13.07.2020

When are two anuli homeomorphic / biholomorphic?

Kreisring / Anulus:

$$A_{r^-,r^+}(z_0) = \{ z \in \mathbb{C} \mid r^- < |z - z_0| < r^+ \}$$

$z_0 \in \mathbb{C}$, $\quad 0 \leq r^- < r^+ \leq +\infty$

$$A_{r^-,r^+} := A_{r^-,r^+}(0) = \{ z \in \mathbb{C} \mid r^- < |z| < r^+ \}$$

$$\begin{aligned} A_{r^-,r^+} &\longrightarrow A_{r^-,r^+}(z_0) \\ z &\longmapsto z + z_0 \end{aligned}$$

Biholomorphismus

- $r^- = 0$: $A_{0,r^+} = B_{r^+}(0) \setminus \{0\}$ punktierte Umgebung

- $0 < r^- < r^+ < +\infty$: A_{r^-,r^+}

- $r^+ = +\infty$: $A_{r^-,+\infty} = \mathbb{C} \setminus \overline{B_{r^-}(0)}$

Special case: $r^- = 0$, $r^+ = +\infty$ $A_{0,+\infty} = \mathbb{C}^*$.

$0 \le r^- < r^+ \le +\infty, \quad s \in \mathbb{R}_{>0}$

$$A_{r^-,r^+} \longrightarrow A_{sr^-,sr^+} \qquad \text{Biholomorphismus}$$
$$z \longmapsto sz$$

<u>Theorem</u> $0 \le r^- < r^+ \le +\infty, \quad 0 \le \rho^- < \rho^+ \le +\infty.$

1) A_{r^-,r^+} und A_{ρ^-,ρ^+} sind homöomorph.

2) A_{r^-,r^+} und A_{ρ^-,ρ^+} sind biholomorph $\iff \underbrace{\dfrac{r^+}{r^-} = \dfrac{\rho^+}{\rho^-} \in [0,+\infty]}_{\text{adjust the statement for } \mathbb{C}^* \; r^+ = +\infty,\ r^- = 0}$

<u>Pf</u> 1) good exercise.

2) $\Leftarrow$) easy $\quad \Rightarrow$) not easy. But we saw some examples.

Some examples:

- $A_{0,r^+} = B_r(0) \setminus \{0\}$ und A_{ρ^-,ρ^+} mit $0 < r^+ < +\infty$, $0 < \rho^- < \rho^+ < +\infty$
  sind nicht biholomorph
  Aufgabe 10.3

- $\mathbb{C}^*$ und A_{ρ^-,ρ^+} mit $0 \leq \rho^- < \rho^+ < +\infty$
  sind nicht biholomorph.

By contradiction $\exists f : \mathbb{C}^* \longrightarrow A_{\rho^-,\rho^+}$ biholomorphism.
$\rho^+ < +\infty \Rightarrow A_{\rho^-,\rho^+}$ is bounded $\Rightarrow f$ is bounded $\Rightarrow$
f has a removable singularity at $0 \Rightarrow \exists \tilde{f} \in \mathcal{O}(\mathbb{C})$
s.t. $\tilde{f}|_{\mathbb{C}^*} = f$. $\tilde{f}(\mathbb{C}) \subseteq A_{\rho^-,\rho^+}$ is bounded $\overset{\text{Liouville}}{\Longrightarrow} \tilde{f}$ is constant
$\Rightarrow f$ is constant $\lightning$

$U, V \subseteq \mathbb{C}$ connected open subsets

U and V are biholomorphic $\Longrightarrow$ U and V are homeomorphic $\Longrightarrow$ U and V are homotopy equivalent

$\Downarrow$

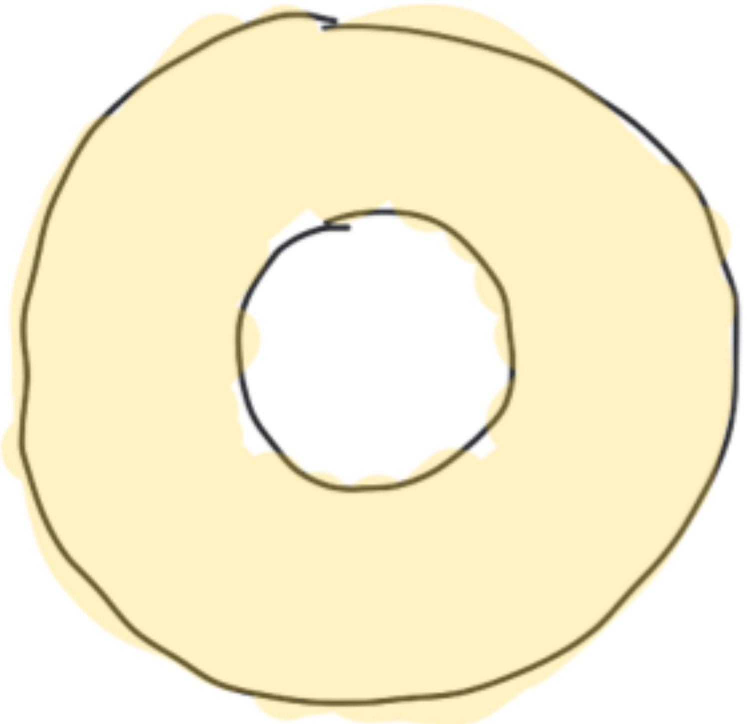
U and V have the same fundamental group:
$\exists$ group isomorphism $\pi_1(U) \xrightarrow{\sim} \pi_1(V)$

not homeomorphic $\Longrightarrow$ not biholomorphic

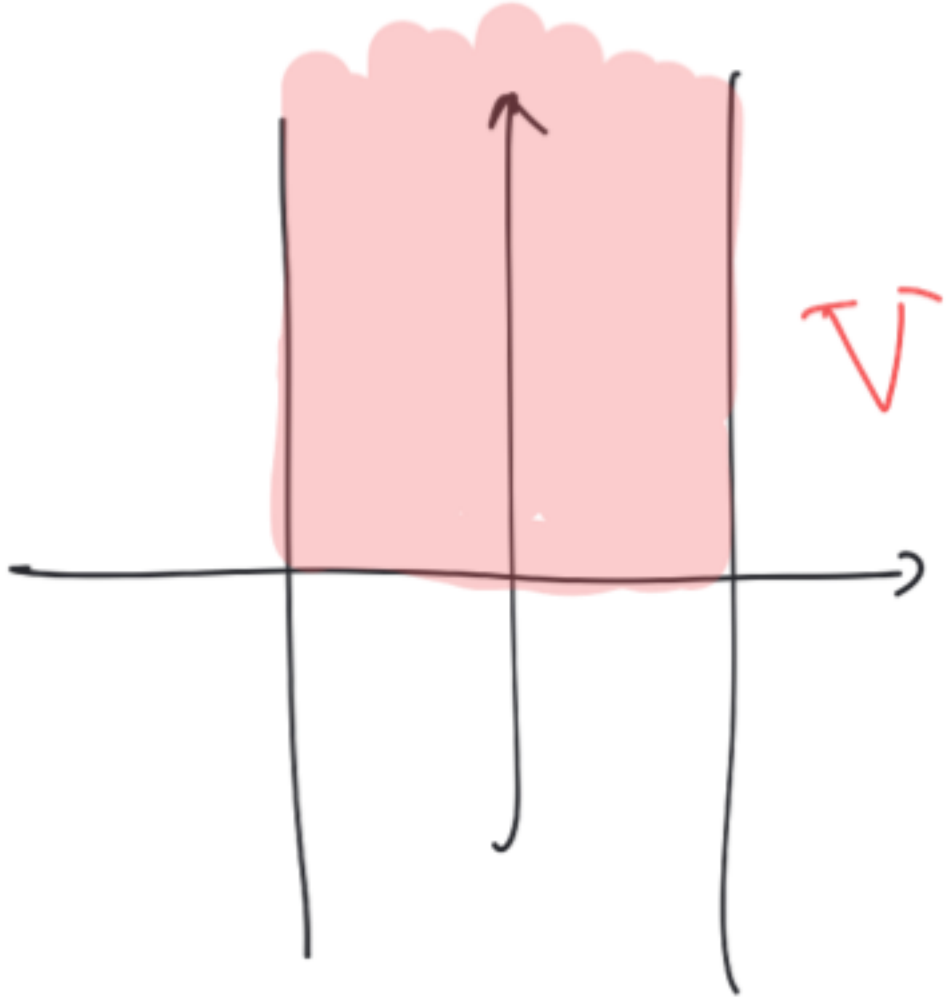
$\mathbb{C}$ und $B_1(0)$ sind homöomorph, aber nicht biholomorph.

$A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$

$V = \{ z \in \mathbb{C} \mid |\operatorname{Re} z| < 1, \operatorname{Im} z > 0 \}$



A



A is homotopy equivalent to $S^1 \Rightarrow \pi_1(A) \simeq \mathbb{Z}$

V is convex, starshaped, contractible $\Rightarrow \pi_1(V) = \{0\}$

$\Rightarrow$ A and V are not homotopy equivalent, so not homeomorphic

S^2 is not contractible
but is simply connected

Theorem (easy version of Riemann uniformisation)

$U, V \subseteq \mathbb{C}$ open connected subsets.

$U \neq \mathbb{C}$, $V \neq \mathbb{C}$,

U and V are simply connected

$\Longrightarrow$ U and V are biholomorphic

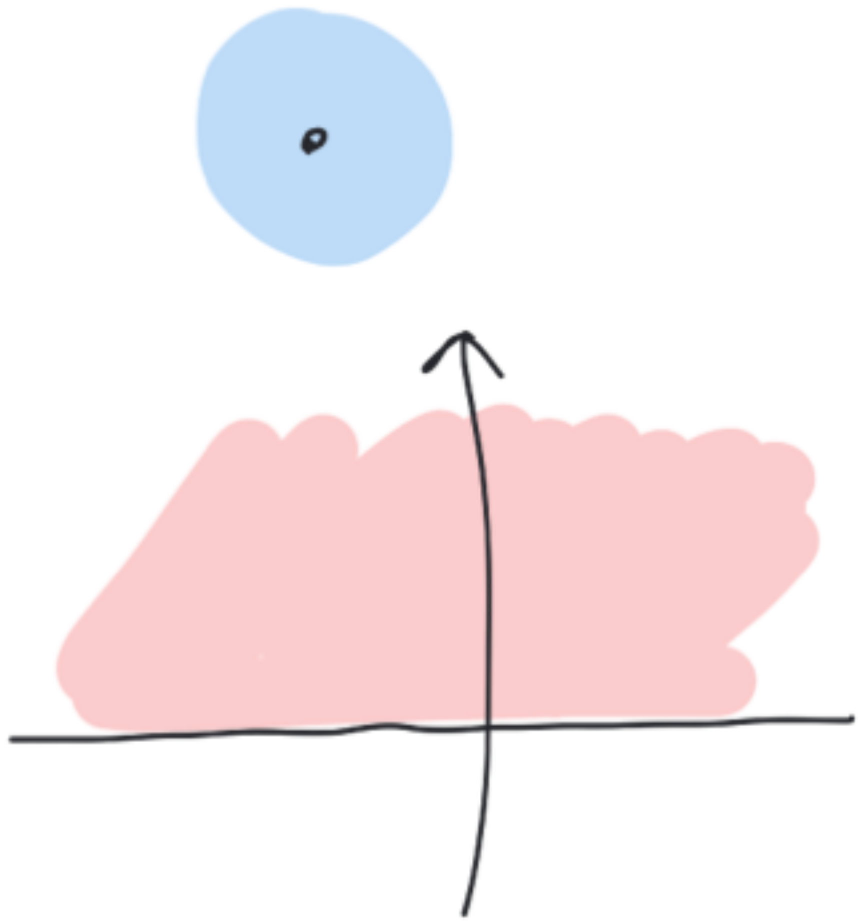
You are **not** allowed to use this theorem

Sie können **nicht** diesen Satz benutzen

Example $B_1(0) = \{ z \in \mathbb{C} \mid |z| < 1 \}$

$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

are biholomorphic.



What is an explicit biholomorphism

$B_1(0) \longrightarrow \mathbb{H}$?

Aufgabe 11.5 $\qquad f(z) = \dfrac{z-1}{iz+i}$

Topological properties

$f: X \to Y$ homeomorphism between two top. spaces

X Hausdorff $\Rightarrow$ Y Hausdorff

X connected $\Rightarrow$ Y connected

X compact $\Rightarrow$ Y compact

X path connected $\Rightarrow$ Y path connected

X simply connected $\Rightarrow$ Y simply connected

Give an example:

X, Y top. spaces

X compact, Y not compact

X and Y have the same fundamental group.

$X = \{z \in \mathbb{C} \mid |z| \leq 1\}$, $Y = \{z \in \mathbb{C} \mid |z| < 1\}$

$X = \{0\}$, $Y = \mathbb{R}$

$X = S^1$, $Y = \mathbb{C}^*$

- open
- closed
- interior point
- closure
- neighborhood

These are not topological properties.

They depend on the ambient top. space you choose.

Example

$[0,1]$ is not open in $\mathbb{R}$

$[0,1]$ is open in itself

$[0,1)$ is not open in $\mathbb{R}$

$[0,1)$ is open in $[0,1]$

$B_1(0) \longrightarrow$

$z \longmapsto \dfrac{z}{1-|z|}$

$w = \dfrac{z}{1-|z|}$ I want to express z in terms of w.

$(1-|z|)\cdot w = z$

$(1-|z|)\,|w| = |z|$

$|w| - |z||w| = |z|$

$|w| = |z||w| + |z|$

$|w| = |z|(1+|w|)$

$|z| = \dfrac{|w|}{1+|w|}$

$1-|z| = 1 - \dfrac{|w|}{1+|w|} = \dfrac{1+|w|-|w|}{1+|w|} = \dfrac{1}{1+|w|} \Rightarrow z = w\cdot\dfrac{1}{1+|w|}$

10.3

$U = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$

$V = \{z \in \mathbb{C} \mid 2 < |z| < 3\}$

$$U \xrightarrow{f} V$$

$$re^{i\theta} \longmapsto (r+2)e^{i\theta} \qquad r > 0,\ \theta \in (0, 2\pi)$$

This expression doesn't say that this function is continuous

$$z = re^{i\theta} \qquad r = |z|,\ e^{i\theta} = \frac{z}{|z|}$$

$$f(z) = (|z|+2)\frac{z}{|z|}$$ continuous!