Problem 13a A ring, $I \subseteq A$ ideal, M A-module

Show that there is a canonical isomorphism
$M \underset{A}{\otimes} A/I \simeq M/IM$.

1st Proof

$$M \times A/I \longrightarrow M/IM$$

$$(x, a+I) \longmapsto ax + IM$$

is a well defined A-bilinear form. So it induces, by the universal property, an A-linear map

$$\varphi : M \otimes_A A/I \longrightarrow M/IM$$

$$x \otimes (a+I) \longmapsto ax + IM$$

This is the definition on the generators of $M \otimes_A A/I$. But one can actually prove that every element of $M \otimes_A A/I$ has the form $x \otimes (a+I)$

Let's consider

$$\psi : M/IM \longrightarrow M \underset{A}{\otimes} A/I$$

$$x + IM \longmapsto x \otimes (1+I)$$

ψ is well defined: if $y \in M$ is such that $x - y \in IM$, then $x - y = \sum_j a_j z_j$ with $a_j \in I$, $z_j \in M$. Therefore $\psi(y+IM) = y \otimes (1+I)$ $= (x + \sum_j a_j z_j) \otimes (1+I) = x \otimes (1+I) + \sum_j z_j \otimes \cancel{(a_j + I)} = x \otimes (1+I)$.

ψ is A-linear: easy

ψ and φ are inverse of each other:

$$(\varphi \circ \psi)(x + IM) = \varphi(x \otimes (1+I)) = x + IM$$

$$(\psi \circ \varphi)(x \otimes (a+I)) = \psi(ax + IM) = ax \otimes (1+I) = x \otimes (a+I)$$

2nd Proof We will show that M/IM corepresents the functor

$$\mathrm{Bil}_A(M, A/I; \cdot) : (\mathrm{Mod}_A) \longrightarrow (\mathrm{Set})$$

$$P \longmapsto \{A\text{-bilinear forms } M \times A/I \to P\}$$

In other words, we have to show there exists an isomor phism of functors

$$\Phi : \mathrm{Bil}_A(M, A/I ; \cdot) \xrightarrow{\sim} \mathrm{Hom}_A(M/IM , \cdot)$$

Let's define

$$\Phi_P : \mathrm{Bil}_A(M, A/I ; P) \longrightarrow \mathrm{Hom}_A(M/IM , P)$$

for every A-module P: if $b : M \times A/I \to P$ is A-bilinear,

$$\Phi_P(b) : M/IM \longrightarrow P$$
$$x + IM \longmapsto b(x, 1+I)$$

$\Phi_P(b)$ is well defined: if $y \in M$ s.t. $x + IM = y + IM$, then $x - y = \sum_j a_j z_j$ with $a_j \in I$, $z_j \in M$; then

$$\Phi_P(b)(y + IM) = b(y, 1+I) = b(x + \sum_j a_j z_j, 1+I) =$$
$$= b(x, 1+I) + \sum_j b(z_j, \cancel{a_j+I}) = \Phi_P(b)(x + IM).$$

$\Phi_P(b)$ is a hom. of A-modules: easy

Let's construct the inverse of Φ_P:

$$\Psi_P : \mathrm{Hom}_A(M/IM , P) \longrightarrow \mathrm{Bil}_A(M, A/I ; P)$$

$$f \longmapsto \left[\begin{array}{l} M \times A/I \longrightarrow P \\ (x, a+I) \longmapsto f(ax + IM) \end{array} \right]$$

$\Psi_P(f)$ is well defined: if $c \in A$ s.t. $a - c \in I$ then $f(ax + IM) - f(cx + IM) = f((a-c)x + IM) = f(0 + IM) = 0$.

$\Psi_P(f)$ is a bilinear form: easy.

Φ_P and Ψ_P are inverse of each other:

$$((\Phi_P \circ \Psi_P)(f))(x + IM) = \Psi_P(f)(x, 1+I) = f(x + IM)$$

$$((\Psi_P \circ \Phi_P)(b))(x, a+I) = \Phi_P(b)(ax + IM) = b(ax, 1+I) =$$
$$= b(x, a+I)$$

So for every A-module P we have constructed a bijection

$$\Phi_P : \mathrm{Bil}_A(M, A/I; P) \xrightarrow{\sim} \mathrm{Hom}_A(M/IM, P)$$

We need to show that the collection $\{\Phi_P\}_P$ is a natural transformation; let $P \xrightarrow{g} Q$ be a hom. of A-modules, show that

$$\begin{tikzcd} \mathrm{Bil}_A(M, A/I; P) \arrow[r, "\Phi_P"] \arrow[d, "\mathrm{Bil}_A(M, A/I; g)"'] & \mathrm{Hom}_A(M/IM, P) \arrow[d, "\mathrm{Hom}_A(M/IM, g)"] \\ \mathrm{Bil}_A(M, A/I; Q) \arrow[r, "\Phi_Q"] & \mathrm{Hom}_A(M/IM, Q) \end{tikzcd}$$

commutes: fix $b \in \mathrm{Bil}_A(M, A/I; P)$ and $x \in M$

$$\big((\Phi_Q \circ \mathrm{Bil}(g))(b)\big)(x + IM) = \Phi_Q\big((\mathrm{Bil}(g))(b)\big)(x + IM) =$$
$$= \big((\mathrm{Bil}(g))(b)\big)(x, 1+I)$$
$$= g(b(x, 1+I))$$

$$\big((\mathrm{Hom}(g) \circ \Phi_P)(b)\big)(x + IM) = g(\Phi_P(b)(x + IM)) =$$
$$= g(b(x, 1+I)).$$

3rd Proof We need the following remark

Remark If A is a ring, $I \subseteq A$ is an ideal and Q is an A-module, then there is a canonical bijection

$$\begin{aligned} \mathrm{Hom}_A(A/I, Q) &\xrightarrow{\sim} \{q \in Q \mid \forall a \in I, aq = 0\} \\ f &\longmapsto f(1+I) \\ \begin{bmatrix} A/I \to Q \\ a+I \mapsto aq \end{bmatrix} &\longleftarrow q \end{aligned}$$

④ For every A-module P we have bijections:

$$\mathrm{Bil}_A(M, A/I; P) \overset{\text{lectures}}{\cong} \mathrm{Hom}_A(A/I, \mathrm{Hom}_A(M,P))$$

$$\overset{\text{remark}}{\cong} \{h \in \mathrm{Hom}_A(M,P) \mid \forall a \in I \ \ ah = 0\}$$

$$= \{h \in \mathrm{Hom}_A(M,P) \mid \forall a \in I, \forall x \in M, \ h(ax) = 0\}$$

$$= \{h \in \mathrm{Hom}_A(M,P) \mid h|_{IM} = 0\}$$

$$\cong \mathrm{Hom}_A(M/IM, P)$$

homomorphism theorem

All these bijections are natural in P: check!

This gives another proof of the fact that there is an isomorphism of functors

$$\mathrm{Bil}_A(M, A/I; \cdot) \cong \mathrm{Hom}_A(M/IM, \cdot).$$

A ring, M A-module, $I \subseteq A$ ideal.

$$0 \to I \to A \to A/I \to 0 \quad \text{exact}$$

Let's apply $- \otimes_A M$; get

$$\begin{tikzcd} I \otimes_A M \arrow[r] & A \otimes_A M \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & A/I \otimes_A M \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & M & M/IM & \end{tikzcd}$$

from 13a

On the other hand, we have

$$0 \to IM \to M \to M/IM \to 0$$

What is the relation between IM and $I \otimes_A M$?

There is a surjective A-linear map $I \otimes_A M \to IM$ which is induced by the bilinear form

$$I \times M \longrightarrow IM$$
$$(a,x) \longmapsto ax$$

The kernel of $I \otimes_A M \longrightarrow IM$, which coincides with the kernel of $I \otimes_A M \longrightarrow A \otimes_A M = M$, is denoted $\mathrm{Tor}_1^A(A/I, M)$.

It can be $\neq 0$:

$A = \mathbb{Z}$
$I = 2\mathbb{Z}$
$M = \mathbb{Z}/2\mathbb{Z}$

$IM = 0$

as $I \simeq \mathbb{Z}$ as $\mathbb{Z}$-modules, $I \otimes_A M \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$

Problem 13b $I, J \subseteq A$ ideals

$$A/I \otimes_A A/J = (A/J) \Big/ I \cdot (A/J) = (A/J) \Big/ \left(I+J/J \right) = A/I+J$$

Useful facts:

- M, N_λ, $\lambda \in \Lambda$ A-modules $\Rightarrow M \otimes_A \left(\bigoplus_{\lambda \in \Lambda} N_\lambda \right) \simeq \bigoplus_{\lambda \in \Lambda} M \otimes_A N_\lambda$

- M A-module $\Rightarrow$ $\begin{aligned} M &\longrightarrow M \otimes_A A \\ x &\longmapsto x \otimes 1 \end{aligned}$ is an isomorphism

Combining these two, one can see how to tensor with free modules:

M A-module $\Rightarrow M \otimes_A \left(\bigoplus_{\lambda \in \Lambda} A \right) \simeq \bigoplus_{\lambda \in \Lambda} M$

In particular $A^{\oplus m} \otimes_A A^{\oplus n} \simeq A^{\oplus m \cdot n}$

Problem 13c

$\mathbb{Q}$ is a field.
A $\mathbb{Q}$-module (i.e. $\mathbb{Q}$-vector space) is always free because it has a basis. In particular $\mathbb{R} \simeq \bigoplus_{\lambda \in \Lambda} \mathbb{Q}$ as $\mathbb{Q}$-v.sp. where $|\Lambda| = \dim_{\mathbb{Q}} \mathbb{R}$. Now

$$\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} = \Big(\bigoplus_{\lambda\in\Lambda} \mathbb{Q}\Big) \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{\lambda \in \Lambda} \mathbb{R}$$ is a v.sp. over $\mathbb{R}$ of dimension $|\Lambda|$.

$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R}$ is a v.sp. over $\mathbb{R}$ of dimension 1.

There is a map $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$. This comes from the more general situation:

$A \xrightarrow{\varphi} B$ ring homomorphism
M, N B-modules. Let's think them also as A-modules via φ. [In Problem 14 these A-module structure is denoted M_A, N_A; but let's not be pedantic here]

Let's consider:

$$\begin{tikzcd} M \times N \arrow[r, "\delta_B"] \arrow[d, "\delta_A"'] & M \otimes_B N \\ M \otimes_A N & \end{tikzcd}$$

the universal B-bilinear form $\delta_B : M \times N \longrightarrow M \otimes_B N$. This is of course A-bilinear via φ; therefore by the universal property of the universal A-bilinear form $\delta_A : M \times N \to M \otimes_A N$ we get a homomorphism of A-modules

$$M \otimes_A N \longrightarrow M \otimes_B N.$$

This is surjective, because $M \otimes_A N$ (resp. $M \otimes_B N$) is generated as $\mathbb{Z}$-module by elements $x \otimes_A y$ (resp. $x \otimes_B y$).
It may not be injective as the example above shows.
However, if $A \to B$ is surjective then it is an isom. (exercise)
(see also $\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ in d)

Problem 13d

$$\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \overset{b}{=} \mathbb{Z}/4\mathbb{Z}+6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/12\mathbb{Z} \,\big/\, 4\mathbb{Z}/12\mathbb{Z}$$

$$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/12\mathbb{Z} \,\big/\, 6\mathbb{Z}/12\mathbb{Z}$$

so

$$\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/12\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \overset{b}{=} \mathbb{Z}/12\mathbb{Z} \Big/ \frac{4\mathbb{Z}}{12\mathbb{Z}} + \frac{6\mathbb{Z}}{12\mathbb{Z}} = \mathbb{Z}/12\mathbb{Z} \,\big/\, 2\mathbb{Z}/12\mathbb{Z} =$$

$$= \mathbb{Z}/2\mathbb{Z}.$$

If $A \to B$, $A \to C$ are ring homom., then $B \otimes_A C$ has the structure of ring:

$$\Big(\sum_i b_i \otimes c_i\Big) \cdot \Big(\sum_j b_j' \otimes c_j'\Big) = \sum_{i,j} b_i b_j' \otimes c_i c_j'$$

with unit $1 \otimes 1$

and

$$B \to B \otimes_A C \qquad\qquad C \to B \otimes_A C$$
$$b \mapsto b \otimes 1 \qquad\qquad c \mapsto 1 \otimes c$$

are ring homomorphisms.

$$\begin{tikzcd} & B \otimes_A C & \\ B \arrow[ur] & & C \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

is a coproduct in the category of rings