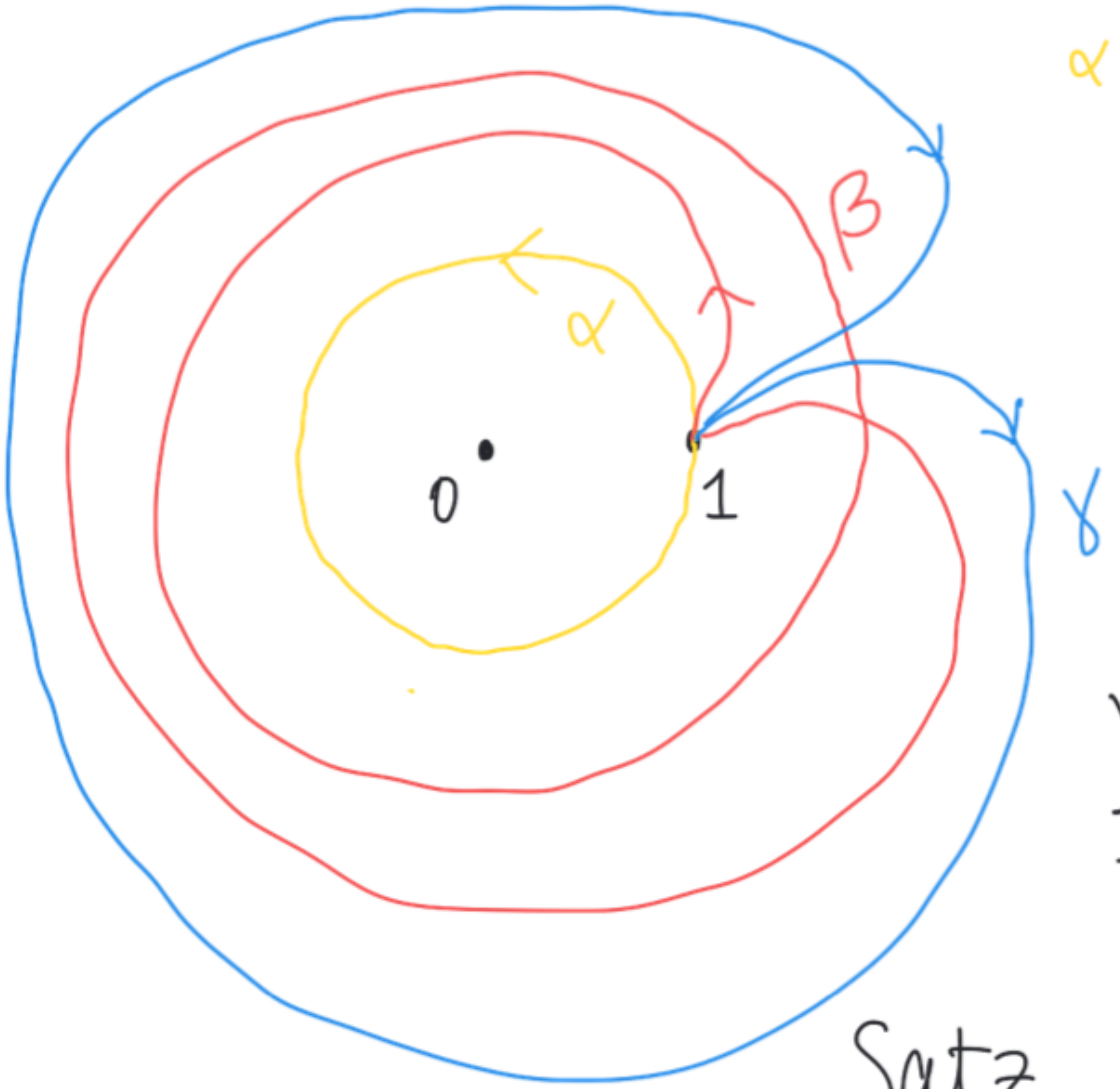
$X \subseteq \mathbb{R}^n$ convex, $x_0 \in X \implies \pi_1(X, x_0)$ trivial

z. B.: $X = \mathbb{R}^n$, $X = B_r(x_0)$, $X = [0,1]$,

$X = [0, +\infty)$, $X = [0,1] \times \mathbb{R}$,

$X = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

$X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \; x_0 = 1.$



$\alpha : [0,1] \longrightarrow \mathbb{C}^*$
$\quad t \longmapsto e^{2\pi i t}$

$\beta \simeq \alpha * \alpha \Rightarrow [\beta] = [\alpha * \alpha] = [\alpha][\alpha] = [\alpha]^2$

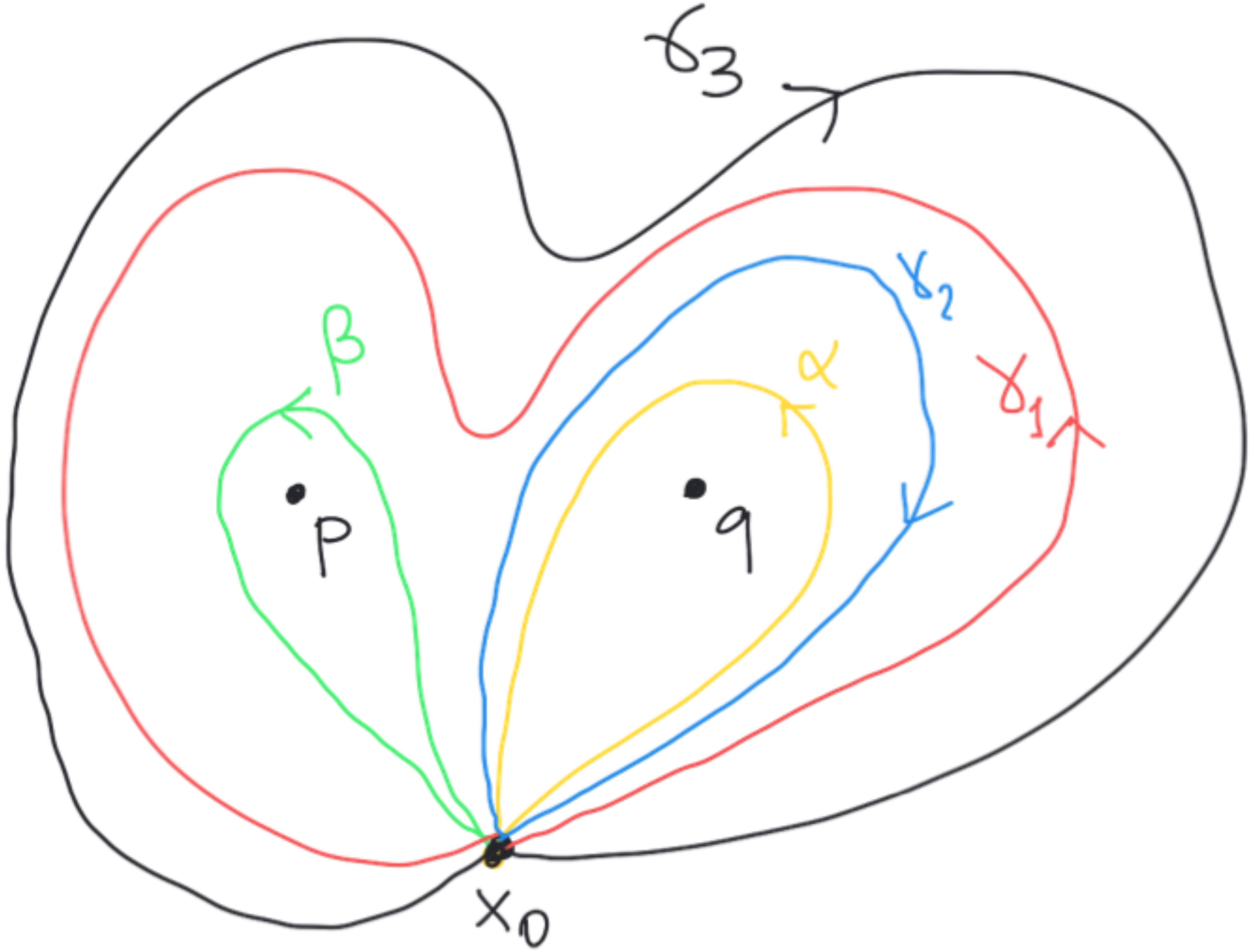
$\gamma \simeq i(\alpha) \Rightarrow [\gamma] = [\iota(\alpha)] = [\alpha]^{-1}$

$\forall \gamma : [0,1] \longrightarrow X$ Weg mit $\gamma(0) = \gamma(1) = 1,$
$\exists! \, k \in \mathbb{Z} : [\gamma] = [\underbrace{\alpha * \cdots * \alpha}_{k \text{ mal}}] = [\alpha]^k$

<u>Satz</u> $\mathbb{Z} \longrightarrow \pi_1(\mathbb{C}^*, 1)$
$\quad k \longmapsto [\alpha]^k$ Gruppenisomorphismus

<u>Bws</u> fakultativ.

$p, q, x_0 \in \mathbb{C}$, $p \neq q$, $p \neq x_0$, $q \neq x_0$, $X = \mathbb{C} \setminus \{p, q\}$, $\pi_1(X, x_0) = ?$

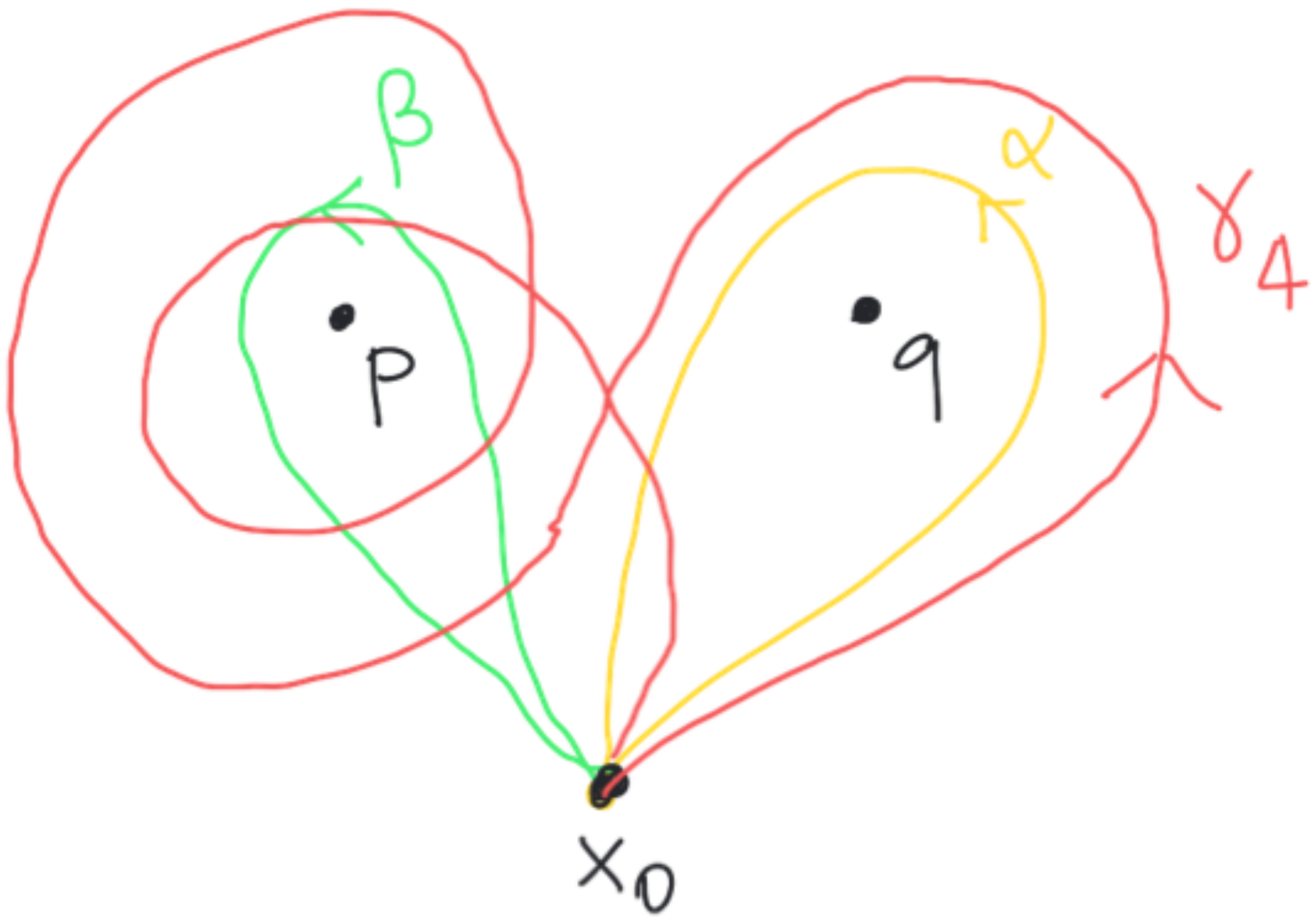


$[\gamma_1] = [\alpha][\beta]$

$[\gamma_2] = [\iota(\alpha)] = [\alpha]^{-1}$

$[\gamma_3] = [i(\beta) * i(\alpha)] = [\beta]^{-1}[\alpha]^{-1}$

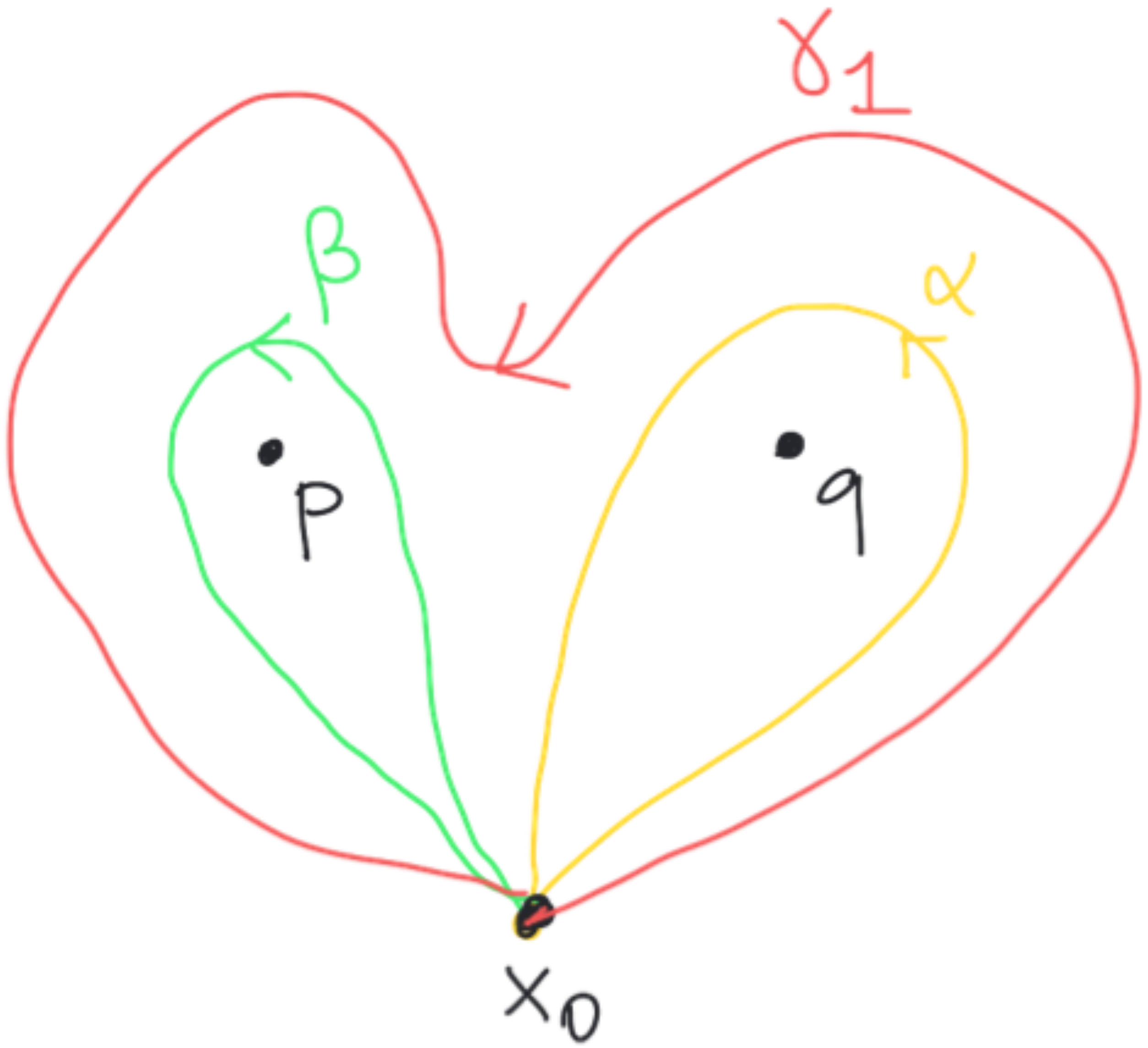
$p, q, x_0 \in \mathbb{C}$, $p \neq q$, $p \neq x_0$, $q \neq x_0$, $X = \mathbb{C} \setminus \{p, q\}$, $\pi_1(X, x_0) = ?$



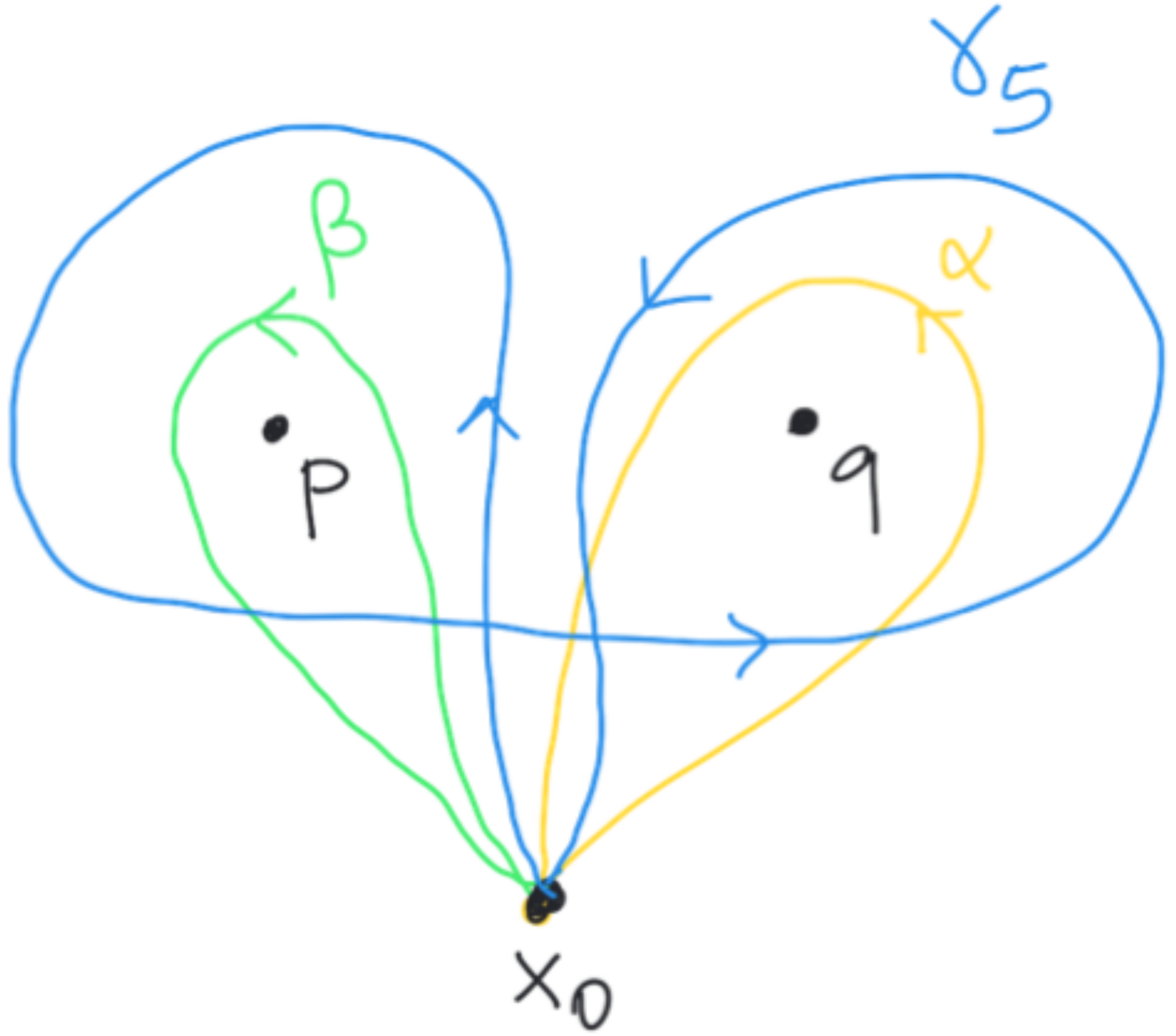
$\gamma_4 \simeq \alpha * i(\beta) * i(\beta)$

$\Rightarrow [\gamma_4] = [\alpha][\beta]^{-2}$

$p, q, x_0 \in \mathbb{C}$, $p \neq q$, $p \neq x_0$, $q \neq x_0$, $X = \mathbb{C} \setminus \{p,q\}$, $\pi_1(X, x_0) = ?$



$[\gamma_1] = [\alpha][\beta]$

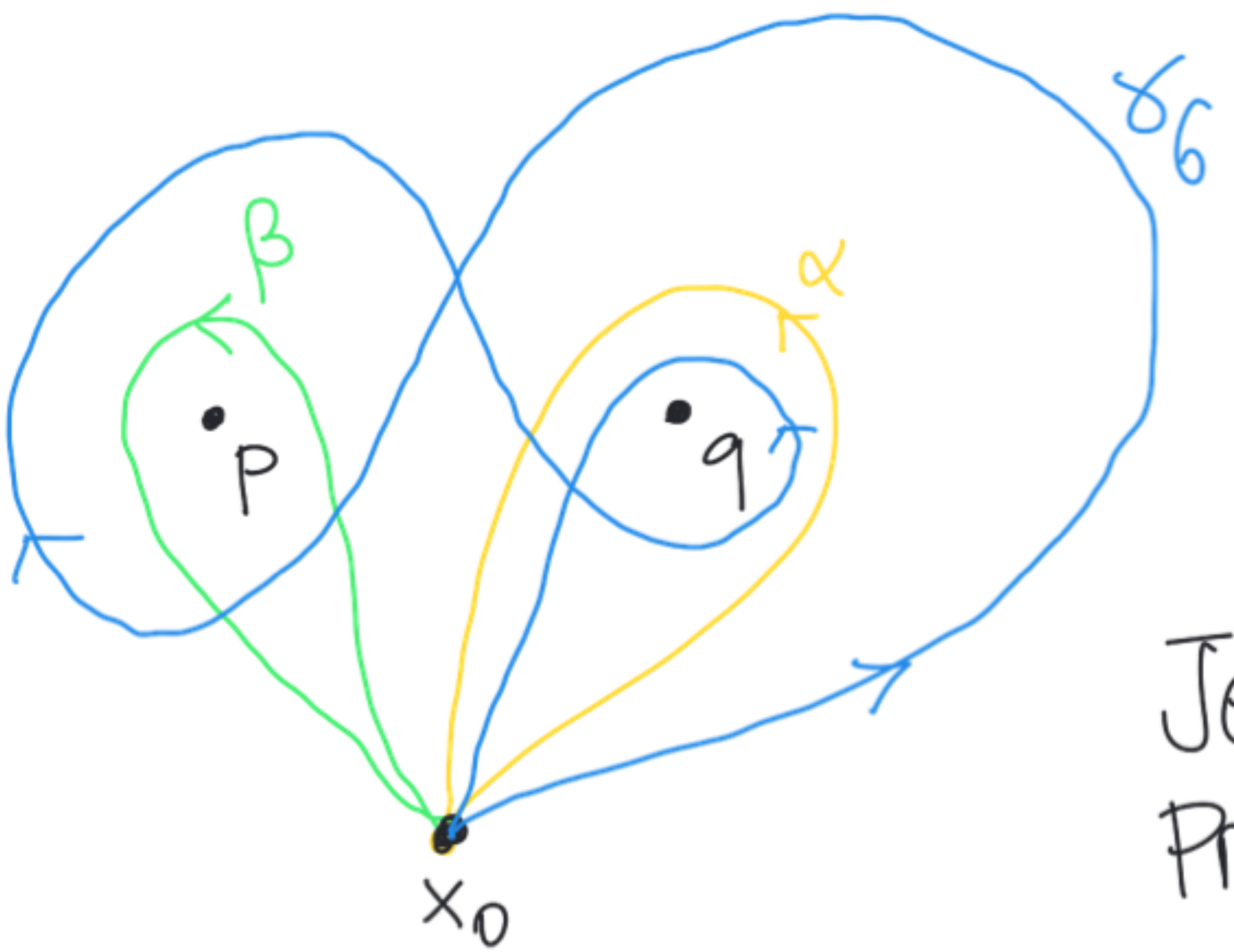


$\gamma_5 \simeq \beta * \alpha$

$[\gamma_5] = [\beta][\alpha]$

$[\alpha][\beta] \neq [\beta][\alpha] \Longrightarrow \pi_1(X, x_0)$ ist nicht abelsch

$p, q, x_0 \in \mathbb{C}$, $p \neq q$, $p \neq x_0$, $q \neq x_0$, $X = \mathbb{C} \setminus \{p, q\}$, $\pi_1(X, x_0) = ?$



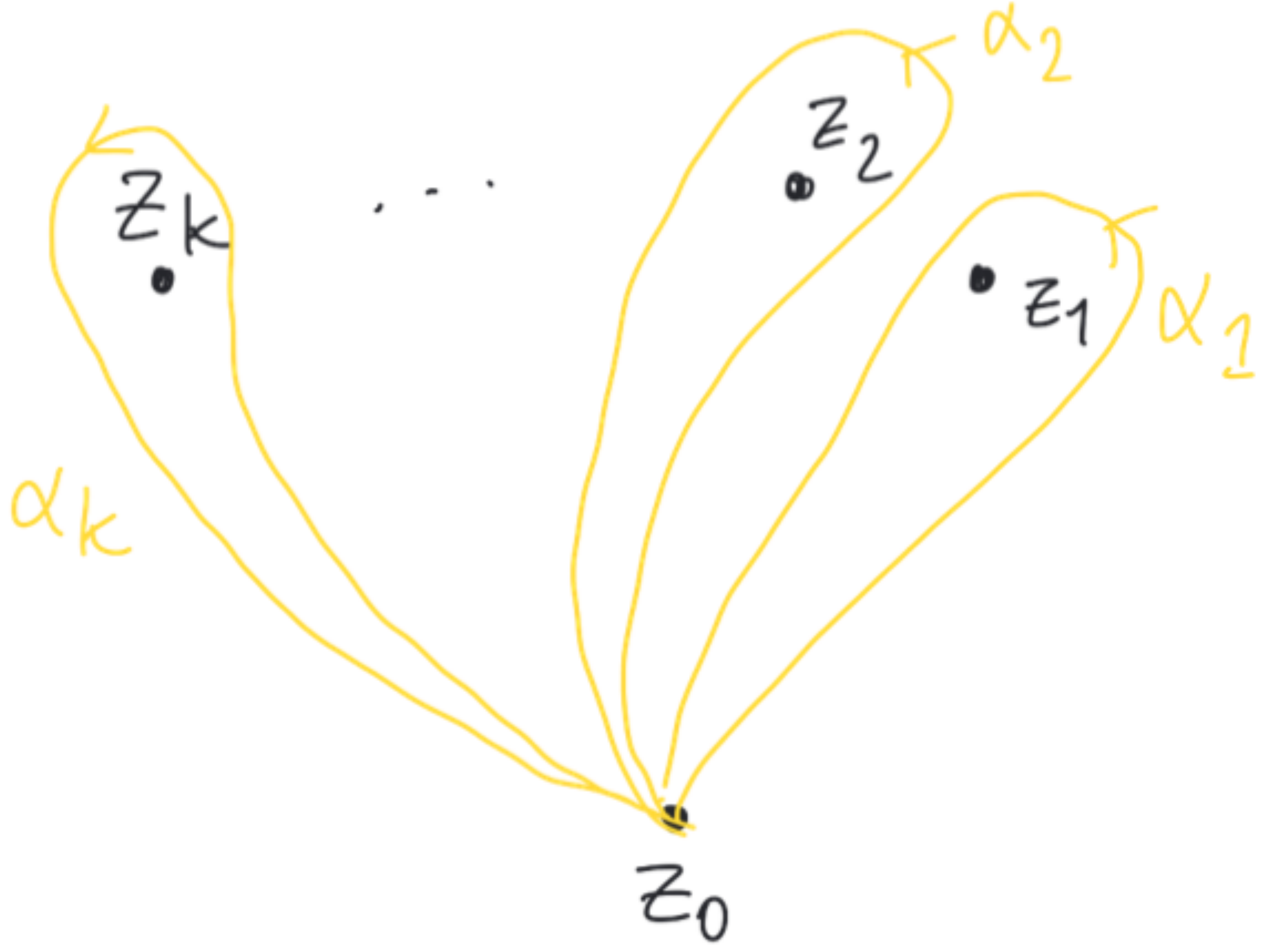
$\gamma_6 \simeq \alpha * i(\beta) * \alpha$

$[\gamma_6] = [\alpha][\beta]^{-1}[\alpha] \neq [\alpha]^2[\beta]^{-1}$

Jedes Element von $\pi_1(X, x_0)$ ist das Produkt von $[\alpha], [\beta], [\alpha]^{-1}, [\beta]^{-1}$.

$\Rightarrow \pi_1(X, x_0)$ ist die $\boxed{\text{freie Gruppe}}$ über $[\alpha], [\beta]$

$z_0, \dots, z_k \in \mathbb{C}$ different, $X = \mathbb{C} \setminus \{z_1, \dots, z_k\}$

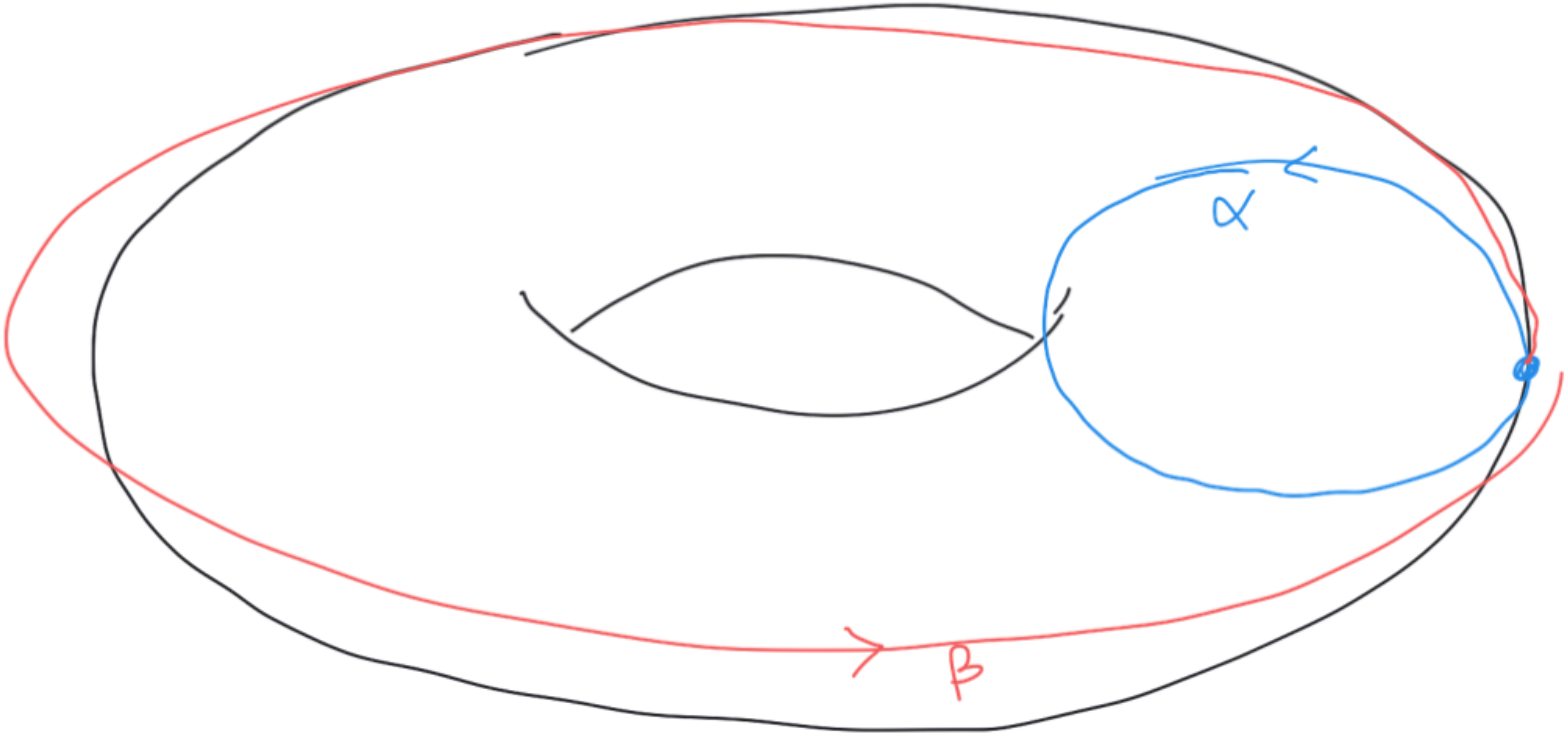


$\pi_1(X, z_0)$ ist die frei Gruppe über $[\alpha_1], \dots, [\alpha_k]$

$$\pi_1(X, z_0) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ mal}}$$

$0 < r < R$, $X = \{((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta) \mid 0 \le \varphi, \theta \le 2\pi\} \subseteq \mathbb{R}^3$

TORUS $= \{(x, y, r\sin\theta) \in \mathbb{R}^3 \mid x^2 + y^2 = (R + r\cos\theta)^2, 0 \le \theta \le 2\pi\}$



$x_0 = (R + r, 0, 0)$

$\pi_1(X, x_0)$ ist die freie abelsche Gruppe über $[\alpha], [\beta]$

$\Rightarrow \pi_1(X, x_0) \cong \mathbb{Z}^2$

$\mathbb{Z} \times \mathbb{Z}$

$\alpha : [0, 2\pi] \longrightarrow X$

$\theta \longmapsto (R + r\cos\theta, 0, r\sin\theta)$

$\beta : [0, 2\pi] \longrightarrow X$

$\varphi \longmapsto ((R + r)\cos\varphi, (R + r)\sin\varphi, 0)$