FUNKTIONENTHEORIE (COMPLEX ANALYSIS)

Freie Universität Berlin, Fachbereich Mathematik und Informatik
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This document contains the summary of the lectures and the problem sheets of the course Complex Analysis (Funktionentheorie), which is taught at Freie Universität Berlin in the Summer Semester 2019.

Preliminary syllabus

The prerequisites are linear algebra, calculus in one variable, calculus in several variables, elementary topology. We will cover the following topics:

(1) Complex numbers, complex exponential and complex logarithm. Complex-differentiable or holomorphic functions (in one variable), Cauchy–Riemann equations. Formal power series, convergent power series, analytic functions in one complex variable. [Car95, I], [Lan93, I,II]
(2) Introduction to homotopy theory: homotopic maps, homotopy equivalence, (deformation) retracts, path homotopies, fundamental group, covering spaces. [Man15, §10-12], [Hat02, §1]
(3) Differential 1-forms on open subsets of $\mathbb{R}^n$: integration along paths, exact and closed forms, an elementary formulation of de Rham theorem. [1]
(4) 1-forms on open subsets of $\mathbb{C}$, winding numbers, Cauchy's theorem. Every holomorphic function is analytic. Properties of holomorphic functions: Cauchy's inequalities, Liouville's theorem. Laurent series and isolated singularities. Residues and evaluation of improper integrals in one real variable. [Car95, II,III], [Lan93, III,VI]

I will mainly follow the books [Car95, Lan93, Man15, Hat02]. The book [Hat02] can be freely downloaded at `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`. There exists an older version of [Car95] in French: [Car61]. The topics (1) and (4) can be found in any text in complex analysis (Funktionentheorie), such as [RKG13, BN10, FL80, FB09, FB93, Pri03]; some of these books are in German. An introduction to the topics (1) and (4) with beautiful pictures can be found in [Nee97, Nee01, Weg12]. The topic (3) is somehow between calculus and differential geometry: unfortunately I do not know a precise reference, because all differential geometry books I know are too advanced; but I have written the note [1] which can be downloaded at `https://userpage.fu-berlin.de/petracci/2019Complex/Forms.pdf`.

Lectures

8th April 2019. Definition of complex numbers. Sum and product of complex numbers. Norm, conjugate, real part, imaginary part of complex numbers and their properties. $\mathbb{C}$ is a field extension of $\mathbb{R}$ with Galois group $\{\mathrm{id}_{\mathbb{C}}, \bar{\cdot}\}$. Exponential of complex numbers: $e^{x+iy} := e^x(\cos y + i \sin y)$. The exponential is a group homomorphism $\mathbb{C} \to \mathbb{C}^*$. Polar form of complex numbers. Roots of unity. [Lan93, I, §1-2]

Definitions of topological space. Open and closed subsets in a topological space. The discrete and indiscrete topologies. Definition of metric space. Definition of ball in a metric space. A metric space is naturally a topological space. Definition of normed vector space. A normed vector space is naturally a metric space. Equivalence of norms on $\mathbb{R}^n$. Examples of balls in $\mathbb{R}^2$ with respect to some norms. Definition of interior part and of closure of a subset in a topological space. Definition of neighbourhood of a point in a topological space. The subspace topology. $[0, 1]$ is not open in $\mathbb{R}$, but is open in itself. $[0, 1)$ is open in $[0, 1]$.

Open balls in a metric space are open. Definition of continuous function. A function between topological spaces is continuous iff the preimage of every open (resp. closed) subset is open (resp. closed). Composition of continuous functions is continuous.

The product topology. The product topology on $\mathbb{R} \times \cdots \times \mathbb{R}$ coincides with the euclidean topology on $\mathbb{R}^n$. If X and Y are topological spaces, then the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are continuous. If X, Y, Z are topological spaces, then a function $f : Z \to X \times Y$ is continuous iff $\mathrm{pr}_X \circ f$ and $\mathrm{pr}_Y \circ f$ are continuous. If V is a normed vector space, then the sum $V \times V \to V$ and the scalar multiplication $\mathbb{R} \times V \to V$ are continuous. If X is a topological space and V is a normed vector space, then the set of continuous functions $X \to V$ is an $\mathbb{R}$-vector space. If X is a topological space, then the set of continuous functions $X \to \mathbb{R}$ (resp. $X \to \mathbb{C}$) is an $\mathbb{R}$-algebra (resp. $\mathbb{C}$-algebra).

Definition of homeomorphism. An example of a bijective continuous function which is not a homeomorphism: the identity from the discrete topology to the indiscrete topology. The exponential gives a homeomorphism between $\mathbb{R}$ and $(0, +\infty)$.

Definition of Hausdorff topological space. Metric spaces are Hausdorff. Definition of connected topological space. A subspace of $\mathbb{R}$ is connected iff it is an interval (without proof). Product and images (via continuous functions) of connected topological spaces are connected (without proof). Definition of open covers and refinements. Definition of compact topological space. $\mathbb{R}^n$ is not compact. Product and images (via continuous functions) of compact topological spaces are compact (without proof). A subset of $\mathbb{R}^n$ is compact iff it is open and closed in $\mathbb{R}^n$ (without proof). $[0,1]$ is not homeomorphic to $[0,1)$. $[0,1)$ is not homeomorphic to $(0,1)$. Definition of the n-sphere and of the n-disk. The stereographic projection is a homeomorphism between $\mathbb{R}^n$ and S^n minus a point. [Man15, §3-4]

9th April 2019. Definition of basis of a topological space. The open rectangles form a basis for the product topology. Definition of local basis (fundamental system of neighbourhoods) at a point. In a metric space the balls centred at a point form a local basis. Definition of first-countable and second-countable topological space. A metric space is first-countable.

Sequences in a topological space: convergence, limit (accumulation) points, uniqueness of the limit in a Hausdorff space, existence of a subsequence converging to a limit point. Definition of a sequentially compact topological space. In a compact topological space, every sequence has some limit points (without proof). A second-countable sequentially compact topological space is compact (without proof).

Sequences in metric spaces. Definition of Cauchy sequence, of complete metric space, of totally bounded metric space. Characterisation of compact a metric spaces: a metric space is compact iff it is sequentially compact iff it is complete and totally bounded (without proof). [Man15, §6]

Sequences of functions from a set to a metric space: definition pointwise convergence, of uniform convergence, of being Cauchy uniform. Uniform convergence implies pointwise convergence and being Cauchy uniform. If the metric space is complete, then being Cauchy uniform implies uniform convergence.

If X is a topological space, Y is a metric space, and $(f_n \colon X \to Y)_n$ is a sequence of continuous functions which uniformly converges to $f \colon X \to Y$, then f is continuous.

Series of functions from a set to a normed vector space: definition of total convergence. If X is a set, Y is a normed vector space, and $f_n \colon X \to Y$ is a sequence of functions such that the series of functions $\sum_{n \geq 0} f_n$ converges totally, then the sequence of partial sums is Cauchy uniform.

If X is a topological space, Y is a Banach space (i.e. a complete normed vector space), and $(f_n \colon X \to Y)_n$ is a sequence of continuous functions whose series converges totally (i.e. $\sum_{n=0}^{+\infty} \sup_{x \in X} \|f_n(x)\| < +\infty$), then the series $\sum_{n=0}^{+\infty} f_n$ uniformly converges to a continuous function $X \to Y$.

15th April 2019. Definition of holomorphic functions and of complex derivative. Examples and non-examples of holomorphic functions. The set of holomorphic functions $\mathcal{O}(U)$ on an open subset $U \subseteq \mathbb{C}$ is a $\mathbb{C}$-algebra. Composition and quotient of holomorphic functions are holomorphic. Recap on partial derivatives and differentiability for maps from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. Cauchy–Riemann equations. Definition of biholomorphism between open subsets of $\mathbb{C}$. [Lan93, I, §5-6] [Car95, II.2.1-2]

Formal power series with coefficient in a field $\mathbb{K}$. $\mathbb{K}[\![T]\!]$ is a $\mathbb{K}$-algebra and is an extension of $\mathbb{K}[T]$. The geometric series $\sum_{n \geq 0} T^n$ is the multiplicative inverse of $1 - T$. A power series has a multiplicative inverse if and only if its constant term is non-zero. Definition of order of a power series. Properties of ord: $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$; $\operatorname{ord}(f+g) \geq \min\{\operatorname{ord}(f), \operatorname{ord}(g)\}$; if $\operatorname{ord}(f) \neq \operatorname{ord}(g)$ then $\operatorname{ord}(f+g) = \min\{\operatorname{ord}(f), \operatorname{ord}(g)\}$. The ring $\mathbb{K}[\![T]\!]$ is an integral domain. Definition of formal derivative of a power series and properties with respect to the sum and the product. If $\operatorname{char}(\mathbb{K}) = 0$ and $f \in \mathbb{K}[\![T]\!]$, then $f' = 0$ if and only if $f \in \mathbb{K}$. [Lan93, II, §1] [Car95, I.1.1-3, I.1.5-6]

The radius of convergence of a power series $\sum_{n \geq 0} a_n T^n \in \mathbb{C}[\![T]\!]$ is $1/\limsup_n \sqrt[n]{|a_n|}$. If (a_n) is a sequence of positive real numbers such that $\lim_n \frac{a_{n+1}}{a_n} = L \in [0, +\infty]$, then $\lim_n \sqrt[n]{|a_n|} = L$. A power series converges absolutely on the ball of convergence and does not converge in the interior part of the complement of the ball of convergence. The function defined on the ball of convergence by a power series is continuous. If $f, g \in \mathbb{C}[\![T]\!]$ have convergence radii $\geq r$, then $f + g$ (resp. fg) has convergence radius $\geq r$ and the function defined by $f + g$ on $B_r(0)$ is the sum (resp. product) of the functions defined by f and g. The set $\mathbb{C}\{T\}$ of power series with positive radius convergence is a subring of $\mathbb{C}[\![T]\!]$. The formal derivative of a power series $f \in \mathbb{C}[\![T]\!]$ has the same convergence radius of f. If $f \in \mathbb{C}[\![T]\!]$ has convergence radius $R > 0$, then the function $B_R(0) \to \mathbb{C}$ defined by f is holomorphic and has complex derivative equal to the function defined by f' (beginning of the proof). [Lan93, II, §2-3] [Car95, I.2.3-4, I.2.7]

16th April 2019 (Übung)**.** Solution of Problem 1c. Description of each set of Problem 2 as preimage of easy open or closed subsets of $\mathbb{R}$ via continuous functions. Solution of Problem 4. Certain subgroups of $\mathrm{GL}_2(\mathbb{R})$; the conformal 2×2 real matrices form a group between $\mathrm{O}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$. Biholomorphisms of open subsets of $\mathbb{C}$ preserve the orientation and the angles. All open intervals of $\mathbb{R}$ are homeomorphic to each other. The interval $[0, 1]$ is not homeomorphic to S^1.

23rd April 2019. Conclusion of the proof from last week. The function defined by a convergent power series on the convergence ball is infinitely many times $\mathbb{C}$-differentiable.

Definition of analytic function. An analytic function is holomorphic and infinitely many times $\mathbb{C}$-differentiable. Uniqueness of power series expansion: relation between the coefficients of the power series expansion and the formal derivatives. The set of analytic functions on an open subset of $\mathbb{C}$ is a $\mathbb{C}$-algebra. The function defined by a convergent power series on the convergence ball is analytic.

Principle of analytic continuation: an analytic function on a connected open subset of $\mathbb{C}$ is zero iff the set of zeroes is not discrete iff there is a point where all the derivatives vanish. The ring of analytic functions on a connected open subset of $\mathbb{C}$ is an integral domain. [Lan93, II, §4-5] [Car95, I.4.1-4]

29th April 2019. Paths in topological spaces: reparametrisation, constant paths, inverse path, concatenation of paths. Being joined by a path is an equivalence relation on a topological space; definition of path-connected topological space and of path-connected components of a topological space. The image of a path-connected topological space via a continuous function is path-connected. The functor π_0 from the category of topological spaces with continuous functions to the category of sets. A path-connected topological space is connected. Every two points in a connected open subset of $\mathbb{R}^n$ are the endpoints of a piecewise affine path with segments parallel to the coordinate axes; every connected open subset of $\mathbb{R}^n$ is path-connected.

Definition of homotopic maps (with respect to a subset). If $Y \subseteq \mathbb{R}^n$ is convex subset, then all continuous maps from a topological space to Y are homotopic to each other. Being homotopic with respect to a subset is an equivalence relation. Definition of homotopy equivalence. Definition of homotopy equivalent topological spaces. Definition of contractible topological space. Definition of retraction and deformation retraction. S^{n-1} is a deformation retract of $\mathbb{R}^n \smallsetminus \{0\}$. If A is a deformation retract of X, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Definition of path-homotopy. Two paths in a convex subset of $\mathbb{R}^n$ are path-homotopic. Properties of being path-homotopic with respect to concatenation, inverse, constant paths. Definition of fundamental group. The fundamental group of a convex subset of $\mathbb{R}^n$ is trivial. The fundamental group of S^1 at 1 is isomorphic to $\mathbb{Z}$ (without proof). The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Dependence of the fundamental group on the base point. Definition of simply connected topological space. A convex subset of $\mathbb{R}^n$ is simply connected. [Man15, §10 and §11.1-2] [Hat02, §1.1]

30th April 2019 (Übung)**.** Solution of Problems 6, 7, 11. The limit $\lim_n \frac{|a_n|}{|a_{n+1}|}$ might not exist, whereas $\limsup_n \sqrt[n]{|a_n|}$ always exists and is equal to the inverse of the convergence radius of $\sum_n a_n T^n$. Solution of Problem 11. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = e^{-1/x^2}$ for $x \neq 0$ is C^∞, but is not real-analytic, because its Taylor series at the origin is zero. Solution of Problem 9. Quick solution of Problem 12.

6th May 2019. Lebesgue's number lemma. [Man15, Theorem 11.23]

The map $p\colon \mathbb{R} \to S^1$ given by $t \mapsto \mathrm{e}^{2\pi i t}$ is a covering space and a group homomorphism. Lifting of loops with respect to p. Lifting of path-homotopies with respect to p. The fundamental group of S^1 is isomorphic to $\mathbb{Z}$. [Hat02, §1.1]

The fundamental group of the product of two topological spaces is the product of the fundamental groups (without proof). Homotopy equivalences induce group isomorphisms on the fundamental groups (without proof). If A is a retract (resp. deformation retract) of X, then the inclusion of A into X induces an injection (resp. isomorphism) on the fundamental groups. [Man15, §11.3]

6th May 2019 (Übung)**.** Solutions of Problem 13 and 14. π_0 is a functor from the category (HTop) of topological spaces up to homotopy to the category of sets. π_1 is a functor from the category (HTop_*) of pointed topological spaces up to homotopy to the category of groups.

Discussion about Problem 15. Relationship among the following properties: convex, star-shaped, deformation-retractable onto a point, contractible, simply connected. If A is a deformation retract of X and $i\colon A \hookrightarrow X$ is the inclusion, then for every point $a_0 \in A$ the homomorphism $\pi_1(i, a_0)\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism.

Quick solution of Problem 16. The fundamental group of $\mathbb{R}^2 \setminus \{n \text{ points}\}$ is the free group on n generators (without proof).

7th May 2019. Definition of derived subgroup and of abelianisation of a group. The universal property of the abelianisation of a group. The abelianisation of the free group with n generators is $\mathbb{Z}^n$, which is the free abelian group with n generators.

The isomorphism between the abelianisation of $\pi_1(X, x_0)$ and the abelianisation of $\pi_1(X, x_1)$ does not depend on the choice of the path from x_0 to x_1. Definition of 1st homology group of a path-connected topological space as the abelianisation of the fundamental group. $\mathrm{H}_1(-;\mathbb{Z})$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups. Two loops in a topological space are called homologous if their homology classes are the same. Examples of path-homotopic and homologous loops in $\mathbb{R}^2 \setminus \{2 \text{ points}\}$. Definition of winding number of a loop around a point in $\mathbb{C}$. Two loops are homologous in $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ if and only if they have the same winding number at each of $z_1, \dots, z_n$.

Definition of 1-forms on open subsets of $\mathbb{R}^n$. Exact/locally exact/closed 1-forms. Definition of the 1st de Rham cohomology group. Existence and uniqueness (up to an additive constant) of the primitive of a locally exact 1-form along a path. Definition of the integral of 1-form along a path. [1]

13th May 2019. The integral of a 1-form along a path does not depend on reparametrisations of the path. Physical interpretation of exact/closed 1-forms and their integrals in terms of conservative/irrotational vector fields and their work along a trajectory. Bilinearity of the integral of a 1-form along a path. A closed 1-form on a star-shaped open subset of $\mathbb{R}^n$ is exact. A 1-form is closed if and only if it is C^1 and locally exact. A 1-form is exact if and only its integral along every loop vanishes. The form $(-y\ \mathrm{d}x + x\ \mathrm{d}y)/(x^2 + y^2)$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is closed, locally exact, non-exact. Homotopy invariance of the integral of locally exact forms. [1]

13th May 2019 (Übung)**.** Solution of Problem 18. Definition of complex logarithm on an open subset of $\mathbb{C}^*$. A complex logarithm is holomorphic (without proof) and its derivative is $\frac{1}{z}$. Three solutions of Problem 17: there is no complex logarithm defined on $\mathbb{C}^*$. Solution of Problem 19. If on an open subset $U \subseteq \mathbb{C}^*$ there is a complex logarithm, then on U there is also an nth root for each $n \in \mathbb{Z}$. Examples of open subsets of $\mathbb{C}^*$ where a complex logarithm exists or does not exist.

Solution of Problem 20: the topological space X deformation retracts onto a torus; so X is homotopy equivalent to $S^1 \times S^1$. Discussion about the direct product and the free product of two groups.

14th May 2019. Homology invariance of the integral of locally exact forms. A 1-form on an open subset U of $\mathbb{R}^n$ is exact if and only if it is locally exact and its integrals along a set of generators of $\mathrm{H}_1(U;\mathbb{Z})$ vanishes. A locally exact 1-form on a simply connected open subset of $\mathbb{R}^n$ is exact. The integration of 1-forms along loops gives an injective $\mathbb{R}$-linear map from the 1st de Rham cohomology to the $\mathbb{R}$-dual of the 1st homology. A 1-form on an open subset U of $\mathbb{R}^2$ is locally exact if and only if its integral is zero along the boundary of every rectangle contained in U. [1]

$\mathbb{C}$-valued 1-forms on open subsets of $\mathbb{C}$. The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. The 1-forms $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. The formula $\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}$. A function f is holomorphic if and only if it is $\mathbb{R}$-differentiable and $\frac{\partial f}{\partial \bar{z}} = 0$. If f is holomorphic, then $f' = \frac{\partial f}{\partial z}$. If g is continuous, then the 1-form $g(z)\mathrm{d}z$ is exact if and only if there exists a holomorphic function f such that $f' = g$. If f is holomorphic and C^1, then the 1-form $f(z)\mathrm{d}z$ is closed (left as an exercise). The 1-form $\frac{\mathrm{d}z}{z}$ on $\mathbb{C}^*$. A complex logarithm is a primitive of $\frac{\mathrm{d}z}{z}$. [Car95, II.2.3]

20th May 2019. Goursat's theorem: if f is holomorphic then the 1-form $f(z)\mathrm{d}z$ is locally exact. A holomorphic function defined on a simply connected open subset of $\mathbb{C}$ has a primitive. Slight generalisation of Goursat's theorem: if $U \subseteq \mathbb{C}$ is open, $L \subseteq \mathbb{C}$ is a horizontal real line and $f\colon U \to \mathbb{C}$ is continuous and such that $f|_{U \setminus L}$ is holomorphic, then the 1-form $f(z)\mathrm{d}z$ is locally exact on U. If $U \subseteq \mathbb{C}$ is open and $f\colon U \to \mathbb{C}$ is continuous such that $f|_{U \setminus \{z_0\}}$ is holomorphic for some $z_0 \in U$, then $f(z)\mathrm{d}z$ is locally exact on U. [Car95, II.2.4]

Cauchy's integral formula: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, $f \in \mathcal{O}(U)$, γ is a loop in $U \setminus \{z_0\}$ which is null-homologous in U, then $W(\gamma, z_0) \cdot f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z-z_0} \mathrm{d}z$. [Car95, II.2.5]

Mean value property: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, $f \in \mathcal{O}(U)$, $r > 0$ is such that $\overline{B_r(z_0)} \subset U$, then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}t}) \mathrm{d}t$. [Car95, III.2.1]

Cauchy's inequality: if $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, $f \in \mathcal{O}(U)$, $r > 0$ is such that $\overline{B_r(z_0)} \subset U$, then $|f(z_0)| \leq \max_{\partial B_r(z_0)} |f|$.

20th May 2019 (Übung)**.** Solution of Problem 24. Solution of Problem 23, with an extra solution of i) without using partitions. Solution of Problem 21. For every abelian group G, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, G)$ is naturally isomorphic to G^n. Solution of Problem 22.

21st May 2019. A holomorphic function on an open ball has a power series expansion on that ball. A function on an open subset of $\mathbb{C}$ is holomorphic if and only if it is analytic. A holomorphic function is C^∞. Integral formula for the coefficients of the power series expansion of a holomorphic function at a point. Cauchy inequalities. The convergence radius of the power series expansion of a holomorphic function $f \in \mathcal{O}(U)$ at the point $z_0 \in U$ is at least the radius of the biggest open ball centred in z_0 and contained in U. Morera's theorem: if f is a continuous function such that $f(z)\mathrm{d}z$ is locally exact, then f is holomorphic. [Car95, II.2.6-7, III.1.1] [Lan93, III-IV]

Relation between the Fourier series and power series expansion of holomorphic functions on the unitary disk.

Definition of entire function. Liouville's theorem: a bounded entire function is constant. An entire function whose real part is bounded above is constant. A proof of the fundamental theorem of algebra. [Car95, III.1.2]

27th May 2019. Maximum modulus principle for functions which satisfy the mean value property on spheres. A function with the mean value property on spheres attains its maximum at the boundary of the bounded open set where it is defined. This last result may be false if the open set is unbounded. [Car95, III.2.2]

Laurent series. Laurent series form a $\mathbb{C}$-vector space, but not a ring. The ring $\mathbb{C}(\!(T)\!)$ of Laurent series with finitely many non-zero coefficients of negative degree. $\mathbb{C}[\![T]\!]$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}(\!(T)\!)$. Convergence of Laurent series: a Laurent series induces a holomorphic function on the anulus of convergence. [Car95, III.4.1]

27th May 2019 (Übung)**.** Solution of Problems 25 and 26. Proof of the equality $\int_{-\infty}^{+\infty} \frac{x+3}{(x^2-2x+2)(x^2+1)} \mathrm{d}x = \frac{7}{5}\pi$, along the lines of Problem 27.

28th May 2019. Laurent expansion of holomorphic functions on anuli. Expression of the coefficients of the Laurent expansion as the integral along a circle. Cauchy's inequalities for the coefficients of the Laurent expansion. The Laurent expansion depends on the anulus: the two Laurent expansions of $\frac{1}{1-z}$ on the anuli $\{|z| < 1\}$ and $\{|z| > 1\}$. [Car95, III.4.2-3]

Classification of isolated singularities: removable, pole, essential. The inverse of a non constantly zero holomorphic function is either holomorphic or has a pole. An isolated singularity is removable if and only if the function is bounded in a punctured neighbourhood. An isolated singularity is a pole if and only if the modulus of the function tends to $+\infty$. [Car95, III.4.4]

3rd June 2019. Casorati–Weierstrass theorem: the image of every punctured neihbourhood of an essential singularity is dense in $\mathbb{C}$. Picard's theorem (without proof). [Car95, III.4.4]

Definition of residues. Practical calculation of residues: simple and multiple poles. The residue of the logarithmic derivative. [Car95, III.5.2-4]

Definition of meromorphic function on an open subset of $\mathbb{C}$. The set $\mathcal{M}(U)$ of meromorphic functions on an open subset $U \subseteq \mathbb{C}$ is a $\mathbb{C}$-algebra. The injective $\mathbb{C}$-algebra homomorphism $\mathcal{O}(U) \hookrightarrow \mathcal{M}(U)$. Comparison among $\mathcal{O}(\mathbb{C}), \mathcal{O}(\mathbb{C}^*), \mathcal{M}(\mathbb{C}), \mathcal{M}(\mathbb{C}^*)$: $\frac{1}{z} \in \mathcal{O}(\mathbb{C}^*) \cap \mathcal{M}(\mathbb{C})$, $\exp(\frac{1}{z}) \in \mathcal{O}(\mathbb{C}^*) \setminus \mathcal{M}(\mathbb{C})$. If $f \in \mathcal{M}(U)$, then $f' \in \mathcal{M}(U)$. If $U \subseteq \mathbb{C}$ is open and connected, then $\mathcal{M}(U)$ is the fraction field of $\mathcal{O}(U)$: proof only of the fact that $\mathcal{M}(U)$ is a field which contains $\mathcal{O}(U)$.

Residue theorem.

3rd June 2019 (ÜBUNG)**.** Solution of Problem 29. Solution of Problem 30: definition of harmonic conjugate. Solution of Problem 32. Quick solution of Problem 31.

4th June 2019. Residue theorem with respect to a compact set with non-empty interior part and piecewise C^1 boundary, for a meromorphic function and for the logarithmic derivative of a meromorphic function. Rouché theorem. Example of application of Rouché: all the zeroes of $z^5 + z^4 + 6z + 1$ are in $B_2(0)$, and only one is in $B_1(0)$. [Car95, Exercise III.19]

Evaluation of definite integrals with the method of residues. Evaluation of $\int_0^{2\pi} R(\cos t, \sin t)\mathrm{d}t$ where $R(x, y)$ is a rational function without poles on $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Example $\int_0^{2\pi} \frac{\mathrm{d}t}{2+\cos t} = \frac{2\pi}{\sqrt{3}}$. Evaluation of $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$, where $P, Q \in \mathbb{R}[x]$ are polynomials such that $\deg Q \geq \deg P + 2$ and Q does not have real zeroes. [Car95, III.6]

11th June 2019 (ÜBUNG)**.** Solution of problems 33 and 35. Solution of Problem 36a,b,c,d.

17th June 2019. Other methods to evaluate real integrals with the method of residues: $\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{\mathrm{i}x}\mathrm{d}x$ where f is a holomorphic function on the closed upper half-plane with the exception of finitely many points in the open upper half-plane. $\int_0^{+\infty} x^\alpha \frac{P(x)}{Q(x)}\mathrm{d}x$ where P, Q are polynomials and $\alpha \in \mathbb{R}$ such that $-1 < \alpha < -1+\deg Q-\deg P$. [Car95, III.6]

If a holomorphic function f is such that $f'(z_0) \neq 0$, then there exists an open neighbourhood of z_0 where the restriction of f is a biholomorphism onto an open neighbourhood of $f(z_0)$ [Lan93, Theorem VI.1.7]. Examples of local biholomorphisms: $\exp\colon \mathbb{C} \to \mathbb{C}^*$ and $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$. [Car95, VI.1.1]

$\mathbb{C}$ and $B_1(0)$ are homeomorphic but not biholomorphic [Car95, VI.2.1]. $B_1(0)$ and $\mathbb{H}$ are biholomorphic [Lan93, Theorem VII.3.1].

17th June 2019 (ÜBUNG)**.** Solution of Problem 38, 40, 39, 37-2. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then the projective space $\mathbb{P}^n(\mathbb{K})$ is compact and path-connected. Playing with homogeneous coordinates: intuition of points at infinity in $\mathbb{P}^2(\mathbb{R})$; $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to S^1; $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to S^2.

24th June 2019. Local behaviour of a non-constant holomorphic map. A non-constant holomorphic map from a connected open subset of $\mathbb{C}$ to $\mathbb{C}$ is open. A holomorphic map is a biholomorphism onto its image if and only if it is injective. [Car95, VI.1.2-3] [Lan93, II, §6]

Definition of charts, atlases, complex structure, Riemann surfaces. Every open subset of $\mathbb{C}$ is a Riemann surface. The Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is a compact connected Riemann surface. Definition of holomorphic map between Riemann surfaces. Definition of holomorphic/meromorphic function on a Riemann surface. Every holomorphic function on the Riemann sphere is constant. [Car95, VI.4.1-5, VI.5.1]

If $F, G \in \mathbb{C}[x_0, x_1]_d$ are homogeneous polynomials of degree $d \geq 1$ such that for every $(\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{0\}$ we have either $F(\lambda_0, \lambda_1) \neq 0$ or $G(\lambda_0, \lambda_1) \neq 0$, then the map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by $[x_0 : x_1] \mapsto [F(x_0, x_1) : G(x_0, x_1)]$ is holomorphic. Definition of Möbius transformations of the Riemann sphere (also called homographies or fractional linear transformations).s Möbius transformations are self-biholomorphisms of the Riemann sphere and form a group under composition (without proof). Möbius transformations send Möbius lines to Möbius lines (without proof). [Lan93, VII, §5]

24th June 2019 (ÜBUNG)**.** Solution of Problems 41.2-4, 42, 43, 44.

1st July 2019. Projective transformations of $\mathbb{P}^n(\mathbb{K})$, where $\mathbb{K}$ is an arbitrary field. The group $\mathrm{PGL}_{n+1}(\mathbb{K})$ is the quotient of $\mathrm{GL}_{n+1}(\mathbb{K})$ by $\mathbb{K}^*$. Affine transformations of $\mathbb{K}^n$ are exactly the projective transformations of $\mathbb{P}^n(\mathbb{K})$ which leave the hyperplane at infinity invariant. The group $\mathrm{PGL}_2(\mathbb{K})$ is generated by $\mathrm{Aff}(\mathbb{K})$ and the involution $\mathrm{inv}\colon z \mapsto z^{-1}$. [FFP16, 1.2, 1.3]

Möbius transformations are exactly the elements of $\mathrm{PGL}_2(\mathbb{C})$. The complex affine transformations $\mathrm{Aff}(\mathbb{C})$ are exactly the real affine transformations $\mathrm{Aff}(\mathbb{R})$ whose linear part preserves the orientation and the angles. The geometry of transformations in $\mathrm{Aff}(\mathbb{C})$. Möbius transformations send Möbius lines to Möbius lines [Lan93, Theorem VII.5.2].

Behaviour at ∞ of a holomorphic function defined on an anulus $\{z \in \mathbb{C} \mid |z| > r\}$ for some $r > 0$. The meromorphic functions on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ are exactly the rational functions: $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(z)$.

Definition of the automorphism group $\mathrm{Aut}(X)$ of a Riemann surface X. The biholomorphisms $\mathbb{C} \to \mathbb{C}$ are of the form $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$; $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(\mathbb{C})$ [Car95, VI.2.3].

1st July 2019 (Übung)**.** Solution of Problem 45. Statement of the Riemann uniformization theorem. Solutions of Problems 48. Solution of Problem 46b and review of the principle of analytic continuation. Solution of Problem 46c and 47a.

2nd July 2019. The biholomorphisms $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ are exactly the Möbius transformations: $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = \mathrm{PGL}_2(\mathbb{C})$. [Car95, VI.2.4]

Recap on de Rham cohomology. If U is an open subset of $\mathbb{R}^n$ such that $\mathrm{H}_1(U;\mathbb{Z})$ is a finite group, then every locally exact 1-form on U is exact, i.e. $\mathrm{H}^1_{\mathrm{dR}}(U) = 0$.

If $n \geq 2$, then the fundamental group and the 1st homology group of $\mathbb{P}^n(\mathbb{R})$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The space $\mathbb{P}^1(\mathbb{R})$ (resp. $\mathbb{P}^1(\mathbb{C})$) is the one-point-compactification (also called Alexandrov compactification) of $\mathbb{R}$ (resp. $\mathbb{C}$), so it is homeomorphic to S^1 (resp. S^2). Study of points at infinity in $\mathbb{P}^2(\mathbb{R})$: $\mathbb{P}^2(\mathbb{R})$ is $\mathbb{R}^2$ plus a point for every non-oriented direction. Projective closure of lines and conics in $\mathbb{R}^2$. All non-singular conics in $\mathbb{P}^2(\mathbb{R})$ are projectively equivalent. [FFP16, Man15]

Definition of genus of a compact connected Riemann surface. If X is a compact connected Riemann surface of genus g, then $\mathrm{H}_1(X;\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. If $F \in \mathbb{C}[x_0, x_1, x_2]_d$ is a homogeneous polynomial of degree $d \geq 1$ such that for every $(a_0, a_1, a_2) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ at least one among $\frac{\partial F}{\partial x_0}(a_0, a_1, a_2)$, $\frac{\partial F}{\partial x_1}(a_0, a_1, a_2)$, $\frac{\partial F}{\partial x_2}(a_0, a_1, a_2)$ is non zero, then $X = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) \mid F(x_0, x_1, x_2) = 0\}$ is a compact connected Riemann surface of genus $(d-1)(d-2)/2$. Definition of M_g, the moduli space of compact connected Riemann surfaces of genus g. Riemann uniformisation implies that M_0 is just a point. M_1 has complex dimension 1. If $g \geq 2$, M_g has complex dimension $3g - 3$.

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Funktionentheorie (Complex Analysis) — Problem sheet 1
To hand in on 15th April 2019 at noon

Problem 1. a) Write the following complex numbers in the polar form: $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $1+i$, $\sqrt{3}+i$.
b) Write the following complex numbers in the form $x+iy$ with $x,y\in\mathbb{R}$: $(1+i)/(1-i)$, $(1+i)^{100}$.
c) Set $\zeta = e^{2\pi i/5}$ and $\alpha = \zeta+\bar{\zeta}$. Prove that $\alpha = (\sqrt{5}-1)/2$ and deduce that $\cos(2\pi/5) = (\sqrt{5}-1)/4$.

Problem 2. Draw the following subsets of $\mathbb{C}$ and say if they are open, closed, bounded, connected, compact.

- $A = \{z\in\mathbb{C} \mid \operatorname{Im} z \geq 0\}$
- $B = \{z\in\mathbb{C} \mid \operatorname{Re} z > 0\}$
- $C = \{z\in\mathbb{C} \mid 0 < |z| < 1\}$
- $D = \{z\in\mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| < 1\}$
- $E = \{z\in\mathbb{C} \mid |z-i| < 1\}$
- $F = \{z\in\mathbb{C} \mid |z-1| = |z+1|\}$
- $G = \{z\in\mathbb{C} \mid |z-i| \geq |z-1|\}$
- $H = \{z\in\mathbb{C} \mid |z|^3 \geq |z|\}$
- $I = \{z\in\mathbb{C} \mid e^z = i\}$
- $J = \left\{z\in\mathbb{C} \,\middle|\, \left|\frac{1}{z+1}\right| < 1\right\}$

Problem 3. a) Let (X,d) be a metric space. Show that for all $x_0\in X$ and $r>0$ the ball $B_r(x_0)$ is an open subset.
b) Let (X,d_X) and (Y,d_Y) be two metric spaces and let $f\colon X\to Y$ be a map. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\inf_n r_n = 0$. Prove that the following statements are equivalent:

(i) f is continuous,
(ii) $\forall x_0\in X, \forall\varepsilon>0, \exists\delta>0 : \forall x\in B_\delta(x_0), f(x)\in B_\varepsilon(f(x_0))$,
(iii) $\forall x_0\in X, \forall\varepsilon>0, \exists\delta>0 : B_\delta(x_0)\subseteq f^{-1}(B_\varepsilon(f(x_0)))$,
(iv) $\forall x_0\in X, \forall n\in\mathbb{N}, \exists\delta>0 : B_\delta(x_0)\subseteq f^{-1}(B_{r_n}(f(x_0)))$,
(v) $\forall x_0\in X, \forall\varepsilon>0$, the set $f^{-1}(B_\varepsilon(f(x_0)))$ is open in X,
(vi) $\forall U\subseteq Y$ open, $f^{-1}(U)$ is open in X,
(vii) $\forall C\subseteq Y$ closed, $f^{-1}(C)$ is closed in X.

Problem 4. (a) The complex field $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$ with basis $\{1,i\}$. Let $\alpha = a+ib\in\mathbb{C}$ with $a,b\in\mathbb{R}$. Consider the map $m_\alpha\colon\mathbb{C}\to\mathbb{C}$ defined by $z\mapsto\alpha\cdot z$. It is a linear endomorphism of the $\mathbb{R}$-vector space $\mathbb{C}$. What is the matrix associated to m_α with respect to the basis $\{1,i\}$? What is the determinant of m_α?
(b) Let $\mathbb{R}^2$ be equipped with the standard scalar product. Fix $A\in\mathrm{GL}_2(\mathbb{R})$ and consider the linear map $L_A\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by $L_A(v) = Av$. Prove that the following conditions are equivalent:

(i) L_A preserves angles, i.e. for all $v_1,v_2\in\mathbb{R}^2\smallsetminus\{0\}$ the (non-oriented) angle between Av_1 and Av_2 is equal to the (non-oriented) angle between v_1 and v_2;
(ii) L_A preserves perpendicularity, i.e. whenever $v_1\in\mathbb{R}^2$ is orthogonal to $v_2\in\mathbb{R}^2$ then Av_1 is orthogonal to Av_2;
(iii) Ae_1 is orthogonal to Ae_2 and $A(e_1-e_2)$ is orthogonal to $A(e_1+e_2)$, where $\{e_1,e_2\}$ is the standard basis of $\mathbb{R}^2$;
(iv) there exist $a,b\in\mathbb{R}$, with $a^2+b^2\neq 0$, such that

$$\text{either}\qquad A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad\text{or}\qquad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

(c) Identifying $\mathbb{C}$ with $\mathbb{R}^2$ with the basis $\{1,i\}$, prove that m_α preserves angles for each $\alpha\in\mathbb{C}\smallsetminus\{0\}$.

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Funktionentheorie (Complex Analysis) — Problem sheet 2
To hand in on 23rd April 2019 at noon

Problem 5. For each of the following functions $f\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$, determine the set on which f is holomorphic and find the derivative of f at all holomorphic points.

a) e^z b) $x^2y + ixy^2$ c) $x^2 - y^2 + 2ixy$

d) $\bar{z}$ e) $|z|^2$

Problem 6. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a holomorphic function.

a) Let $V \subseteq \mathbb{C}$ be an open subset. If $f(U) \subseteq V$ and $g\colon V \to \mathbb{C}$ is a holomorphic function, show that $g \circ f\colon U \to \mathbb{C}$ is holomorphic and compute its derivative.

b) If $g\colon U \to \mathbb{C} \smallsetminus \{0\}$ is a holomorphic function, show that $z \mapsto \frac{f(z)}{g(z)}$ is a holomorphic function on U with derivative $\frac{f'g - fg'}{g^2}$.

Problem 7. a) Let (F_n) be the Fibonacci sequence, i.e. the sequence of natural numbers defined by

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \forall n \geq 1. \end{cases}$$

Consider the power series $g = \sum_{n\geq 0} F_{n+1} T^n$. Explicitly find the multiplicative inverse of g in $\mathbb{C}[\![T]\!]$.

b) Consider the following two formal power series

$$\cos T := \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} T^{2n} \qquad \text{and} \qquad \sin T := \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} T^{2n+1}.$$

Prove that the equality $(\cos T)^2 + (\sin T)^2 = 1$ holds in $\mathbb{C}[\![T]\!]$. [Hint: avoid long calculations!]

Problem 8. Find the radius of convergence of the following power series.

a) $\displaystyle\sum_{n\geq 0} \frac{(-i)^n}{n!} T^n$ b) $\displaystyle\sum_{n\geq 0} (n + 2^n) T^n$

c) $\displaystyle\sum_{n\geq 0} \frac{n!}{n^n} T^n$ d) $\displaystyle\sum_{n\geq 0} n^m c_n T^n$ for $m \in \mathbb{N}$ in terms of the radius of convergence R of $\displaystyle\sum_{n\geq 0} c_n T^n$

e) $\displaystyle\sum_{n\geq 1} n^{\log n} T^n$

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Funktionentheorie (Complex Analysis) — Problem sheet 3
To hand in on 29th April 2019 at noon

Problem 9. Let $A \subseteq \mathbb{R}$ be an open subset; a function $\varphi\colon A \to \mathbb{R}$ is called *real-analytic* if for every point $x_0 \in A$ the following condition is satisfied: there exist a real number $\varepsilon > 0$ and a power series $\sum_{n\geq 0} a_n T^n \in \mathbb{R}[\![T]\!]$ with real coefficients such that

- the convergence radius of $\sum_{n\geq 0} a_n T^n$ is $\geq \varepsilon$,
- $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq A$,
- $\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon),\ \varphi(x) = \sum_{n\geq 0} a_n (x - x_0)^n$.

a) Let $U \subseteq \mathbb{C}$ be an open subset and $f\colon U \to \mathbb{C}$ be an analytic function such that $f(U \cap \mathbb{R}) \subseteq \mathbb{R}$. Prove that the power series expansion of f at each point of $U \cap \mathbb{R}$ has real coefficients and that $f|_{U\cap\mathbb{R}}\colon U \cap \mathbb{R} \to \mathbb{R}$ is real-analytic.

b) Let $A \subseteq \mathbb{R}$ be an open subset and let $\varphi\colon A \to \mathbb{R}$ be a real-analytic function. Prove that there exist an open subset $U \subseteq \mathbb{C}$ and an analytic function $f\colon U \to \mathbb{C}$ such that $U \cap \mathbb{R} = A$ and $f|_A = \varphi$.

Problem 10. Consider the real number $\varphi = (1 + \sqrt{5})/2$ and the function $r\colon B_{1/\varphi}(0) \to \mathbb{C}$ defined by

$$\forall z \in B_{1/\varphi}(0), \qquad r(z) = \frac{z}{1 - z - z^2}.$$

Prove that r is analytic and compute the power series expansion of r at the origin. [Hint: consider the Fibonacci sequence (F_n), i.e. the sequence of natural numbers defined by

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \forall n \geq 1. \end{cases}$$

You can assume that the equality $\lim_n \frac{F_{n+1}}{F_n} = \varphi$ is well known and you do not need to prove this.]

Problem 11. Fix $a \in \mathbb{C}$ and consider the function $f\colon \mathbb{C} \smallsetminus \{a\} \to \mathbb{C}$ defined by $f(z) = \frac{1}{z-a}$. For each point $z_0 \in \mathbb{C} \smallsetminus \{a\}$, explicitly find a power series expansion of f around z_0 and compute its convergence radius. Deduce that f is analytic.

Problem 12. a) Exhibit an explicit homeomorphism between the topological spaces $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ and $\mathbb{R} \times S^1$.

b) Let $0 \leq k < n$ be integers and let A be a k-dimensional affine subspace of $\mathbb{R}^n$. Prove that $\mathbb{R}^n \smallsetminus A$ is homeomorphic to $\mathbb{R}^{k+1} \times S^{n-k-1}$.

c) Consider the sans-serif capital alphabet ABCDEFGHIJKLMNOPQRSTUVWXYZ. Think of each letter as a compact connected subspace of $\mathbb{R}^2$ which is the union of finitely many C^∞ curves. Which letters are homeomorphic to each other? Present the partition of the alphabet into homeomorphism classes. [For this part you do not need to give proofs. Just give the results.]

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Funktionentheorie (Complex Analysis) — Problem sheet 4
To hand in on 6th May 2019 at noon

Problem 13. Let X and Y be topological spaces and let $f\colon X \to Y$ be a map. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of subsets of X such that $\bigcup_{\lambda\in\Lambda} A_\lambda = X$. Assume that, for every $\lambda \in \Lambda$, the restriction $f|_{A_\lambda}\colon A_\lambda \to Y$ is continuous.

a) If A_λ is open in X for every $\lambda \in \Lambda$, show that f is continuous.

b) If Λ is finite and A_λ is closed in X for every $\lambda \in \Lambda$, show that f is continuous.

c) Give an example where f is not continuous and A_λ is closed in X for every $\lambda \in \Lambda$.

Problem 14. a) Let X, Y, Z be three topological spaces and let $A \subseteq X$ be a subset. Consider four continuous maps $f_0\colon X \to Y$, $f_1\colon X \to Y$, $g_0\colon Y \to Z$ and $g_1\colon Y \to Z$. If f_0, f_1 are homotopic with respect to A and g_0, g_1 are homotopic with respect to $f_0(A)$, then prove that $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic with respect to A.

b) Let X, Y be two topological spaces and let $f_0\colon X \to Y$ and $f_1\colon X \to Y$ be two continuous map which are homotopic. Prove that $\pi_0(f_0) = \pi_0(f_1)\colon \pi_0(X) \to \pi_0(Y)$. [This implies that π_0 is a functor from the category of topological spaces up to homotopy to the category of sets.]

Problem 15. a) Let X be a topological space. Consider the following three conditions:

(i) X is contractible,

(ii) $\mathrm{id}_X\colon X \to X$ is homotopic to a constant map $X \to X$,

(iii) there exists a point $x_0 \in X$ such that $\{x_0\}$ is a deformation retract of X.

Prove the following implications: (iii)$\Rightarrow$(i)$\Leftrightarrow$(ii). [Unfortunately there exist topological spaces which are contractible but such that there is no point onto which they deformation retract. See [Hat02, Chapter 0, Exercises 6 and 7].]

b) A subset $X \subseteq \mathbb{R}^n$ is called *star-shaped* if there exists a point $x_0 \in X$ such that for all $x \in X$ the segment with endpoints x_0 and x is contained in X. Prove that every star-shaped subset of $\mathbb{R}^n$ is contractible and simply connected. Give an example of a contractible subset of $\mathbb{R}^2$ which is not star-shaped.

Problem 16. a) Consider the sans-serif capital alphabet ABCDEFGHIJKLMNOPQRSTUVWXYZ. Think of each letter as a compact connected subspace of $\mathbb{R}^2$ which is the union of finitely many C^∞ curves. Which letters are homotopy equivalent to each other? Present the partition of the alphabet into homotopy equivalence classes. [For this part you do not need to give proofs. Just give the results.]

b) Fix integers $0 \leq k < n$. Let A be a k-dimensional affine subspace of $\mathbb{R}^n$. Prove that $X = \mathbb{R}^n \smallsetminus A$ is homotopy equivalent to S^{n-k-1}.

c) Make a drawing in order convince yourself and me that $\mathbb{R}^2 \smallsetminus \{2 \text{ points}\}$ is homotopy equivalent to the topological space $\{z \in \mathbb{C} \mid (|z-1|-1)(|z+1|-1) = 0\}$. What about $S^2 \smallsetminus \{3 \text{ points}\}$?

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Funktionentheorie (Complex Analysis) — Problem sheet 5
To hand in on 13th May 2019 at noon

Problem 17. Prove that there is no continuous function $f \colon \mathbb{C}^* \to \mathbb{C}$ such that $\forall z \in \mathbb{C}^*,\ e^{f(z)} = z$.

Problem 18. i) Consider the open subset $U = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Explicitly exhibit a holomorphic function $f \colon U \to \mathbb{C}$ such that $\forall z \in U,\ e^{f(z)} = z$.

ii) Consider the open subset $V = \mathbb{C} \smallsetminus \{x \in \mathbb{R} \mid x \leq 0\}$. Explicitly exhibit a holomorphic function $f \colon V \to \mathbb{C}$ such that $\forall z \in V,\ e^{f(z)} = z$.

Problem 19. Fix an integer $n \in \mathbb{Z} \smallsetminus \{-1, 0, 1\}$.

a) What is the group homomorphism $\pi_1(\mathbb{C}^*, 1) \to \pi_1(\mathbb{C}^*, 1)$ induced by the continuous map $\mathbb{C}^* \to \mathbb{C}^*$ defined by $z \mapsto z^n$?

b) Prove that there is no continuous function $g \colon \mathbb{C}^* \to \mathbb{C}^*$ such that $\forall z \in \mathbb{C}^*,\ g(z)^n = z$.

c) Consider the open subset $V = \mathbb{C} \smallsetminus \{x \in \mathbb{R} \mid x \leq 0\} \subseteq \mathbb{C}$. Prove that there exists a holomorphic function $g \colon V \to \mathbb{C}$ such that $\forall z \in V,\ g(z)^n = z$.

Problem 20. Consider the topological space $X = \mathbb{R}^3 \smallsetminus (C \cup L)$, where $C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $L = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$. Compute the fundamental group of X. Make a picture of X where you draw a set of generators of $\pi_1(X, x_0)$, where x_0 is your favourite point in X. [Hint: show that X is homeomorphic to $S^1 \times \big(((0, +\infty) \times \mathbb{R}) \smallsetminus \{(1, 0)\} \big)$.]

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Funktionentheorie (Complex Analysis) — Problem sheet 6
To hand in on 20th May 2019 at noon

Problem 21. Consider the open subset $U = \mathbb{R}^3 \setminus (C \cup L)$ of $\mathbb{R}^3$, where $C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $L = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$. Consider the following two 1-forms on U

$$\omega = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{r^2} \qquad \text{and} \qquad \eta = -\frac{z}{(r-1)^2 + z^2}\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{r} + \frac{r-1}{(r-1)^2 + z^2}\,\mathrm{d}z$$

where $r \colon U \to \mathbb{R}$ is defined by $r(x, y, z) = \sqrt{x^2 + y^2}$. Consider the following two loops γ and β in U defined by

$$\gamma(t) = (\cos t, \sin t, 1) \qquad t \in [0, 2\pi],$$

$$\beta(t) = \left(1 + \frac{1}{2}\cos t,\ 0,\ \frac{1}{2}\sin t\right) \qquad t \in [0, 2\pi].$$

a) Prove that ω and η are locally exact.
b) Compute $\int_\gamma \omega$, $\int_\beta \omega$, $\int_\gamma \eta$, $\int_\beta \eta$.
c) Prove that the $\mathbb{R}$-vector space $\mathrm{H}^1_{\mathrm{dR}}(U)$ has dimension 2.

Problem 22. Consider the open subset $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ of $\mathbb{R}^2$.
a) Let $\varphi \colon (0, +\infty) \to \mathbb{R}$ be a C^1 function. Consider the 1-form $\eta = \varphi(x^2 + y^2) \cdot (x\,\mathrm{d}x + y\,\mathrm{d}y)$ on U. Prove that η is closed. Is η exact?
b) Let $C^1(U)$ be the set of C^1 functions $U \to \mathbb{R}$. For each $u \in C^1(U)$, we consider the 1-form

$$\omega_u = u(x, y)\,\mathrm{d}x + \frac{y^3}{x^2 + y^4}\,\mathrm{d}y$$

on U. For which $u \in C^1(U)$ is the 1-form ω_u locally exact? For which $u \in C^1(U)$ is the 1-form ω_u exact?

Problem 23. i) Let $a, b \in \mathbb{R}$ such that $a < b$ and let $f \colon [a, b] \to \mathbb{C}$ be a continuous function. We define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + \mathrm{i}\int_a^b v(t)\mathrm{d}t,$$

where $u \colon [a, b] \to \mathbb{R}$ and $v \colon [a, b] \to \mathbb{R}$ are such that $f = u + \mathrm{i}v$. Prove the following inequality:

$$\left| \int_a^b f(t)\mathrm{d}t \right| \leq \int_a^b |f(t)|\mathrm{d}t.$$

Hint: you need to remember the definition of integrals of real continuous functions in one variable, as follows. Let $g \colon [a, b] \to \mathbb{R}$ be a continuous function. Consider $\mathcal{P}$ the set of finite partitions of $[a, b]$, i.e. finite sequences $\sigma = (t_0, t_1, \ldots, t_n)$ such that $n \geq 1$ and $a = t_0 < t_1 < \cdots < t_n = b$. For every $\sigma = (t_0, t_1, \ldots, t_n) \in \mathcal{P}$ we define $I(\sigma, g) := \sum_{j=1}^n (t_j - t_{j-1}) g(t_j)$ and $\delta(\sigma) := \sup_{1 \leq j \leq n}(t_j - t_{j-1})$. Then

$$\int_a^b g(t)\mathrm{d}t := \lim_{\delta(\sigma) \to 0} I(\sigma, g).$$

ii) Let $U \subseteq \mathbb{C}$ be an open subset and $f \colon U \to \mathbb{C}$ be a continuous function. If $\gamma \colon [a, b] \to U$ is a piecewise C^1 path in U, prove the inequality

$$\left| \int_\gamma f(z)\mathrm{d}z \right| \leq L(\gamma) \cdot \max_{\gamma([a,b])} |f|,$$

where $L(\gamma) = \int_a^b |\gamma'(t)|\mathrm{d}t$ is the length of γ.

Problem 24. i) Let $U \subseteq \mathbb{C}$ be an open subset and $f \colon U \to \mathbb{C}$ be a C^1 holomorphic function. Prove that the 1-form $f(z)\mathrm{d}z$ on U is closed.
ii) Fix $m \in \mathbb{Z}$ and $r \in \mathbb{R}_{>0}$. Consider the loop $\gamma \colon [0, 2\pi] \to \mathbb{C}^*$ defined by $\gamma(t) = r\mathrm{e}^{\mathrm{i}t}$ for every $t \in [0, 2\pi]$. Evaluate the integral $\int_\gamma z^m \mathrm{d}z$.

iii) Fix $z_0 \in \mathbb{C}$ and a loop γ in $\mathbb{C} \smallsetminus \{z_0\}$. Let $W(\gamma, z_0)$ be the winding number of γ around z_0. Prove the formula

$$W(\gamma, z_0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z - z_0}.$$

[Hint: recall that the winding number of γ around z_0 is the unique integer $W(\gamma, z_0) \in \mathbb{Z}$ such that the equality $[\gamma] = W(\gamma, z_0) \cdot [\alpha]$ holds in $\mathrm{H}_1(\mathbb{C} \smallsetminus \{z_0\}; \mathbb{Z})$, where $[\alpha]$ is the homology class of the loop $\alpha\colon [0, 2\pi] \to \mathbb{C} \smallsetminus \{z_0\}$ defined by $\alpha(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$ and r is any positive real number.]

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Funktionentheorie (Complex Analysis) — Problem sheet 7
To hand in on 27th May 2019 at noon

Problem 25. Let $U \subseteq \mathbb{C}^*$ be an open connected subset. Consider the inclusion $i\colon U \hookrightarrow \mathbb{C}^*$. Fix an integer $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and a point $z_0 \in U$. Prove that the following statements are equivalent:

(1) there exists a holomorphic function $f\colon U \to \mathbb{C}$ such that $\forall z \in U,\ e^{f(z)} = z$,
(2) there exists a continuous function $f\colon U \to \mathbb{C}$ such that $\forall z \in U,\ e^{f(z)} = z$,
(3) the homomorphism $\pi_1(i, z_0)\colon \pi_1(U, z_0) \to \pi_1(\mathbb{C}^*, z_0)$ is zero,
(4) the homomorphism $\mathrm{H}_1(i)\colon \mathrm{H}_1(U; \mathbb{Z}) \to \mathrm{H}_1(\mathbb{C}^*; \mathbb{Z})$ is zero,
(5) the 1-form $\frac{\mathrm{d}z}{z}$ is exact on U,
(6) there exists a holomorphic function $g\colon U \to \mathbb{C}^*$ such that $\forall z \in U,\ g(z)^n = z$,
(7) there exists a continuous function $g\colon U \to \mathbb{C}^*$ such that $\forall z \in U,\ g(z)^n = z$,
(8) the image of $\mathrm{H}_1(i)\colon \mathrm{H}_1(U; \mathbb{Z}) \to \mathrm{H}_1(\mathbb{C}^*; \mathbb{Z})$ is contained in the subgroup $\{nc \mid c \in \mathrm{H}_1(\mathbb{C}^*; \mathbb{Z})\}$ of $\mathrm{H}_1(\mathbb{C}^*; \mathbb{Z})$.

[The implication (8)⇒(4) is true, but you can assume it for granted and you do not need to prove it. I would suggest you to prove (4)⇒(5) and (5)⇒(1) among the many possible implications.]

Problem 26. Consider the following open subsets of $\mathbb{C}$: $U = \mathbb{C} \setminus \{\mathrm{i}, -\mathrm{i}\}$, $V = \mathbb{C} \setminus \{\mathrm{i}y \mid -1 \le y \le 1\}$, and $A = \mathbb{C} \setminus \{\mathrm{i}y \mid y \le 1\}$. For every $a, b \in \mathbb{C}$ consider the 1-form $\omega = \frac{az+b}{z^2+1}\mathrm{d}z$ on U.

i) For which $a, b \in \mathbb{C}$ is the form $\omega|_A$ exact on A?
ii) For which $a, b \in \mathbb{C}$ is the form ω exact on U?
iii) For which $a, b \in \mathbb{C}$ is the form $\omega|_V$ exact on V?

[Hint: find $c, d \in \mathbb{C}$ such that $\frac{az+b}{z^2+1} = \frac{c}{z+\mathrm{i}} + \frac{d}{z-\mathrm{i}}$ and use Problem 24iii (or Cauchy's integral formula) for appropriate generators of the 1st homology groups.]

Problem 27. In this exercise you have to pretend that you do not know the values of any primitive of the real function $x \mapsto \frac{1}{x^2+1}$. In other words you cannot use the values of the function arctan, nor the fact that $\lim_{x\to\pm\infty} \arctan(x) = \pm\frac{\pi}{2}$.

For every real number $r > 0$ consider the two paths in $\mathbb{C}$ given by

$$\alpha_r\colon [-r, r] \to \mathbb{R} \qquad t \mapsto t,$$
$$\beta_r\colon [0, \pi] \to \mathbb{C} \qquad t \mapsto r\mathrm{e}^{\mathrm{i}t}.$$

Consider the 1-form $\omega = \frac{\mathrm{d}z}{z^2+1}$ on $\mathbb{C} \setminus \{\pm i\}$.

a) For every $r \in (0, 1) \cup (1, +\infty)$, evaluate the integral

$$\int_{\alpha_r * \beta_r} \omega.$$

[Hint: write $\omega = \frac{a}{z-\mathrm{i}}\mathrm{d}z + \frac{b}{z+\mathrm{i}}\mathrm{d}z$ for some $a, b \in \mathbb{C}$.]

b) Prove that for every $r > 1$

$$\left| \int_{\beta_r} \omega \right| \le \frac{\pi r}{r^2 - 1}.$$

c) Evaluate

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2+1} = \lim_{r\to+\infty} \int_{\alpha_r} \omega.$$

Problem 28. Consider the 1-form

$$\omega = \frac{\mathrm{e}^{-z}}{(z+2)^3}\mathrm{d}z$$

on $\mathbb{C} \setminus \{-2\}$. For each $r > 0$, consider the rectangle $R_r = \{x + \mathrm{i}y \in \mathbb{C} \mid |x| \le r,\ |y| \le 1\}$ and the loop $\gamma_r\colon [0, 2] \to \mathbb{C}$ defined by $\gamma_r(t) = r(\cos(2\pi t^2) + \mathrm{i}\sin(2\pi t^2))$ for every $t \in [0, 2]$. Evaluate the integrals

$$\int_{\partial B_r(0)} \omega, \qquad \int_{\partial R_r} \omega, \qquad \text{and} \qquad \int_{\gamma_r} \omega$$

for every $r \in (0, 2) \cup (2, +\infty)$.

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Funktionentheorie (Complex Analysis) — Problem sheet 8
To hand in on 3rd June 2019 at noon

Problem 29. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Prove the following statements.

a) Let p be a non-negative integer. If there exist real numbers $C, R > 0$ such that $\forall z \in \mathbb{C} \smallsetminus B_R(0),\ |f(z)| \leq C|z|^p$, then f is a polynomial of degree $\leq p$.
b) If there exists $C \in \mathbb{R}$ such that $\forall z \in \mathbb{C},\ \operatorname{Re} f(z) \geq C$, then f is constant.
c) If there exists $C \in \mathbb{R}$ such that $\forall z \in \mathbb{C},\ \operatorname{Im} f(z) \leq C$, then f is constant.
d) If there exists $C \in \mathbb{R}$ such that $\forall z \in \mathbb{C},\ \operatorname{Im} f(z) \geq C$, then f is constant.
e) If $\sup_{z \in \mathbb{C}} |f(z)^2 + 1| < +\infty$, then f is constant.

Problem 30. Let $U \subseteq \mathbb{R}^2$ be an open subset. A function $u\colon U \to \mathbb{R}$ is called *harmonic* if it is C^2 and $\Delta u = 0$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the laplacian operator. Prove the following statements.

a) If $f\colon U \to \mathbb{C}$ is holomorphic, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.
b) If $u\colon U \to \mathbb{R}$ is harmonic, then the 1-form $-\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$ on U is closed.
c) If $u\colon U \to \mathbb{R}$ is harmonic and U is simply connected, then there exists a harmonic function $v\colon U \to \mathbb{R}$ such that $u + \mathrm{i}v\colon U \to \mathbb{C}$ is holomorphic.
d) Exhibit an open connected subset $U \subseteq \mathbb{R}^2$ and a harmonic function $u\colon U \to \mathbb{R}$ such that there is no holomorphic function $f\colon U \to \mathbb{C}$ such that $u = \operatorname{Re} f$.
e) If $u\colon U \to \mathbb{R}$ is harmonic, then u is C^∞.
f) If $u\colon U \to \mathbb{R}$ is harmonic, then $\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\colon U \to \mathbb{C}$ is holomorphic.
g) If $u\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic and non-constant, then $u(\mathbb{R}^2) = \mathbb{R}$.

Problem 31. Consider the holomorphic function $f\colon \mathbb{C} \smallsetminus \{-1, 3\} \to \mathbb{C}$ given by

$$f(z) = \frac{-4}{(z+1)(z-3)}.$$

Explicitly exhibit the Laurent expansions of f around $0 \in \mathbb{C}$ in the following three anuli: $A_{0,1}(0) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, $A_{1,3}(0) = \{z \in \mathbb{C} \mid 1 < |z| < 3\}$ and $A_{3,+\infty}(0) = \{z \in \mathbb{C} \mid |z| > 3\}$. For each positive real number $r \neq 1, 3$, evaluate the integral

$$\int_{\partial B_r(0)} f(z)\mathrm{d}z.$$

Problem 32. There exists a real number $\varepsilon > 0$ such that the following functions are holomorphic on the anulus $\{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\}$:

a) $\dfrac{1}{1-z^2}$
b) $\mathrm{e}^{1/z}$
c) $\dfrac{\mathrm{e}^{-z}}{z(z+1)}$
d) $\dfrac{\sin z}{z}$
e) $\dfrac{z}{\sin z}$
f) $z \sin\left(\dfrac{1}{z}\right)$

where $\sin z := \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})$. For each of these functions, compute the coefficients a_n for $|n| \leq 2$ of their Laurent expansion around 0 in the anulus $\{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\}$.

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Funktionentheorie (Complex Analysis) — Problem sheet 9
To hand in on 11th June 2019 at noon

Problem 33. Evaluate all non-trivial residues of the following meromorphic functions on $\mathbb{C}$:

$$f(z) := \frac{1}{(z^2+1)(z-i)^3} \qquad \text{and} \qquad g(z) := \frac{1}{\exp(z)+1}.$$

Problem 34. Using the method of residues, evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x}{(x^2+1)(x^2-2x+2)} \mathrm{d}x.$$

Problem 35. Fix an integer $n \geq 2$. For each $0 < \varepsilon < 1$ consider the open subset

$$U_\varepsilon = \left\{ r\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ -\frac{\varepsilon}{2} < \theta < 2\pi - \frac{\varepsilon}{2} \right\}$$

of $\mathbb{C}^*$, the holomorphic functions $g_\varepsilon, f_\varepsilon \in \mathcal{O}(U_\varepsilon)$ defined by

$$g_\varepsilon\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = \sqrt[n]{r}\mathrm{e}^{\mathrm{i}\frac{\theta}{n}} \qquad \forall r > 0,\ \theta \in \left(-\frac{\varepsilon}{2}, 2\pi - \frac{\varepsilon}{2}\right)$$

$$f_\varepsilon(z) = \frac{g_\varepsilon(z)}{(z^2+1)^2} \qquad \forall z \in U_\varepsilon,$$

and the paths $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon$ in U_ε defined by:

$$\begin{aligned}
\alpha_\varepsilon &\colon \left[\varepsilon, \varepsilon^{-1}\right] \to U_\varepsilon & t &\mapsto t \\
\beta_\varepsilon &\colon [0, 2\pi - \varepsilon] \to U_\varepsilon & t &\mapsto \varepsilon^{-1}\mathrm{e}^{\mathrm{i}t} \\
\gamma_\varepsilon &\colon \left[\varepsilon, \varepsilon^{-1}\right] \to U_\varepsilon & t &\mapsto t\mathrm{e}^{-\mathrm{i}\varepsilon} \\
\delta_\varepsilon &\colon [0, 2\pi - \varepsilon] \to U_\varepsilon & t &\mapsto \varepsilon\mathrm{e}^{\mathrm{i}t}.
\end{aligned}$$

Set $\zeta = \mathrm{e}^{\frac{\pi \mathrm{i}}{2n}}$ and

$$I = \int_0^{+\infty} \frac{\sqrt[n]{x}}{(x^2+1)^2} \mathrm{d}x.$$

a) For each $0 < \varepsilon < 1$, prove that

$$g'_\varepsilon(z) = \frac{g_\varepsilon(z)}{nz}$$

for all $z \in U_\varepsilon$.

b) For each $0 < \varepsilon < 1$, evaluate the residue of f_ε at i and at $-$i.

c) For each $0 < \varepsilon < 1$, prove the equality

$$\int_{\alpha_\varepsilon * \beta_\varepsilon * i(\gamma_\varepsilon) * i(\delta_\varepsilon)} f_\varepsilon(z)\mathrm{d}z = \frac{\pi}{2}\frac{n-1}{n}\left(\zeta - \zeta^3\right).$$

d) Prove

$$\lim_{\varepsilon \to 0^+} \int_{\alpha_\varepsilon} f_\varepsilon(z)\mathrm{d}z = I \qquad \lim_{\varepsilon \to 0^+} \int_{\beta_\varepsilon} f_\varepsilon(z)\mathrm{d}z = 0$$

$$\lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} f_\varepsilon(z)\mathrm{d}z = \zeta^4 I \qquad \lim_{\varepsilon \to 0^+} \int_{\delta_\varepsilon} f_\varepsilon(z)\mathrm{d}z = 0.$$

e) For every $\varphi \in \mathbb{R} \smallsetminus (1+2\mathbb{Z})\frac{\pi}{2}$, set $w = \mathrm{e}^{\mathrm{i}\varphi}$ and prove the equalities:

$$\frac{1}{1+w^2} = \frac{1}{2}\left(1 - \frac{\sin\varphi}{\cos\varphi}\mathrm{i}\right) \qquad \frac{w}{1+w^2} = \frac{1}{2\cos\varphi}.$$

f) Prove the equality

$$I = \frac{\pi}{4}\frac{n-1}{n}\frac{1}{\cos\frac{\pi}{2n}}.$$

Problem 36. Let $\mathbb{K}$ be a field and let $n \geq 1$ be an integer. On the set $\mathbb{K}^{n+1} \smallsetminus \{0\}$ consider the equivalence relation $\sim$ defined by: $v \sim w$ if and only if there exists $\lambda \in \mathbb{K}^*$ such that $v = \lambda w$. Let $\mathbb{P}^n(\mathbb{K})$ denote the quotient set $(\mathbb{K}^{n+1} \smallsetminus \{0\})/\sim$ and let $\pi\colon \mathbb{K}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n(\mathbb{K})$ be the quotient map. The set $\mathbb{P}^n(\mathbb{K})$ is called the standard n-dimensional *projective space* over $\mathbb{K}$. We denote by $[v] = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{K})$ the equivalence class of the vector $v = (x_0, \ldots, x_n) \in \mathbb{K}^{n+1} \smallsetminus \{0\}$.

For every integer $0 \leq i \leq n$ consider the subset $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{K}) \mid x_i \neq 0\}$ and the map $j_i\colon \mathbb{K}^n \to U_i$ given by $j_i(y_1, \ldots, y_n) = [y_1 : \cdots , y_{i-1} : 1 : y_i : \cdots : y_n]$.

If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we equip $\mathbb{P}^n(\mathbb{K})$ with the topology $\{A \subseteq \mathbb{P}^n(\mathbb{K}) \mid \pi^{-1}(A) \text{ is open in } \mathbb{K}^{n+1} \smallsetminus \{0\}\}$.

a) Prove that $j_i\colon \mathbb{K}^n \to U_i$ is bijective for each $i = 0, \ldots, n$.

b) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, prove that U_i is an open subset of $\mathbb{P}^n(\mathbb{K})$ and that $j_i\colon \mathbb{K}^n \to U_i$ is a homeomorphism for every $i = 0, \ldots, n$.

c) If $n = 1$, consider $U_{01} = U_0 \cap U_1$ and write down the explicit formulae of the map

$$j_1^{-1} \circ j_0|_{j_0^{-1}(U_{01})}\colon j_0^{-1}(U_{01}) \longrightarrow j_1^{-1}(U_{01}).$$

d) Fix integers $0 \leq i < k \leq n$ and consider $U_{ik} = U_i \cap U_k$. Write down the explicit formulae of the map

$$j_k^{-1} \circ j_i|_{j_i^{-1}(U_{ik})}\colon j_i^{-1}(U_{ik}) \longrightarrow j_k^{-1}(U_{ik}).$$

[In order to ease the notation, you can assume $i = 0$ and that k is your favourite integer in $\{1, \ldots, n\}$.]

e) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, prove that $\mathbb{P}^n(\mathbb{K})$ is a path-connected compact topological space. [Hint: consider the restriction of π to the unitary sphere in $\mathbb{K}^{n+1}$.]

f) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, prove that $\mathbb{P}^n(\mathbb{K})$ is Hausdorff.

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Funktionentheorie (Complex Analysis) — Problem sheet 10
To hand in on 17th June 2019 at noon

Problem 37. 1) By using the method of residues, prove the equality

$$\int_0^{+\infty} \frac{\mathrm{d}x}{x^n+1} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$

for each integer $n \geq 2$. [Hint: consider the boundary of the compact set $K_r = \{\rho e^{i\theta} \mid 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{2\pi}{n}\}$ for $r > 0$.]

2) By using the method of residues, evaluate

$$\int_0^{+\infty} \frac{\sqrt[4]{x}}{x^2+2x+4}\mathrm{d}x.$$

Problem 38. i) Does there exist a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(100) = 1$ and $f(\frac{1}{n}) = \frac{1}{n^3}$ for every integer $n \geq 1$? If yes, exhibit an explicit example. If no, prove that such an f cannot exist.

ii) Explicitly exhibit an open subset $U \subseteq \mathbb{C}$ and a holomorphic function $f\colon U \to \mathbb{C}$ such that $f(100) = 1$ and $f(\frac{1}{n}) = \frac{1}{n^3}$ for every integer $n \geq 1$.

iii) Does there exist a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $\forall z \in S^1,\ f(z) = 2\bar{z}$? If yes, exhibit an explicit example. If no, prove that such an f cannot exist.

iv) Does there exist a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $\forall t \in [0,1],\ f(1+ti) = \frac{1-ti}{1+t^2}$? If yes, exhibit an explicit example. If no, prove that such an f cannot exist.

Problem 39. For all $p, q, r, s \in \mathbb{C}$, consider the 1-form

$$\omega = 4z^3 \frac{pz^3 + qz^2 + rz + s}{z^4 - 1}\mathrm{d}z$$

on $A = \{z \in \mathbb{C} \mid z^4 \neq 1\}$. Consider the following open subsets of A:

$$B = \mathbb{C} \smallsetminus (\{\pm 1\} \cup \{iy \mid y \in \mathbb{R},\ |y| \leq 1\}),$$
$$C = \mathbb{C} \smallsetminus \left(\left\{\sqrt{3}i + 2e^{i\theta} \mid -\frac{\pi}{3} \leq \theta \leq \frac{4\pi}{3}\right\} \cup \{iy \mid y \in \mathbb{R},\ |y| \leq 1\}\right),$$
$$D = \{z \in \mathbb{C} \mid |z| > 1\}.$$

a) For which $p, q, r, s \in \mathbb{C}$ is the 1-form ω exact?
b) For which $p, q, r, s \in \mathbb{C}$ is the 1-form $\omega|_B$ exact?
c) For which $p, q, r, s \in \mathbb{C}$ is the 1-form $\omega|_C$ exact?
d) For which $p, q, r, s \in \mathbb{C}$ is the 1-form $\omega|_D$ exact?

Problem 40. Consider the meromorphic function $f \in \mathcal{M}(\mathbb{C})$ given by:

$$f(z) = z(z^3 - 4) + e^{z/2} + \frac{1}{4z-6}.$$

Compute the number of zeroes, counted with their multiplicities, of f which are contained in each of the following sets: $B_2(0)$, $B_1(0)$ and $B_2(0) \smallsetminus B_1(0)$.

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Funktionentheorie (Complex Analysis) — Problem sheet 11
To hand in on 24th June 2019 at noon

Problem 41. Use the method of residues to evaluate the following integrals:

(1)
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+2x+2} \mathrm{d}x,$$
(2)
$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2} \mathrm{d}x,$$
(3)
$$\int_{0}^{2\pi} \frac{\cos t}{2+\cos^2 t} \mathrm{d}t,$$
(4)
$$\int_{0}^{+\infty} \frac{\sqrt[n]{x}}{x^2+1} \mathrm{d}x \qquad \text{for every integer } n \geq 2.$$

Problem 42. Consider the following open subsets of $\mathbb{C}$: $U = \{z \in \mathbb{C} \mid |z| > 1\}$ and $V = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Prove the following facts:

i) U and V are biholomorphic.
ii) $\mathbb{C}^*$ and U are homeomorphic.
iii) $\mathbb{C}^*$ and U are not biholomorphic.

Problem 43. Consider the following subsets of $\mathbb{C}$:
$$U = \{z \in \mathbb{C} \mid 0 < |z| < 1\},$$
$$V = \{z \in \mathbb{C} \mid 2 < |z| < 3\},$$
$$A = \{z \in \mathbb{C} \mid 2 \leq |z| \leq 3\}.$$

i) If $f \in \mathcal{O}(U)$ is holomorphic and such that $f(U) \subseteq A$, then show that the homomorphism
$$\mathrm{H}_1(f) \colon \mathrm{H}_1(U;\mathbb{Z}) \longrightarrow \mathrm{H}_1(A;\mathbb{Z})$$
is zero.
ii) Prove that U and V are not biholomorphic.
iii) Prove that U and V are homeomorphic.

Problem 44. Let $U \subseteq \mathbb{C}$ be an open connected subset and let $h \colon U \to \mathbb{C}^*$ be a holomorphic function. Let S be an infinite set of positive integers. Fix a point $z_0 \in U$. Prove that the following statements are equivalent:

(1) there exists a holomorphic function $f \colon U \to \mathbb{C}$ such that $\forall z \in U,\ e^{f(z)} = h(z)$,
(2) there exists a continuous function $f \colon U \to \mathbb{C}$ such that $\forall z \in U,\ e^{f(z)} = h(z)$,
(3) the homomorphism $\pi_1(h, z_0) \colon \pi_1(U, z_0) \to \pi_1(\mathbb{C}^*, z_0)$ is zero,
(4) the homomorphism $\mathrm{H}_1(h) \colon \mathrm{H}_1(U;\mathbb{Z}) \to \mathrm{H}_1(\mathbb{C}^*;\mathbb{Z})$ is zero,
(5) the 1-form $\frac{h'(z)}{h(z)}\mathrm{d}z$ is exact on U,
(6) for every integer $n \geq 2$, there exists a holomorphic function $g_n \colon U \to \mathbb{C}^*$ such that $\forall z \in U,\ g_n(z)^n = h(z)$,
(7) for every $n \in S$, there exists a continuous function $g_n \colon U \to \mathbb{C}^*$ such that $\forall z \in U,\ g_n(z)^n = h(z)$.

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Funktionentheorie (Complex Analysis) — Problem sheet 12
To hand in on 1st July 2019 at noon

Problem 45. a) Consider the Möbius transformation f defined by

$$f(z) = \frac{z-1}{\mathrm{i}(z+1)}.$$

Write the formula for the inverse of f. Determine the images under f of the points $0, 1, -1, \mathrm{i}, -\mathrm{i} \in \mathbb{C}$ and of the following subsets of $\mathbb{C}$:

$$\begin{aligned}
\mathbb{R} &= \{z \in \mathbb{C} \mid \operatorname{Im} z = 0\},\\
\mathrm{i}\mathbb{R} &= \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\},\\
S^1 &= \{z \in \mathbb{C} \mid |z| = 1\},\\
B_1(0) &= \{z \in \mathbb{C} \mid |z| < 1\},\\
U &= \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im} z > 0\},\\
V &= \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Re} z > 0\}.
\end{aligned}$$

[Hint: use the fact that Möbius transformations send Möbius lines to Möbius lines.]

b) Show that all the following open subsets of $\mathbb{C}$ are biholomorphic:

$$\begin{aligned}
U &= \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im} z > 0\},\\
V &= \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Re} z > 0\},\\
A &= \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, \operatorname{Im} z > 0\},\\
\mathbb{H} &= \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\},\\
B_1(0) &= \{z \in \mathbb{C} \mid |z| < 1\},\\
C &= \{z \in \mathbb{C} \mid |z| < 1\} \smallsetminus \{x \in \mathbb{R} \mid x \geq 0\},\\
D &= \{x + \mathrm{i}y \in \mathbb{C} \mid x, y \in \mathbb{R}, x < 0, 0 < y < \pi\},\\
E &= \{z \in \mathbb{C} \mid 1 < \operatorname{Re} z < 2, \operatorname{Im} z > 0\}.
\end{aligned}$$

[You can use the results of a).]

Problem 46. a) Let X be a topological space, let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X, and let $A \subseteq X$ be a subset. Show that A is open in X if and only if $A \cap U_\lambda$ is open in U_λ for every $\lambda \in \Lambda$. Show that A is closed in X if and only if $A \cap U_\lambda$ is closed in U_λ for every $\lambda \in \Lambda$.

b) Let $f\colon X \to Y$ be a holomorphic map between two Riemann surfaces. Fix $y \in Y$. Show that the set

$$A_y := \{x \in X \mid \exists \text{ open subset } V \subseteq X \text{ such that } x \in V \text{ and } f(V) = \{y\}\}$$

is both open and closed in X. For every $x \in X$, show that $x \in A_y$ if and only if x is a non-isolated point of $f^{-1}(y)$.

c) Let $f\colon X \to Y$ be a holomorphic map between two Riemann surfaces. Assume that X is connected. Prove that the following statements are equivalent:

i) f is open, i.e. for every open subset V of X the image $f(V)$ is open in Y;
ii) $f(X)$ is an open subset of Y;
iii) f is non-constant;
iv) for every $y \in Y$, the set

$$A_y := \{x \in X \mid \exists \text{ open subset } V \subseteq X \text{ such that } x \in V \text{ and } f(V) = \{y\}\}$$

is empty;
v) for every $y \in Y$, $f^{-1}(y)$ is discrete;
vi) for every non-empty open subset V of X, $f|_V$ is non-constant.

[Hint: among the many implications I would suggest you to prove iii)$\Rightarrow$iv), iv)$\Leftrightarrow$v), iv)$\Leftrightarrow$vi), vi)$\Rightarrow$i).]

Problem 47. a) Let X be a compact connected Riemann surface. Show that every holomorphic function $f\colon X \to \mathbb{C}$ is constant. [Hint: use the maximum modulus principle and Problem 46c.]

b) Let $f\colon X \to Y$ be a non-constant holomorphic map between two connected Riemann surfaces. Assume that X is compact. Show that f is surjective and that Y is compact. [You can use the following fact: if S is a compact subspace of a Hausdorff topological space T, then S is closed in T.]

c) Let $f\colon X \to Y$ be a non-constant holomorphic map between two connected compact Riemann surfaces. For every $y \in Y$, show that $f^{-1}(y)$ is finite.

Problem 48. For each of the following holomorphic functions defined on $\mathbb{C}^*$, answer the following questions:

- can it be extended to a holomorphic function on $\mathbb{C}$?
- can it be extended to a meromorphic function on $\mathbb{C}$?
- can it be extended to a holomorphic function on the Riemann sphere?
- can it be extended to a meromorphic function on the Riemann sphere?

$$
\begin{aligned}
a(z) &= -3,\\
b(z) &= z^2 + 5,\\
c(z) &= \frac{1}{z} + 1,\\
d(z) &= \mathrm{e}^{\frac{1}{z}},\\
e(z) &= \mathrm{e}^{z}.
\end{aligned}
$$

Exam 08.07.2019

Problem 1. Consider the power series $f(T) = \sum_{n\geq 0}(n+1)T^n \in \mathbb{C}[\![T]\!]$.

a) Compute the radius of convergence R of f.

b) Compute the multiplicative inverse of f in the ring $\mathbb{C}[\![T]\!]$.

c) Exhibit a meromorphic function g on $\mathbb{C}$ such that $\forall z \in B_R(0),\ g(z) = f(z)$.

Problem 2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that there exist real numbers $C > 0$ and $R > 1$ such that

$$\forall z \in \mathbb{C} \smallsetminus B_R(0), \quad |f(z)| \leq C \log |z|.$$

Prove that f is constant.

Problem 3. Use the method of residues to evaluate the integral

$$I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+4)(x^2-2x+2)}.$$

Problem 4. For all $p, q \in \mathbb{C}$, consider the 1-form

$$\omega = \frac{pz + q\mathrm{e}^{\pi z}}{z^2+1}\mathrm{d}z$$

on the open subset $A = \{z \in \mathbb{C} \mid z^2 + 1 \neq 0\}$ of $\mathbb{C}$. Consider the following open subsets of A:

$$B = \{z \in \mathbb{C} \mid |z| > 1\} \qquad \text{and} \qquad C = \{z \in \mathbb{C} \mid \operatorname{Im} z > 1\}.$$

a) For which $p, q \in \mathbb{C}$ is the 1-form ω exact on A?

b) For which $p, q \in \mathbb{C}$ is the 1-form $\omega|_B$ exact on B?

c) For which $p, q \in \mathbb{C}$ is the 1-form $\omega|_C$ exact on C?

Problem 5. Consider the polynomial $f(z) = z^5 + 7z^2 - 3 \in \mathbb{C}[z]$. Determine the number of zeroes (counted with multiplicity) of f which are contained in the anulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

Problem 6. Consider the following open subsets of $\mathbb{C}$:

$$\begin{aligned} A &= \{z \in \mathbb{C} \mid 1 < |z| < 2\}, \\ U &= \{z \in \mathbb{C} \mid 5 < |z - \mathrm{i}| < 10\}, \\ V &= \{z \in \mathbb{C} \mid |\operatorname{Re} z| < 1, \operatorname{Im} z > 0\}. \end{aligned}$$

Are A and U homeomorphic or biholomorphic? Are A and V homeomorphic or biholomorphic? Are U and V homeomorphic or biholomorphic?

Problem 7. a) Give an example of an open subset $U \subseteq \mathbb{C}$ and of a non-constant holomorphic function $f\colon U \to \mathbb{C}$ such that $f(U)$ is not open in $\mathbb{C}$.

b) Does there exist a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(\mathrm{i}y) = \frac{1}{y}$ for every rational number $y > 1$?

Exam 07.10.2019

Problem 1. Consider the power series $f(T) = \sum_{n\geq 0}(1+(-1)^n)T^n \in \mathbb{C}[\![T]\!]$.
a) What is the radius of convergence R of f?
b) What is the multiplicative inverse of f in the ring $\mathbb{C}[\![T]\!]$?
c) Exhibit a meromorphic function g on $\mathbb{C}$ such that $\forall z \in B_R(0),\ g(z) = f(z)$.

Problem 2. Let $f\colon \mathbb{C} \to \mathbb{C}$ and $g\colon \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Assume that g is non-constant and that $g \circ f$ is bounded on $\mathbb{C}$. Prove that f is constant. [Hint: what kind of topological space is $g^{-1}(c)$ for any $c \in \mathbb{C}$?]

Problem 3. Use the method of residues to evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)}\mathrm{d}x.$$

Problem 4. For all $p, q, r \in \mathbb{C}$, consider the 1-form

$$\omega = \frac{pz^2+qz+r}{z(z^2-1)}\mathrm{d}z$$

on the open subset $A = \mathbb{C} \smallsetminus \{0, 1, -1\}$ of $\mathbb{C}$. Consider the following open subsets of A:

$$B = \mathbb{C} \smallsetminus (\{-1\} \cup \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}) \qquad \text{and} \qquad C = \{z \in \mathbb{C} \mid |z| > 1\}.$$

a) For which $p, q, r \in \mathbb{C}$ is the 1-form ω exact on A?
b) For which $p, q, r \in \mathbb{C}$ is the 1-form $\omega|_B$ exact on B?
c) For which $p, q, r \in \mathbb{C}$ is the 1-form $\omega|_C$ exact on C?

Problem 5. Consider the polynomial $f(z) = z^4 + 8z - 5\mathrm{i} \in \mathbb{C}[z]$. Determine the number of zeroes (counted with multiplicity) of f which are contained in the anulus $\{z \in \mathbb{C} \mid 1 < |z| < 3\}$.

Problem 6. Consider the following open subsets of $\mathbb{C}$:

$$U = \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < 1,\ |\mathrm{Im}\, z| < 1\} \qquad \text{and} \qquad V = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}\, z < 4,\ 0 < \mathrm{Im}\, z < 4\}.$$

a) Is U biholomorphic to V? If yes, explicitly write down a biholomorphism $U \to V$.
b) Is U biholomorphic to $\mathbb{C} \smallsetminus \{5\mathrm{i}\}$? If yes, explicitly write down a biholomorphism $U \to \mathbb{C} \smallsetminus \{5\mathrm{i}\}$.
c) Is U biholomorphic to $\mathbb{C}$? If yes, explicitly write down a biholomorphism $U \to \mathbb{C}$.

Problem 7. Let U be a connected open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Let $u\colon U \to \mathbb{R}$ be the real part of f and let $v\colon U \to \mathbb{R}$ be the imaginary part of f.
a) If u is constant, then show that f is constant.
b) If $u + v$ is constant, then show that f is constant.