5.5.2020

Satz $X \subseteq \mathbb{R}^n$.

$$X \text{ kompakt} \iff \begin{cases} X \text{ abgeschlossen in } \mathbb{R}^n \\ X \text{ beschränkt} \end{cases}$$

Satz (X,d) metrisch

X kompakt $\iff$ X sequentially compact $\iff$ X totally bounded & complete

Example $(0,1] \subseteq \mathbb{R}$ bounded and totally bounded

Satz $X \subseteq \mathbb{R}^n$. X bounded $\iff$ totally bounded

(X,d) $\quad d(x,y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$

X bounded $\quad X = B_2(x_0)$

$0 < \varepsilon < 1$

X nicht endlich $\Rightarrow$ X cannot be covered by finitely many balls of radius ε.

$\Downarrow$

X not totally bounded

X connected $\iff$ $X \neq \emptyset$ & the subsets of X which are both open and closed in X are exactly $\emptyset$ and X

$\Uparrow$

X path-connected $\iff$ $X \neq \emptyset$ & $\forall x, y \in X$, $\exists$ path in X from x to y

$\Uparrow$

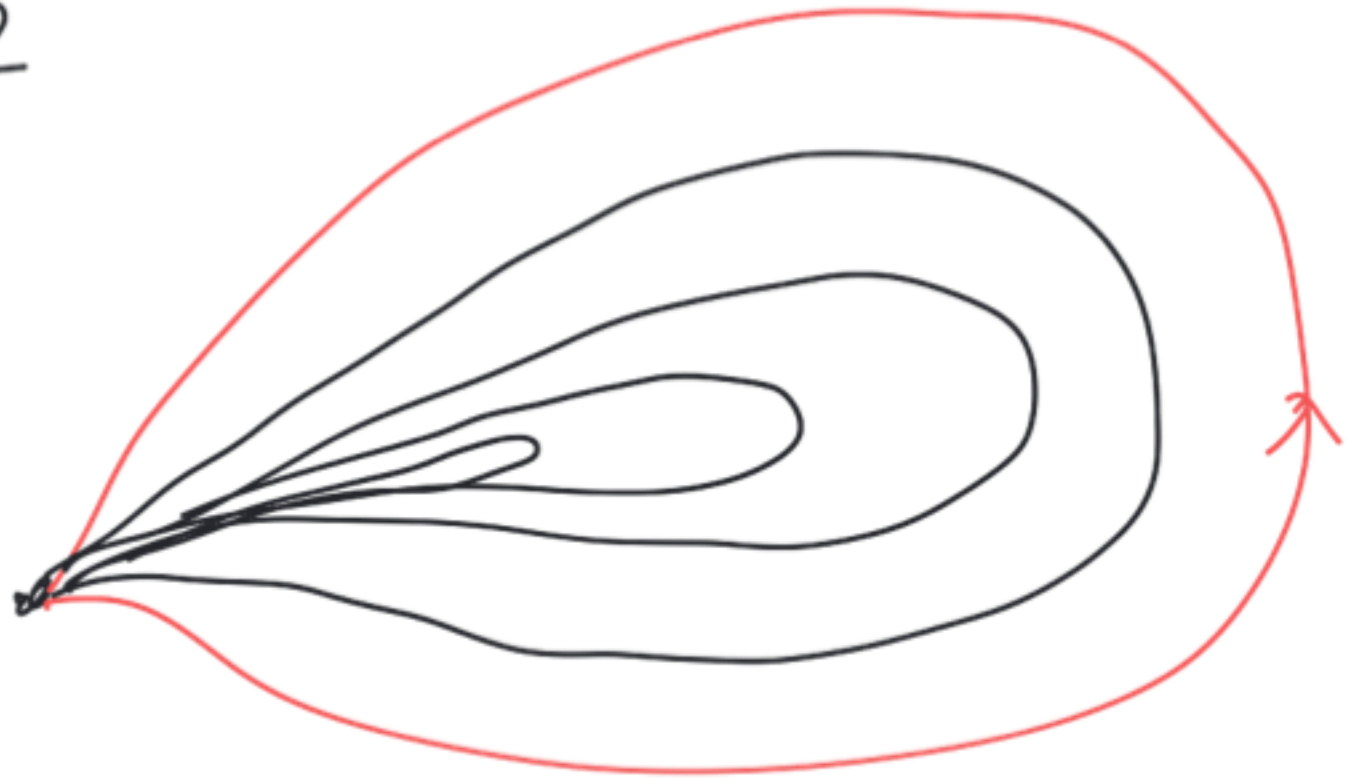
X simply connected $\iff$ X path-connected & every loop in X is path-homotopic to a constant path

$$X = (\{0\} \times [0,1]) \cup \{(x, \sin \tfrac{1}{x}) \mid 0 < x \leq 1\}$$

connected
but not path-connected

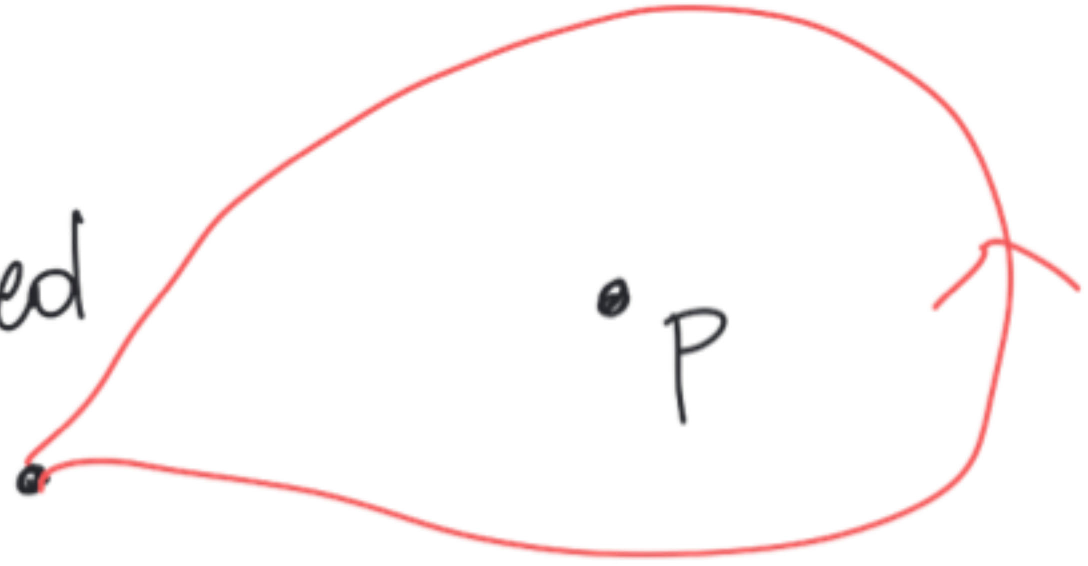
$X = \mathbb{R}^2$

simply
connected



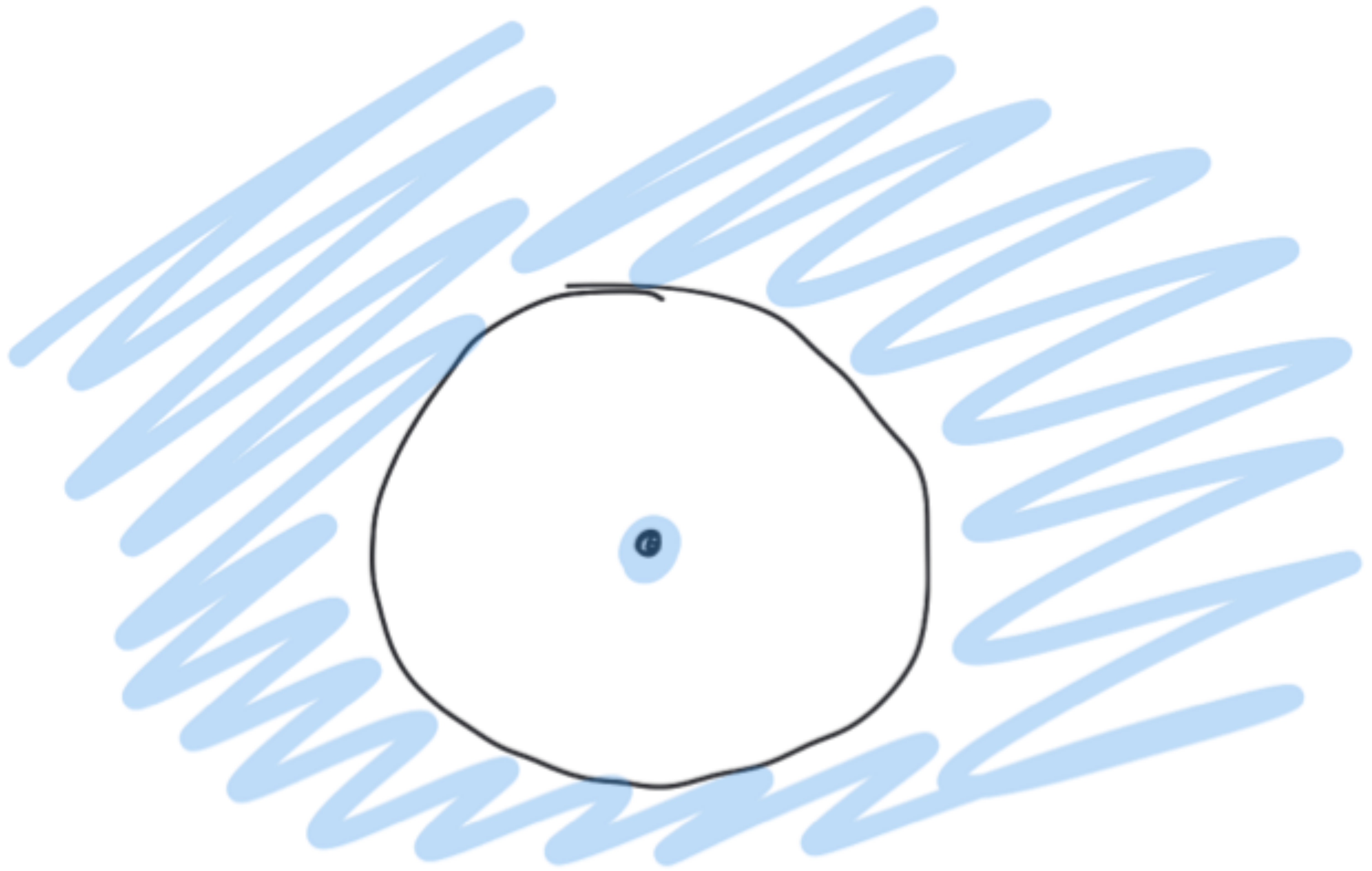
$X = \mathbb{R}^2 \setminus \{p\}$

not
simply
connected



$$\{z \in \mathbb{C} \mid |z|^3 \geqslant |z|\} =$$

$$= \{z \in \mathbb{C} \mid |z| = 0 \text{ or } |z| \geqslant 1\}$$



1.5.h $A = \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geq 1\}$

$f: A \cup \{0\} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x^2 & x = \frac{1}{n}, \ n \text{ even} \\ \sin x & x = \frac{1}{n}, \ n \text{ odd} \\ 0 & x = 0 \end{cases}$$

[$a \in A$, $\{a\}$ is open in $A \cup \{0\}$
$\{a\}$ is a neighborhood of a in $A \cup \{0\}$

$\Rightarrow f$ is continuous in a]

$\{0\}$ is not open in $A \cup \{0\}$.

Prove that f is continuous in 0. Fix $\varepsilon > 0$.

$\mathbb{R} \to \mathbb{R}$, $x \mapsto \sin x$ continuous in $0 \Rightarrow \exists \delta_1 > 0 : \forall x \in \mathbb{R}, |x| < \delta_1 \Rightarrow |\sin x| < \varepsilon$

$\delta_1 = \varepsilon \quad |\sin x| \leq |x|$

$\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ continuous in $0 \Rightarrow \exists \delta_2 > 0 : \forall x \in \mathbb{R}\ |x| < \delta_2 \Rightarrow |x^2| < \varepsilon$

$\delta_2 = \min\{1, \sqrt{\varepsilon}\}$

$\delta = \min\{\delta_1, \delta_2\}. \Rightarrow \forall x \in \mathbb{R}, |x| < \delta \Rightarrow \begin{matrix} |\sin x| < \varepsilon \\ |x^2| < \varepsilon \end{matrix}$

$\Rightarrow \forall x \in A \cup \{0\}, |x| < \delta \Rightarrow |f(x)| < \varepsilon.$

Satz X, Y top Räume, $f: X \to Y$.
X diskret $\Rightarrow$ f stetig

Bws $A \subseteq Y$ offen $\Rightarrow$ $f^{-1}(A) \subseteq X$ offen. $\square$