1-FORMS ON OPEN SUBSETS OF $\mathbb{R}^n$

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If G is a group, the *derived subgroup* of G is the subgroup $G' \subseteq G$ generated by the commutators $ghg^{-1}h^{-1}$, as $g, h \in G$. One can easily show that G' is a normal subgroup of G. The quotient G/G' is called the *abelianisation* of G and is denoted by G^{ab}. It is easy to see that G^{ab} is an abelian group. If G is an abelian group, then the derived subgroup G' is trivial and the quotient projection $G \twoheadrightarrow G^{\mathrm{ab}}$ is an isomorphism. Thanks to the homomorphism theorem, for every abelian group H there is a natural bijection between $\{G \to H \text{ group homomorphism}\}$ and $\{G^{\mathrm{ab}} \to H \text{ group homomorphism}\}$.

Definition 1. Let X be a path-connected topological space. The *1st singular homology group with coefficients in* $\mathbb{Z}$ of X is the abelianisation of the fundamental group of X: $\mathrm{H}_1(X;\mathbb{Z}) := \pi_1(X, x_0)^{\mathrm{ab}}$ for some $x_0 \in X$.

Remark 2. We need to show that this group does not depend on the point x_0. Let's pick another point $x_1 \in X$. Since X is path-connected, there exists a path $\gamma \in \Omega(X, x_0, x_1)$. We consider the group isomorphism

$$\gamma_\sharp \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by $[\alpha] \mapsto [\iota(\gamma) * \alpha * \gamma]$. This induces the isomorphism

$$\gamma_\star \colon \pi_1(X, x_0)^{\mathrm{ab}} \to \pi_1(X, x_1)^{\mathrm{ab}}$$

given by $[\alpha]\pi_1(X, x_0)' \mapsto [\iota(\gamma) * \alpha * \gamma]\pi_1(X, x_1)'$. If we pick another path $\tilde{\gamma} \in \Omega(X, x_0, x_1)$, we need to prove that $\gamma_\star = \tilde{\gamma}_\star$. We need to show that for every $\alpha \in \Omega(X, x_0, x_0)$

$$[\iota(\gamma) * \alpha * \gamma]\pi_1(X, x_1)' = [\iota(\tilde{\gamma}) * \alpha * \tilde{\gamma}]\pi_1(X, x_1)'$$

i.e. the derived subgroup $\pi_1(X, x_1)' \subseteq \pi_1(X, x_1)$ contains

$$[\iota(\gamma) * \alpha * \gamma][\iota(\tilde{\gamma}) * \alpha * \tilde{\gamma}]^{-1} = [\iota(\gamma) * \alpha * \gamma * \iota(\tilde{\gamma}) * \iota(\alpha) * \tilde{\gamma}] = [\iota(\gamma) * \alpha * \gamma][\iota(\tilde{\gamma}) * \gamma][\iota(\gamma) * \alpha * \gamma]^{-1}[\iota(\tilde{\gamma}) * \gamma]^{-1}$$

which is a commutator.

Clearly, if X is simply connected then $\mathrm{H}_1(X;\mathbb{Z}) = 0$.

Definition 3. Let X be a path-connected topological space. A *loop* in X is a path $\alpha \colon [0,1] \to X$ such that $\alpha(0) = \alpha(1)$. Two loops α and $\tilde{\alpha}$ in X are called *homologous* if they induce the same element of $\mathrm{H}_1(X;\mathbb{Z})$, i.e. for some/every path $\gamma \in \Omega(X, \alpha(0), \tilde{\alpha}(0))$ the path-homotopy class of $\alpha * \gamma * \iota(\tilde{\alpha}) * \iota(\gamma)$ lies in the derived subgroup of $\pi_1(X, \alpha(0))$.

If α and β are path-homotopic paths in a path-connected topological space X, then α and β are homologous.

We denote by $(\mathbb{R}^n)^\vee$ the dual of the vector space $\mathbb{R}^n$. Let $dx_1, \dots, dx_n$ be the dual of the standard basis of $\mathbb{R}^n$.

Definition 4. A *1-form* on an open subset $U \subseteq \mathbb{R}^n$ is a continuous map $\omega \colon U \to (\mathbb{R}^n)^\vee$.

We can write a 1-form uniquely as $\omega = \sum_{i=1}^n \omega_i dx_i$, where $\omega_1, \dots, \omega_n$ are continuous functions on U. We say that ω is C^k if the functions $\omega_1, \dots, \omega_n$ are C^k.

If $F \colon U \to \mathbb{R}$ is a C^1 function, then its differential $dF := \sum_{i=1}^n \frac{\partial F}{\partial x_i} dx_i$ is a 1-form on U.

Definition 5. A 1-form ω on an open subset $U \subseteq \mathbb{R}^n$ is called:

- *exact* if there exists a C^1 function $F \colon U \to \mathbb{R}$ such that $\omega = dF$; (such an F is called a *primitive* of ω)
- *locally exact* if there exists an open cover $\{U_\lambda\}_\lambda$ of U such that, for each λ, the restriction $\omega|_{U_\lambda}$ is an exact form on U_λ;
- *closed* if ω is C^1 and for all $1 \leq i, j \leq n$ we have the following equality of functions on U:

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}.$$

If $U \subseteq \mathbb{R}^n$ is a connected open subset and ω is an exact 1-form on U, then the primitive of ω is unique up to an additive constant.

Obviously an exact 1-form is locally exact. In Proposition 15 we will see that being closed is the same as being locally exact and C^1. So, for a C^1 1-form it is very easy to understand if it is locally exact: it is enough to compute some partial derivatives. The discrepancy between being exact and being locally exact depends only on the topology of the open subset $U \subseteq \mathbb{R}^n$, as we will see below.

It is clear that, fixed an open subset $U \subseteq \mathbb{R}^n$, the sets of 1-forms, exact 1-forms, locally exact 1-forms, closed 1-forms on U are real vector spaces of infinite dimension.

Definition 6. The *1st de Rham cohomology group* of an open subset $U \subseteq \mathbb{R}^n$ is the following real vector space:

$$\mathrm{H}^1_{\mathrm{dR}}(U) := \frac{\{\text{locally exact 1-forms on } U\}}{\{\text{exact 1-forms on } U\}}.$$

Given an open subset $U \subseteq \mathbb{R}^n$, we have that $\mathrm{H}^1_{\mathrm{dR}}(U) = 0$ if and only if every locally exact 1-form on U is exact.

Definition 7. Let ω be a locally exact 1-form on the open subset $U \subseteq \mathbb{R}^n$ and let $\gamma\colon [a,b] \to U$ be a path. A *primitive of* ω *along* γ is a continuous function $f\colon [a,b] \to \mathbb{R}$ such that: for every $\tau \in [a,b]$, there exists an open neighbourhood V of $\gamma(\tau)$ in U such that $\omega|_V$ is exact, $F\colon V \to \mathbb{R}$ is a primitive of $\omega|_V$ and $F \circ \gamma|_{\gamma^{-1}(V)} = f|_{\gamma^{-1}(V)}$.

Proposition 8. *Let ω be a locally exact 1-form on the open subset $U \subseteq \mathbb{R}^n$ and let $\gamma\colon [a,b] \to U$ be a path. Then a primitive of ω along γ exists and is unique up to an additive constant.*

Proof. See [Car95, Theorem 1, p. 57] or [Lan93, III, §4].

Existence: we know that ω is locally exact, so there exists an open cover $\{U_\lambda\}_\lambda$ of U such that $\omega|_{U_\lambda}$ is exact for each λ. Since $[a,b]$ is compact and $\gamma\colon [a,b] \to U$ is continuous, by the Lebesgue number there exists a finite sequence of points $a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b$ such that, for each integer $i = 0, \ldots, r$, $\gamma([t_i, t_{i+1}]) \subseteq U_{\lambda_i}$ for some λ_i. Let F_i be a primitive of $\omega|_{U_{\lambda_i}}$. We notice that $\gamma(t_{i+1}) \in U_{\lambda_i} \cap U_{\lambda_{i+1}}$, for each $i = 0, \ldots, r$. Up to adding a constant to F_1, we can assume that $F_0(\gamma(t_1)) = F_1(\gamma(t_1))$. Up to adding a constant to F_2, we can assume that $F_1(\gamma(t_2)) = F_2(\gamma(t_2))$. And so on until $F_{r-1}(\gamma(t_r)) = F_r(\gamma(t_r))$. Now define $f\colon [a,b] \to \mathbb{R}$ by $f(t) = F_i(\gamma(t))$ if $t \in [t_i, t_{i+1}]$, for each $i = 0, \ldots, r$. It is obvious that f is continuous and that f is a primitive of ω along γ.

Uniqueness: let f_1 and f_2 be two primitives of ω along γ. Fix $\tau \in [a,b]$. Then by the definition, we can find an open neighbourhood of τ in $[a,b]$ where $f_1 - f_2$ is constant, thanks to the uniqueness of the primitive of a 1-form. We have proved that $f_1 - f_2$ is locally constant. As $[a,b]$ is connected and $f_1 - f_2$ is continuous, we have that $f_1 - f_2$ is constant. □

Now we want to define the integral of a 1-form along a path.

Definition 9. Let ω be a 1-form on the open subset $U \subseteq \mathbb{R}^n$ and let $\gamma\colon [a,b] \to U$ be a path. The *integral of* ω *along* γ is the real number, denoted by $\int_\gamma \omega$, defined in the following two (overlapping) cases.

- Assume that γ is piecewise C^1. Then
$$\int_\gamma \omega := \int_a^b \omega(\gamma(t))(\gamma'(t))dt = \int_a^b \sum_{i=1}^n \omega_i(\gamma(t))\gamma_i'(t)dt.$$
This definition makes sense because γ' is not defined in at most finitely many points in $[a,b]$.
- Assume that ω is locally exact. Then
$$\int_\gamma \omega := f(b) - f(a)$$
where $f\colon [a,b] \to \mathbb{R}$ is a primitive of ω along γ. This definition makes sense by Proposition 8.

Proposition 10. *The two definitions of $\int_\gamma \omega$ in Definition 9 are compatible.*

Proof. Assume that ω is locally exact and that $\gamma\colon [a,b] \to U$ is piecewise C^1. As in the proof of Proposition 8, there exist open subsets $U_0, \ldots, U_r \subseteq U$ such that $\omega|_{U_0}, \ldots, \omega|_{U_r}$ are exact and there exists a finite sequence of points $a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b$ such that, for each integer $i = 0, \ldots, r$, $\gamma([t_i, t_{i+1}]) \subseteq U_i$. Let $F_i\colon U_i \to \mathbb{R}$ be a primitive of $\omega|_{U_i}$ and assume that $F_i(\gamma(t_{i+1})) = F_{i+1}(\gamma(t_{i+1}))$, for all $i = 0, \ldots, r$. Consider the primitive f of ω along γ given by $f(t) = F_i(\gamma(t))$ if $t \in [t_i, t_{i+1}]$. Then

$$\int_a^b \sum_{j=1}^n \omega_j(\gamma(t))\gamma_j'(t)dt = \sum_{i=0}^r \int_{t_i}^{t_{i+1}} \sum_{j=1}^n \omega_j(\gamma(t))\gamma_j'(t)dt = \sum_{i=0}^r \int_{t_i}^{t_{i+1}} \sum_{j=1}^n \frac{\partial F_i}{\partial x_j}(\gamma(t))\gamma_j'(t)dt =$$

$$= \sum_{i=0}^r \int_{t_i}^{t_{i+1}} (F_i \circ \gamma)'(t)dt = \sum_{i=0}^r (f(t_{i+1}) - f(t_i)) = f(b) - f(a).$$ □

Proposition 11. *Let ω be a 1-form on the open subset $U \subseteq \mathbb{R}^n$ and let $\gamma\colon [a,b] \to U$ be a path. Assume either that γ is piecewise C^1 or that ω is locally exact. Then $\int_\gamma \omega$ does not depend on reparametrizations of γ.*

Proof. Let $\varphi\colon [a_1, b_1] \to [a,b]$ be an increasing homeomorphism.

Assume that γ is piecewise C^1 and that φ is piecewise C^1. Then

$$\int_{\gamma\circ\varphi} \omega = \int_{a_1}^{b_1} \sum_{i=1}^n \omega_i(\gamma(\varphi(s))) \cdot (\gamma_i \circ \varphi)'(s) ds = \int_{a_1}^{b_1} \sum_{i=1}^n \omega_i(\gamma(\varphi(s)))\gamma_i'(\varphi(s))\varphi'(s) ds =$$

$$= \int_a^b \sum_{i=1}^n \omega_i(\gamma(t))\gamma_i'(t) dt = \int_\gamma \omega.$$

Assume that ω is locally exact. If f is a primitive of ω along γ, then $f \circ \varphi$ is a primitive of ω along $\gamma \circ \varphi$. Therefore

$$\int_{\gamma\circ\varphi} \omega = f(\varphi(b_1)) - f(\varphi(a_1)) = f(b) - f(a) = \int_\gamma \omega.$$

□

Thanks to the proposition above we will often assume that paths are defined over $[0,1]$.

Remark 12. Let $\omega = \sum_{i=1}^3 \omega_i dx_i$ be a 1-form on an open subset $U \subseteq \mathbb{R}^3$. Consider the vector field $\mathbf{F} = (\omega_1, \omega_2, \omega_3)\colon U \to \mathbb{R}^3$.

- ω is exact if and only if $\mathbf{F}$ is conservative. A primitive of ω is exactly a potential of $\mathbf{F}$, i.e. a C^1 function $V\colon U \to \mathbb{R}$ such that $\nabla V = \mathbf{F}$.
- Assume that ω is C^1. Recall that the curl of $\mathbf{F}$ is the vector field

$$\nabla \times \mathbf{F} := \left(\frac{\partial \omega_3}{\partial x_2} - \frac{\partial \omega_2}{\partial x_3}, \frac{\partial \omega_1}{\partial x_3} - \frac{\partial \omega_3}{\partial x_1}, \frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} \right).$$

  Therefore ω is closed if and only if $\mathbf{F}$ is irrotational, i.e. its curl $\nabla \times \mathbf{F}$ vanishes.

If $\gamma\colon [a,b] \to U$ is the (piecewise C^1) trajectory of a point in U and $\mathbf{F}$ is a force field, then

$$\int_\gamma \omega = \int_a^b \mathbf{F}(\gamma(t)) \bullet \dot{\gamma}(t) dt$$

(where $\bullet$ denotes the scalar product) is the work done by the force $\mathbf{F}$ along the path γ.

Proposition 13 (Bilinearity of the integral)**.** *Let $U \subseteq \mathbb{R}^n$ be an open subset. Assume that all the integrals below are defined.*

- *Let ω be a 1-form on U and let $\gamma_1, \gamma_2\colon [0,1] \to U$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Consider the concatenation $\gamma_1 * \gamma_2\colon [0,1] \to U$. Then*

$$\int_{\gamma_1 * \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

- *Let ω be a 1-form on U and let $\gamma\colon [0,1] \to U$ be a path. Consider the inverse $\iota(\gamma)\colon [0,1] \to U$ of γ. Then*

$$\int_{\iota(\gamma)} \omega = -\int_\gamma \omega.$$

- *Let ω and η be two 1-forms on U and let $\gamma\colon [0,1] \to U$ be a path. Then*

$$\int_\gamma (\omega + \eta) = \int_\gamma \omega + \int_\gamma \eta.$$

- *Let ω be a 1-form on U, let $\lambda \in \mathbb{R}$, and let $\gamma\colon [0,1] \to U$ be a path. Then*

$$\int_\gamma \lambda\omega = \lambda \int_\gamma \omega.$$

Proof. Left to the reader. □

Proposition 14. *Let $U \subseteq \mathbb{R}^n$ be a star-shaped open subset and let ω be a closed 1-form on U. Then ω is exact.*

Proof. Assume that U is star-shaped with respect to the point $\bar{x} \in U$. For each point $x \in U$ consider the segment $\gamma_x\colon [0,1] \to U$ defined by $t \mapsto \bar{x} + t(x - \bar{x})$. Consider the function $F\colon U \to \mathbb{R}$ defined by

$$F(x) = \int_{\gamma_x} \omega = \int_0^1 \sum_{i=1}^n \omega_i\left(\bar{x} + t(x - \bar{x})\right) \cdot (x_i - \bar{x}_i) dt.$$

We want to show that F is a primitive of ω. Fix $1 \leq k \leq n$ and $x \in U$. Consider the function $G(t) = \omega_k(\bar{x} + t(x - \bar{x}))$ on $[0, 1]$. By the chain rule we have

$$G'(t) = \sum_{i=1}^{n} \frac{\partial \omega_k}{\partial x_i}(\bar{x} + t(x - \bar{x})) \cdot (x_i - \bar{x}_i) = \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x_k}(\bar{x} + t(x - \bar{x})) \cdot (x_i - \bar{x}_i),$$

where we have used that ω is closed. Thus

$$\begin{aligned}
\frac{\partial F}{\partial x_k}(x) &= \int_0^1 \sum_{i=1}^{n} \frac{\partial}{\partial x_k}\left[\omega_i(\bar{x} + t(x - \bar{x})) \cdot (x_i - \bar{x}_i)\right] dt \\
&= \int_0^1 \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial x_k}\left[\omega_i(\bar{x} + t(x - \bar{x}))\right] \cdot (x_i - \bar{x}_i) + \omega_k(\bar{x} + t(x - \bar{x})) \right\} dt \\
&= \int_0^1 \left[G'(t)t + G(t)\right] dt = \int_0^1 \left[\frac{d}{dt}(G(t)t)\right] dt = G(1) = \omega_k(x).
\end{aligned}$$

□

Proposition 15 (Closed = locally exact + C^1)**.** *Let ω be a 1-form on an open subset $U \subseteq \mathbb{R}^n$. Then ω is closed if and only if it is C^1 and locally exact.*

Proof. ⇒) Assume that ω is closed. Since every open ball is star-shaped, by Proposition 14 we have that the restriction of ω to each open ball contained in U is exact. Conclude by choosing an open cover of U made up of open balls.

⇐) Assume that ω is C^1 and locally exact. Let $\{U_\lambda\}_\lambda$ be an open cover of U such that $\omega|_{U_\lambda}$ is exact for each λ. Let $F_\lambda : U_\lambda \to \mathbb{R}$ be a primitive of $\omega|_{U_\lambda}$, i.e. $\partial F_\lambda / \partial x_i = \omega_i|_{U_\lambda}$. Since F_λ is C^2, we have

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial^2 F_\lambda}{\partial x_j \partial x_i} = \frac{\partial^2 F_\lambda}{\partial x_i \partial x_j} = \frac{\partial \omega_j}{\partial x_i}$$

on U_λ. Conclude because $U = \bigcup_\lambda U_\lambda$. □

Theorem 16 (Homotopy invariance of the integral of locally exact forms)**.** *If ω is a locally exact 1-form on an open subset $U \subseteq \mathbb{R}^n$ and let $\gamma_0, \gamma_1 : [0, 1] \to U$ be two paths which are path-homotopic, then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. See [Car95, II.1.6] or [Lan93, III, §5]. Let $\delta : [0, 1] \times [0, 1] \to U$ be the homotopy between γ_0 and γ_1, i.e. a continuous map such that $\delta(\cdot, 0) = \gamma_0$, $\delta(\cdot, 1) = \gamma_1$ and $\delta(0, s) = \gamma_0(0) = \gamma_1(0)$, $\delta(1, s) = \gamma_0(1) = \gamma_1(1)$ for all $s \in [0, 1]$.

In a similar way to Proposition 8 we want to construct "a primitive of ω along δ". Let $\{U_\lambda\}_\lambda$ be an open cover such that $\omega|_{U_\lambda}$ is exact for each λ. Since $[0, 1] \times [0, 1]$ is compact and δ is continuous, by Lebesgue number we can find two finite sequences of points $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = 1$ and $0 = s_0 < s_1 < \cdots < s_r < s_{r+1} = 1$ such that, for all $0 \leq i, j \leq r$, $\delta([t_i, t_{i+1}] \times [s_j, s_{j+1}]) \subseteq U_{\lambda_{i,j}}$ for some $\lambda_{i,j}$. Let $F_{i,j} : U_{\lambda_{i,j}} \to \mathbb{R}$ be a primitive of $\omega|_{U_{\lambda_{i,j}}}$.

Keep j fixed. We can add a constant to $F_{i,j}$ in such a way that $F_{i,j} \circ \delta|_{\{t_{i+1}\} \times [s_j, s_{j+1}]} = F_{i+1,j} \circ \delta|_{\{t_{i+1}\} \times [s_j, s_{j+1}]}$. We construct the continuous function $f_j : [a, b] \times [s_j, s_{j+1}] \to \mathbb{R}$ given by $f_j(t, s) = F_{i,j}(\delta(t, s))$ for $t \in [t_i, t_{i+1}]$. It is clear that for each $s \in [s_j, s_{j+1}]$ the function $f(\cdot, s)$ is a primitive of ω along $\delta(\cdot, s)$.

If we add some constants to the f_js, we can assume that for each $j = 0, \ldots, r$ we have $f_j(\cdot, s_{j+1}) = f_j(\cdot, s_j)$. We can glue all these f_js to a continuous function $f : [0, 1] \times [0, 1] \to \mathbb{R}$, such that for each $s \in [0, 1]$ $f(\cdot, s)$ is a primitive of $\delta(\cdot, s)$. Therefore $\int_{\gamma_0} \omega = f(1, 0) - f(0, 0)$ and $\int_{\gamma_1} \omega = f(1, 1) - f(0, 1)$. But $\delta(\cdot, 0)$ and $\delta(\cdot, 1)$ are constant; therefore $f(1, 0) = f(1, 1)$ and $f(0, 0) = f(0, 1)$. □

Corollary 17 (Homology invariance of the integral of locally exact forms)**.** *Let $U \subseteq \mathbb{R}^n$ be a connected open subset and let γ_1 and γ_2 two loops in U which are homologous. If ω is a locally exact 1-form on U, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.*

Proof. Choose a path $\gamma \in \Omega(U, \gamma_1(0), \gamma_2(0))$. Consider the loop $\tilde{\gamma} := \gamma_1 * \gamma * \iota(\gamma_2) * \iota(\gamma)$. Since γ_1 and γ_2 are homologous, the path-homotopy class $[\tilde{\gamma}]$ lies in the derived subgroup of $\pi_1(X, \gamma_1(0))$. There exist loops $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r \in \Omega(X, \gamma_1(0), \gamma_1(0))$ such that $\tilde{\gamma}$ is path-homotopic to $(\alpha_1 * \beta_1 * \iota(\alpha_1) * \iota(\beta_1)) * \cdots * (\alpha_r * \beta_r * \iota(\alpha_r) * \iota(\beta_r))$. By Theorem 16 and Proposition 13,

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\tilde{\gamma}} \omega = \sum_{i=1}^{r} \int_{\alpha_i * \beta_i * \iota(\alpha_i) * \iota(\beta_i)} \omega = 0.$$

□

If x and y are two points in an open subset U of $\mathbb{R}^n$, we denote by $\Omega(U, x, y)_{C^1}$ (resp. $\Omega(U, x, y)_{\mathrm{aff}}$) the set of paths in U from x to y which are piecewise C^1 (resp. piecewise affine with segments parallel to the coordinate axes).

The following theorem gives a criterion for testing whether a locally exact 1-form is exact.

Theorem 18 (Characterisation of exact forms). *Let ω be a 1-form on a connected open subset $U \subseteq \mathbb{R}^n$. Let $\bar{x} \in U$ be a point. Let $\mathcal{F}$ be a collection of loops in U such that their homology classes form a set a generators of the group $\mathrm{H}_1(U;\mathbb{Z})$. Then the following statements are equivalent:*

(1) ω is exact;
(2) ω is locally exact and $\forall x \in U, \forall \gamma \in \Omega(U,x,x), \int_\gamma \omega = 0$;
(2') $\forall x \in U, \forall \gamma \in \Omega(U,x,x)_{C^1}, \int_\gamma \omega = 0$;
(2") $\forall x \in U, \forall \gamma \in \Omega(U,x,x)_{\mathrm{aff}}, \int_\gamma \omega = 0$;
(3) ω is locally exact and $\forall \gamma \in \Omega(U,\bar{x},\bar{x}), \int_\gamma \omega = 0$;
(3') $\forall \gamma \in \Omega(U,\bar{x},\bar{x})_{C^1}, \int_\gamma \omega = 0$;
(3") $\forall \gamma \in \Omega(U,\bar{x},\bar{x})_{\mathrm{aff}}, \int_\gamma \omega = 0$;
(4) ω is locally exact and $\forall x,y \in U, \forall \gamma_1,\gamma_2 \in \Omega(U,x,y), \int_{\gamma_1} \omega = \int_{\gamma_2} \omega$;
(4') $\forall x,y \in U, \forall \gamma_1,\gamma_2 \in \Omega(U,x,y)_{C^1}, \int_{\gamma_1} \omega = \int_{\gamma_2} \omega$;
(4") $\forall x,y \in U, \forall \gamma_1,\gamma_2 \in \Omega(U,x,y)_{\mathrm{aff}}, \int_{\gamma_1} \omega = \int_{\gamma_2} \omega$;
(5) ω is locally exact and for every $\gamma \in \mathcal{F}$ we have $\int_\gamma \omega = 0$.

Proof. The implications (2) $\Rightarrow$ (2') $\Rightarrow$ (2"), (3) $\Rightarrow$ (3') $\Rightarrow$ (3"), (4) $\Rightarrow$ (4') $\Rightarrow$ (4"), (2) $\Rightarrow$ (3), (2') $\Rightarrow$ (3'), (2") $\Rightarrow$ (3"), and (2) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (4): let $x,y \in U$ and $\gamma_1,\gamma_2 \in \Omega(U,x,y)$. Then $\gamma_1 * \iota(\gamma_2) \in \Omega(U,x,x)$; then $0 = \int_{\gamma_1 * \iota(\gamma_2)} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$.

(4) $\Rightarrow$ (2): let $x \in U$ and $\gamma \in \Omega(U,x,x)$; let c_x be the constant path at x. Therefore $0 = \int_{c_x} \omega = \int_\gamma \omega$.

(2') $\Leftrightarrow$ (4'), (2") $\Leftrightarrow$ (4"): same proofs.

(3) $\Rightarrow$ (2): let $x \in U$ and $\gamma \in \Omega(U,x,x)$. Since U is connected, there exists $\beta \in \Omega(U,\bar{x},x)$. Then $\beta * \gamma * \iota(\beta) \in \Omega(U,\bar{x},\bar{x})$. Therefore $0 = \int_{\beta*\gamma*\iota(\beta)} \omega = \int_\beta \omega + \int_\gamma \omega - \int_\beta \omega = \int_\gamma \omega$.

(3') $\Rightarrow$ (2'), (3") $\Rightarrow$ (2"): same proof.

(1) $\Rightarrow$ (2): let $F \in C^1(U)$ such that $\omega = dF$. Fix a loop $\gamma\colon [0,1] \to U$ based at an arbitrary point $x \in U$. Then $F \circ \gamma$ is a primitive of ω along γ. Therefore $\int_\gamma \omega = F(\gamma(1)) - F(\gamma(0)) = F(x) - F(x) = 0$.

(5) $\Rightarrow$ (2): without loss of generality, up to enlarge $\mathcal{F}$, we assume that whenever a loop is in $\mathcal{F}$ also its inverse is in $\mathcal{F}$. Let $\gamma \in \Omega(U,x,x)$. Then there exist $\gamma_1,\dots,\gamma_r \in \mathcal{F}$ such that the equality $[\gamma] = [\gamma_1] + \cdots + [\gamma_r]$ holds in the group $\mathrm{H}_1(U;\mathbb{Z})$. This implies that there exist $\alpha_i \in \Omega(U,x,\gamma_i(0))$ such that the path-homotopy class of the loop

$$\beta := \iota(\gamma) * \alpha_1 * \gamma_1 * \iota(\alpha_1) * \cdots * \alpha_r * \gamma_1 * \iota(\alpha_r) \in \Omega(U,x,x)$$

lies in the derived subgroup of $\pi_1(U,\bar{x})'$, i.e. $[\beta] \in \pi_1(U,\bar{x})'$. By Corollary 17 we have

$$0 = \int_\beta \omega = \int_{\iota(\gamma)*\alpha_1*\gamma_1*\iota(\alpha_1)*\cdots*\alpha_r*\gamma_r*\iota(\alpha_r)} \omega = -\int_\gamma \omega + \int_{\gamma_1} \omega + \cdots + \int_{\gamma_r} \omega = -\int_\gamma \omega.$$

(4") $\Rightarrow$ (1): we define the function $F\colon U \to \mathbb{R}$ as $F(x) = \int_{\gamma_x} \omega$, where $\gamma_x \in \Omega(U,\bar{x},x)_{\mathrm{aff}}$. The function F is well defined because of (4"). We will show that, in every point of U, the partial derivatives of F are equal to the components of ω.

Fix $x \in U$ and $1 \le k \le n$. Choose $h \in \mathbb{R} \setminus \{0\}$ such that the closed ball $\overline{B_{|h|}(x)}$ is contained in U. Consider the path $\gamma\colon [0,1] \to U$ defined by $t \mapsto x + the_k$. So $\gamma_x * \gamma \in \Omega(U,\bar{x},x)_{\mathrm{aff}}$. Therefore $F(x+he_k) = \int_{\gamma_x*\gamma} \omega = \int_{\gamma_x} \omega + \int_\gamma \omega$ and then $F(x+he_k) - F(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_k(x+the_k)h\,dt = \int_0^h \omega_k(x+se_k)ds$. Since ω_k is continuous, there exists $\xi \in \mathbb{R}$ between 0 and h such that $\frac{1}{h}[F(x+he_k) - F(x)] = \frac{1}{h}\int_0^h \omega_k(x+se_k)ds = \omega_k(x+\xi e_k)$. As $h \to 0$ we have that also $\xi \to 0$, and then $x + \xi e_k \to x$. So by the continuity of ω_k we get $\frac{\partial F}{\partial x_k}(x) = \omega_k(x)$. $\square$

If G and H are abelian groups, we denote by $\mathrm{Hom}_{\mathbb{Z}}(G,H)$ the abelian group whose elements are the group homomorphisms from G to H. For every abelian group G, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{R})$ has a natural structure of vector space over $\mathbb{R}$.

Theorem 19 (de Rham). *If $U \subseteq \mathbb{R}^n$ is a connected open subset, then the integration of 1-forms along paths gives an injective $\mathbb{R}$-linear map $\mathrm{H}^1_{\mathrm{dR}}(U) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(U;\mathbb{Z}),\mathbb{R})$.*

Remark 20. Actually the $\mathbb{R}$-linear map $\mathrm{H}^1_{\mathrm{dR}}(U) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(U;\mathbb{Z}),\mathbb{R})$ is bijective, but we will not prove this. This is the de Rham theorem for 1-forms on open subsets of $\mathbb{R}^n$; it can be generalised to p-forms on smooth manifolds.

Proof of Theorem 19. Fix $\bar{x} \in U$. We consider the map

$$\{\text{locally exact 1-forms on } U\} \times \Omega(U,\bar{x},\bar{x}) \longrightarrow \mathbb{R}$$

given by $(\omega,\gamma) \mapsto \int_\gamma \omega$. By Theorem 16 we get a map

$$\{\text{locally exact 1-forms on } U\} \times \pi_1(U,\bar{x}) \longrightarrow \mathbb{R}.$$

By Corollary 17 we get a bilinear map

$$\{\text{locally exact 1-forms on } U\} \times \mathrm{H}_1(U;\mathbb{Z}) \longrightarrow \mathbb{R}.$$

By Theorem 18(1)⇒(3) we get a bilinear map

$$\mathrm{H}^1_{\mathrm{dR}}(U;\mathbb{R}) \times \mathrm{H}_1(U;\mathbb{Z}) \longrightarrow \mathbb{R}.$$

This gives a $\mathbb{R}$-linear map $\mathrm{H}^1_{\mathrm{dR}}(U) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(U;\mathbb{Z}),\mathbb{R})$, which is injective by the implication (5)⇒(1) of Theorem 18. □

Corollary 21. *Let $U \subseteq \mathbb{R}^n$ be a connected open subset. If the abelianisation of the fundamental group of U is a finite group (e.g. if U is simply connected), then every locally exact 1-form on U is exact.*

A *(compact) rectangle* in $\mathbb{R}^2$ is a subset $R = [a,b] \times [c,d]$ for some $a,b,c,d \in \mathbb{R}$ such that $a < b$ and $c < d$. We denote by ∂R the loop which covers the boundary of R, more precisely ∂R is the map $[0,4] \to \mathbb{R}^2$ given by

$$t \mapsto \begin{cases} (b,(1-t)c+td) & \text{if } 0 \le t \le 1, \\ ((2-t)b+(t-1)a,d) & \text{if } 1 \le t \le 2, \\ (a,(3-t)d+(t-2)c) & \text{if } 2 \le t \le 3, \\ ((4-t)a+(t-3)b,c) & \text{if } 3 \le t \le 4. \end{cases}$$

Proposition 22. *Let ω be a 1-form on an open subset $U \subseteq \mathbb{R}^2$. Then: ω is locally exact if and only if $\int_{\partial R} \omega = 0$ for every rectangle $R \subset U$.*

Proof. ⇒) Fix a rectangle $R \subset U$. We can find an open rectangle V such that $R \subset V \subseteq U$. The form $\omega|_V$ is locally exact. As V is simply connected, $\omega|_V$ is exact. Then $\int_{\partial R} \omega = \int_{\partial R} \omega|_V = 0$ by Theorem 18(1)⇒(2").

⇐) Fix an open ball $B \subseteq U$. We want to show that $\omega|_B$ is exact. Let γ be a loop in B which is piecewise affine and all the segments are parallel to the coordinate axes. Since B is a ball, it is possible to find a finite number of rectangles $R_1,\dots,R_r$ such that $\int_\gamma \omega = \sum_{j=1}^r \int_{\partial R_j} \omega$. By the assumption we get that $\int_\gamma \omega$. As γ was arbitrary, we get that $\omega|_B$ is exact by Theorem 18(2")⇒(1). □

Another proof under an additional assumption. Let us assume that ω is C^1. Write $\omega = Pdx + Qdy$, where P,Q are two real C^1 functions on U. By Green's theorem we get that

$$\int_{\partial R} \omega = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

for every rectangle $R \subset U$.

⇒) Since ω is locally exact and C^1, we have that ω is closed, i.e. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on U. By the formula above we get that $\int_{\partial R} \omega = 0$ for every rectangle $R \subset U$.

⇐) Assume by contradiction that ω is not closed. Then there exists a point $p \in U$ such that $\frac{\partial Q}{\partial x}(p) - \frac{\partial P}{\partial y}(p) > 0$. (The case $\frac{\partial Q}{\partial x}(p) - \frac{\partial P}{\partial y}(p) < 0$ is completely analogous.) By continuity of $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, we can find a small rectangle R around p such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0$ on R. By the formula above we get $\int_{\partial R} \omega > 0$, which is absurd. □

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