e is the extended ideal. The $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C} \times \mathbb{C}$. The $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2})$ is isomorphic to the direct product of two fields. If $K' \supseteq K$ is a field extension and $L \supseteq K$ is a finite separable field extension, then $K' \otimes_K L$ is a product of fields (left as an exercise); counterexample when $L \supseteq K$ is not separable: if $K = \mathbb{F}_p(t)$ and $L = \mathbb{F}_p(\sqrt[p]{t})$ then $L \otimes_K L$ is isomorphic to $L[x]/(x^p)$.
- Mi 21. Nov. If $\{M_\lambda\}_{\lambda \in \Lambda}$ is a family of A-modules, then the direct sum $\bigoplus_{\lambda \in \Lambda} M_\lambda$ represents the functor $\prod_{\lambda \in \Lambda} \mathrm{Hom}_A(M_\lambda, \cdot)$ and the direct product $\prod_{\lambda \in \Lambda} M_\lambda$ represents the functor $\prod_{\lambda \in \Lambda} \mathrm{Hom}_A(\cdot, M_\lambda)$. Relationship between tensor product and direct product/sum of modules. Correction of Problem 17. If $A \to B$ a ring homomorphism, then the natural map $B \otimes_A A[\![x]\!] \to B[\![x]\!]$ needn't be injective nor surjective; the subrings $A[\![x]\!][y]$ and $A[y][\![x]\!]$ of $A[\![x, y]\!]$ are different; if I is a non finitely generated ideal of the ring A, then the extended ideal of I to $A[\![x]\!]$ can be strictly smaller than $I[\![x]\!]$. Base change of an algebra of finite type: if $A \to B$ is a ring homomorphism and I is an ideal in $A[x_1, ..., x_n]$, then $B \otimes_A A[x_1, ..., x_n]/I$ is isomorphic to $B[x_1, ..., x_n]/I^e$, where I^e is the extended ideal. The $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C} \times \mathbb{C}$. The $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2})$ is isomorphic to $\mathbb{Q}(\sqrt[3]{2}) \times \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. If $K' \supseteq K$ is a field extension and $L \supseteq K$ is a finite separable field extension, then $K' \otimes_K L$ is a product of fields (left as an exercise); counterexample when $L \supseteq K$ is not separable: if $K = \mathbb{F}_p(t)$ and $L = \mathbb{F}_p(\sqrt[p]{t})$ then $L \otimes_K L$ is isomorphic to $L[x]/(x^p)$. Definition of torsion submodule for a module over a domain. If A is an integral domain, M is

an A-module and K is the fraction field of A, then the torsion submodule M^{tor} is the kernel of the natural map $M \to M \otimes_A K$.

- Mo 26. Nov. Exercises 3.7 and 3.8 in the book by Atiyah and MacDonald; definition of a saturated multiplicative subset in a ring; a subset of a ring is multiplicative and saturated if and only if its complement is a union of prime ideals; if $S \subseteq T \subseteq A$ are multiplicative subsets, then the natural ring homomorphism $S^{-1}A \to T^{-1}A$ is bijective if and only if T is contained in the saturation of S. Explicit study of all multiplicative subsets and all localisations of the ring $\mathbb{Z}/6\mathbb{Z}$; correction of Problem 19a. If A and B are two rings, then all ideals of $A \times B$ are of the form $I \times J$ for $I \subseteq A$, $J \subseteq B$ ideals; $(A \times B)/(I \times J) \simeq A/I \times B/J$; description of prime ideals of $A \times B$; the spectrum of $A \times B$ is homeomorphic to the disjoint union of $\operatorname{Spec} A$ and $\operatorname{Spec} B$; if $\mathfrak{p} \in \operatorname{Spec} A$ then $(A \times B)_{\mathfrak{p} \times B} \simeq A_{\mathfrak{p}}$. Definition of torsion submodule for a module over a domain. If A is a domain, M is an A-module and K is the fraction field of A, then the torsion submodule M^{tor} is the kernel of the natural map $M \to M \otimes_A K$. If $(A, \mathfrak{m}, k)$ is a local ring and M is a finite A-module, then every minimal set of generators of M has cardinality $\dim_k M \otimes_A k$. Definition of residue field of a prime ideal in a ring: $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Frac}(A/\mathfrak{p}) = A_{\mathfrak{p}} \otimes_A A/\mathfrak{p}$. If $\mathfrak{m}$ is a maximal ideal in the ring A and M is an A-module such that $\mathfrak{m} \subseteq \operatorname{ann}_A(M)$, then the localisation map $M \to M_{\mathfrak{m}}$ is an isomorphism. If $\mathfrak{m}$ is a maximal ideal in the ring A, then for each integer $i \geq 0$ the natural map $\mathfrak{m}^i/\mathfrak{m}^{i+1} \to \mathfrak{m}^i A_{\mathfrak{m}}/\mathfrak{m}^{i+1}A_{\mathfrak{m}}$ is an isomorphism. Correction of Problem 12c. If $f \in \mathbb{C}[x, y]$ is a non-zero polynomial such that $f(0, 0) = 0$, $A = \mathbb{C}[x, y]/(f)$ and $\mathfrak{m} = (x, y)/(f) \subseteq A$, then every minimal set of generators of the ideal $\mathfrak{m}A_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ has cardinality equal to 1 (resp. 2) if $\nabla f(0, 0) \neq (0, 0)$ (resp. $\nabla f(0, 0) = (0, 0)$) (left as an exercise). Quick geometric interpretation about the last exercise and smoothness of plane algebraic curves.
- Mi 28. Nov. Exercises 3.7 and 3.8 in the book by Atiyah and MacDonald; definition of a saturated multiplicative subset in a ring; a subset of a ring is multiplicative and saturated if and only if its complement is a union of prime ideals; if $S \subseteq T \subseteq A$ are multiplicative subsets, then the natural ring homomorphism $S^{-1}A \to T^{-1}A$ is bijective if and only if T is contained in the saturation of S. Explicit study of all multiplicative subsets and all localisations of the ring $\mathbb{Z}/6\mathbb{Z}$; correction of Problem 19a. A localisation $S^{-1}A$ is zero if and only if $0 \in S$. If $S \subseteq A$ is multiplicative and $f\colon A \to B$ is a ring homomorphism such that $f(S) \subseteq B^*$, then the induced homomorphism $g\colon S^{-1}A \to B$ is such that $\ker g = S^{-1} \ker f$. If A and B are two rings, then all ideals of $A \times B$ are of the form $I \times J$ for $I \subseteq A$, $J \subseteq B$ ideals; $(A \times B)/(I \times J) \simeq A/I \times B/J$; description of prime ideals of $A \times B$; the spectrum of $A \times B$ is homeomorphic to the disjoint union of $\operatorname{Spec} A$ and $\operatorname{Spec} B$; if $\mathfrak{p} \in \operatorname{Spec} A$ then $(A \times B)_{\mathfrak{p} \times B} \simeq A_{\mathfrak{p}}$. If $(A, \mathfrak{m}, k)$ is a local ring and M is a finite A-module, then every minimal set of generators of M has cardinality $\dim_k M \otimes_A k$. Definition of residue field of a prime ideal in a ring: $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Frac}(A/\mathfrak{p}) = A_{\mathfrak{p}} \otimes_A A/\mathfrak{p}$. If $\mathfrak{m}$ is a maximal ideal in the ring A and M is an A-module such that $\mathfrak{m} \subseteq \operatorname{ann}_A(M)$, then the localisation map $M \to M_{\mathfrak{m}}$ is an isomorphism. If $\mathfrak{m}$ is a maximal ideal in the ring A, then for each integer $i \geq 0$ the natural map $\mathfrak{m}^i/\mathfrak{m}^{i+1} \to \mathfrak{m}^i A_{\mathfrak{m}}/\mathfrak{m}^{i+1}A_{\mathfrak{m}}$ is an isomorphism. Correction of Problem 12c. If $f \in \mathbb{C}[x, y]$ is a non-zero polynomial such that $f(0, 0) = 0$, $A = \mathbb{C}[x, y]/(f)$ and $\mathfrak{m} = (x, y)/(f) \subseteq A$, then every minimal set of generators of the ideal $\mathfrak{m}A_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ has cardinality equal to 1 (resp. 2) if $\nabla f(0, 0) \neq (0, 0)$ (resp. $\nabla f(0, 0) = (0, 0)$) (left as

an exercise). Quick geometric interpretation about the last exercise and smoothness of plane algebraic curves.

- Mo 3. Dez. Correction of Problem 21. If A is a noetherian ring, then $A[\![x]\!]$ is a noetherian ring. If p is a prime number, then the ring $\mathbb{Z}_p := \mathbb{Z}[\![x]\!]/(x-p)$ of p-adic numbers is noetherian. Correction of Problem 23. If A is a noetherian domain and M is a finite A-module with $\mathrm{Ass}_A(M) = \{0\}$, then M is clean (i.e. there exists a filtration where each quotient of two consecutive submodules is isomorphic to the quotient of A with respect to an associated prime of M) if and only if M is free. Definition of split short exact sequence: existence of a compatible isomorphism to the direct sum, existence of a section of the surjection, existence of a retraction of the injection. A short exact sequence splits if the third module is free. Correction of Problem 28. Definition of monomial ideals: generated by monomials, or such that if a polynomial lies in the ideal then every term lies in the ideal. Two monomial ideals coincide if and only if they contain the same monomials. If $I = (m_i)_i$ and $J = (n_j)_j$ are monomial ideals in a polynomial ring over a field with finitely many variables, then $I + J = (m_i, n_j)_{i,j}$, $IJ = (m_i n_j)_{i,j}$, $I \cap J = (\mathrm{lcm}(m_i, n_j))_{i,j}$, $\sqrt{I} = (\sqrt{m_i})_i$, and $(I : J) = \bigcap_j (m_i/\gcd(m_i, n_j))_i$.
- Mi 5. Dez. Correction of Problem 21. Definition of split short exact sequence: existence of a compatible isomorphism to the direct sum, existence of a section of the surjection, existence of a retraction of the injection. A short exact sequence splits if the third module is free. A finite module M over a noetherian ring A is called clean if there exists a filtration $M = M_n \supseteq M_{n-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$ such that M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Ass}\, M$. If A is a noetherian domain and M is a finite A-module with $\mathrm{Ass}_A(M) = \{0\}$, then M is clean if and only if M is free. If A is a ring and $I \subseteq A$ is an ideal which is not principal, then I is not a free A-module. If A is a noetherian domain and $I \subseteq A$ is an ideal which is not principal, then I is not a clean A-module. Correction of Problem 23. Correction of Problem 28. If A is a noetherian ring, then $A[\![x]\!]$ is a noetherian ring (without proof). If p is a prime number, then the ring $\mathbb{Z}_p := \mathbb{Z}[\![x]\!]/(x-p)$ of p-adic numbers is noetherian. There is a natural injective ring homomorphism $\mathbb{Z}_{(p)} \to \mathbb{Z}_p$, where $\mathbb{Z}_{(p)}$ is the localisation of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$ (left as exercise). Recap about monomial ideals: an ideal $I \subseteq k[x_1, \dots, x_n]$ is generated by some monomials if and only if whenever $f \in I$ all monomials appearing in f lie in I. If $\mathcal{S}$ is a set of monomial generators of a monomial ideal I and u is a monomial, then $u \in I$ iff there exists $s \in \mathcal{S}$ such that $s|u$. Two monomial ideals coincide if and only if they contain the same monomials. Existence and uniqueness of the minimal monomial basis of a monomial ideal. From any set of monomial generators of a monomial ideal it is possible to extract the minimal monomial basis of that ideal. If I and J are monomial ideals, then $I + J$, IJ, $I \cap J$, $\sqrt{I}$ and $(I : J)$ are monomial ideals; more precisely if $\{m_i\}_i$ is a set of monomial generators of I and $\{n_j\}_j$ is a set of monomial generators of J, then $I + J = (m_i, n_j)_{i,j}$, $IJ = (m_i n_j)_{i,j}$, $I \cap J = (\mathrm{lcm}(m_i, n_j))_{i,j}$, $\sqrt{I} = (\sqrt{m_i})_i$, and $(I : J) = \bigcap_j (m_i/\gcd(m_i, n_j))_i$.
- Mo 10. Dez. Recap on operations of monomial ideals. Two monomial ideals coincide if and only if they contain the same monomials. Existence and uniqueness of the minimal monomial basis of a monomial ideal. If I is a monomial ideal in $k[x_1, \dots, x_n]$ with minimal monomial basis $\mathcal{B}$ then: I is maximal $\Leftrightarrow \mathcal{B} = \{x_1, \dots, x_n\}$; I is prime $\Leftrightarrow \mathcal{B} \subseteq \{x_1, \dots, x_n\}$; I is radical

$\Leftrightarrow \mathcal{B} = \{0,1\}^n$, i.e. in each monomial in $\mathcal{B}$ every variable appears with exponent 0 or 1; I is irreducible $\Leftrightarrow \mathcal{B}$ consists of powers of some variables (the proof of $\Leftarrow$ is omitted); I is primary $\Leftrightarrow \mathcal{B} = \{x_{i_1}^{a_1}, \dots, x_{i_r}^{a_r}, m_1, \dots, m_s\}$ where $1 \leq i_1 < \cdots < i_r \leq n$, $a_1, \dots, a_r \geq 1$ and each m_j is a monomial in $x_{i_1}, \dots, x_{i_r}$. If $\mathfrak{m}$ is a maximal ideal in a ring A and $I \subseteq A$ is an ideal such that $\sqrt{I} = \mathfrak{m}$, then I is $\mathfrak{m}$-primary. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$. If $m_1, \dots, m_r, u, v$ are monomials such that $\gcd(u,v) = 1$, then $(m_1, \dots, m_r, uv) = (m_1, \dots, m_r, u) \cap (m_1, \dots, m_r, v)$. Algorithm to express a monomial ideal as intersection of irreducible monomial ideals. Primary decomposition of monomial ideals. Correction of Problem 30. If A is a noetherian ring, then $\dim A[x] = 1 + \dim A$ (proof omitted). Irreducible closed subsets of $\operatorname{Spec} A$ are in one-to-one correspondence with points of $\operatorname{Spec} A$ (proof omitted). Irreducible components of $\operatorname{Spec} A$ are in one-to-one correspondence with minimal primes of A. If A is a noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are its minimal primes, then $\dim A = \max_{1 \leq i \leq n} \dim A/\mathfrak{p}_i$ (the dimension of $\operatorname{Spec} A$ is the maximum of the dimension of its irreducible components). Examples of computing the dimension and the associated/minimal/embedding primes of the quotient of a polynomial ring over a field modulo a monomial ideal. Geometric interpretation.

- Mi 12. Dez. If I is a monomial ideal in $k[x_1, \dots, x_n]$ with minimal monomial basis $\mathcal{B}$ then: I is maximal $\Leftrightarrow \mathcal{B} = \{x_1, \dots, x_n\}$; I is prime $\Leftrightarrow \mathcal{B} \subseteq \{x_1, \dots, x_n\}$; I is radical $\Leftrightarrow \mathcal{B} = \{0,1\}^n$, i.e. in each monomial in $\mathcal{B}$ every variable appears with exponent 0 or 1; I is irreducible $\Leftrightarrow \mathcal{B}$ consists of powers of some variables (the proof of $\Leftarrow$ is omitted); I is primary $\Leftrightarrow \mathcal{B} = \{x_{i_1}^{a_1}, \dots, x_{i_r}^{a_r}, m_1, \dots, m_s\}$ where $1 \leq i_1 < \cdots < i_r \leq n$, $a_1, \dots, a_r \geq 1$ and each m_j is a monomial in $x_{i_1}, \dots, x_{i_r}$. If $\mathfrak{m}$ is a maximal ideal in a ring A and $I \subseteq A$ is an ideal such that $\sqrt{I} = \mathfrak{m}$, then I is $\mathfrak{m}$-primary. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$. If $m_1, \dots, m_r, u, v$ are monomials such that $\gcd(u,v) = 1$, then $(m_1, \dots, m_r, uv) = (m_1, \dots, m_r, u) \cap (m_1, \dots, m_r, v)$. Algorithm to express a monomial ideal as intersection of irreducible monomial ideals. Primary decomposition of monomial ideals. If A is a noetherian ring, then $\dim A[x] = 1 + \dim A$ (proof omitted). Irreducible closed subsets of $\operatorname{Spec} A$ are in one-to-one correspondence with points of $\operatorname{Spec} A$ (proof omitted). Irreducible components of $\operatorname{Spec} A$ are in one-to-one correspondence with minimal primes of A. If A is a noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are its minimal primes, then $\dim A = \max_{1 \leq i \leq n} \dim A/\mathfrak{p}_i$ (the dimension of $\operatorname{Spec} A$ is the maximum of the dimension of its irreducible components). Examples of computing the dimension and the associated/minimal/embedding primes of the quotient of a polynomial ring over a field modulo a monomial ideal. Geometric interpretation.
- Mo 17. Dez. Correction of Problem 36; if $A \subseteq B$ is an integral extension, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and closed. Definition of normal domain. Correction of Problem 34: a UFD is a normal domain. Definition of semigroup algebra associated to a commutative monoid. Definition of the Grothendieck group associated to a commutative monoid. The intersection of normal domains with the same fraction field is a normal domain. Correction of Problem 35. A proposition on the integral closure in a quadratic extension: if A is a UFD with fraction field K, $2 \neq 0$ in A, α is an element of the algebraic closure of K such that α^2 is equal to an element $a \in A$ which is not a square in A and is square-free, then the integral closure B of A in $K(\alpha)$ is such that $A[\alpha] \subseteq B \subseteq \{c_0 + c_1\alpha \mid c_0, c_1 \in K \text{ s.t. } 2c_0, 2c_1, c_0^2 - ac_1^2 \in A\}$;

if in addition 2 is invertible in A, then $B = A[\alpha]$. If k is a field of characteristic different from 2 and $f \in k[x]$ is a non-constant polynomial which is square-free, then $k[x,y]/(y^2 - f(x))$ is a normal domain. If k is a field, then the domain $k[x,y]/(y^2 - x^3)$ is not normal. Integral closure of $\mathbb{Z}$ in the quadratic field extensions of $\mathbb{Q}$ (left as an exercise).

- Mi 19. Dez. Correction of Problem 36; if $A \subseteq B$ is an integral extension, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and closed. Definition of normal domain. Correction of Problem 34: a UFD is a normal domain. Definition of semigroup algebra associated to a commutative monoid. The intersection of normal domains with the same fraction field is a normal domain. Correction of Problem 35.
- Mo 7. Jan. Recap on finitely presented modules and projective modules. Correction of Problem 40. If A is non-zero ring and $A^n \to A^m$ is an injective (resp. surjective) A-linear morphism, then $n \leq m$ (resp. $n \geq m$). Correction of Problem 38. Correction of Problem 37a.
- Mi 9. Jan. Recap on projective modules. If A is a ring and M is a finitely presented module, then the following conditions are equivalent: M is projective, M is flat, there exist $f_1, \dots, f_r \in A$ such that $(f_1, \dots, f_r) = A$ and M_{f_i} is a free A_{f_i}-module for each $i = 1, \dots, r$, for every $\mathfrak{p} \in \operatorname{Spec} A$ $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, for every $\mathfrak{m} \in \operatorname{Specm} A$ $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module (statement and the proof of some of the implications). A finite projective module over a local ring is free. Correction of Problem 40a and 40b. In a ring two non-zero elements are always linearly dependent. Correction of Problem 38a and 38b.
- Mo 14. Jan. Recap on the connecting homomorphism in the long exact sequence in homology and on the cone complex. Correction of Problems 44 and 42. Computation of $\operatorname{Tor}_i^{\mathbb{Z}}(M,N)$ when M is a finitely generated abelian group and N is an arbitrary abelian group.
- Mi 16. Jan. Recap on properties of Tor. Computation of $\operatorname{Tor}_i^{\mathbb{Z}}(M,N)$ when M and N are finitely generated abelian groups. Recap on the connecting homomorphism in the long exact sequence in homology and on the cone complex. Correction of Problem 44a,b.
- Mo 21. Jan. Correction of Problems 46 and 47. If A is a PID, then every submodule of a free A-module is free (without proof). If A is a PID, then every A-module has a projective resolution of length ≤ 1. If A is a PID and M and N are A-modules, then $\operatorname{Tor}_i^A(M,N) = 0$ and $\operatorname{Ext}_A^i(M,N) = 0$ for each $i \geq 2$. If $(A, \mathfrak{m}, k)$ is a noetherian local ring of dimension n, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$ iff every (finite) A-module has a projective resolution of length $\leq n$; if this is the case the ring A is called regular (without proof). Computation of $\operatorname{Ext}_{\mathbb{Z}}^1(M,N)$ when M is a finitely generated abelian group and N is an arbitrary abelian group.
- Mi 23. Jan. Computation of $\operatorname{Ext}_{\mathbb{Z}}^1(M,N)$ when M is a finitely generated abelian group and N is an arbitrary abelian group. Correction of Problem 46. If A is a PID, then every submodule of a free A-module is free (without proof). If A is a PID, then every A-module has a projective resolution of length ≤ 1. If A is a PID and M and N are A-modules, then $\operatorname{Tor}_i^A(M,N) = 0$ and $\operatorname{Ext}_A^i(M,N) = 0$ for each $i \geq 2$. Correction of Problem 47. If $(A, \mathfrak{m}, k)$ is a noetherian local ring of dimension n, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$ iff every (finite) A-module has a projective resolution of length $\leq n$; if this is the case the ring A is called regular (without proof). Quick discussion of the geometric meaning of regular rings: the local ring at the origin of the cuspidal cubic is not regular.

- Mo 28. Jan. (extra session). If $A \subseteq B$ is an integral extension, then $\dim A = \dim B$. Correction of Problem 27. Recap on associated primes. Correction of Problem 25. The local lengths of a finite module over an artinian ring; explicit example of $\mathbb{Z}/72\mathbb{Z}$. Characterization of artinian modules over a noetherian ring.
- Mo 28. Jan. Correction of Problem 49. Quick comment on Problem 50. Correction of Problem 51: characterization of prime/primary homogeneous ideals in an $\mathbb{N}$-graded ring. If S is an $\mathbb{N}$-graded ring, then S is noetherian iff S_0 is noetherian and S is a finitely generated S_0-algebra. If S is an $\mathbb{N}$-graded ring which is finitely generated over S_0 and M is a finite $\mathbb{Z}$-graded S-module, then M_n is a finite S_0-module for each $n \in \mathbb{Z}$. Correction of Problem 52. Recap on Hilbert series, Hilbert polynomial, Hilbert function. Example of $k[x,y]/(x^2, xy)$ with standard grading. Statement of the theorem of the dimension of a local noetherian ring.
- Mi 30. Jan. Correction of Problem 49. Quick comment on Problem 50. Correction of Problem 51: characterization of prime/primary homogeneous ideals in an $\mathbb{N}$-graded ring. If S is an $\mathbb{N}$-graded ring, then S is noetherian iff S_0 is noetherian and S is a finitely generated S_0-algebra. If S is an $\mathbb{N}$-graded ring which is finitely generated over S_0 and M is a finite $\mathbb{Z}$-graded S-module, then M_n is a finite S_0-module for each $n \in \mathbb{Z}$. Correction of Problem 52. Computation of the Hilbert function, of the Hilbert series, and of the Hilbert polynomial of $k[x,y]/(x^2, xy)$ with the standard grading.
- Mi 4. Feb. Comparison between the Rees algebra $\mathrm{Bl}_I A$ and $\mathrm{Gr}_I A$. Discussion about the blowup of $\mathbb{A}^2$ at the origin. Correction of Problems 53, 55, 56. Computation of $\mathrm{Gr}_P R$, where P is the ideal generated by x and y in the ring $R = k[x,y]/(y^2 - x^3)$: the normal cone of the origin in the cuspidal elliptic curve is $k[X,Y]/(Y^2)$.
- Mo 6. Feb. Comparison between the Rees algebra $\mathrm{Bl}_I A$ and $\mathrm{Gr}_I A$. Correction of Problems 53, 54, 55, 56.
- Mi 13. Feb. Klausureinsicht.