4.6.2020

X top. Raum. $Y \subseteq X$ heißt DICHT in X wenn $\overline{Y}^X = X$.

(X,d) met. Raum. $Y \subseteq X$.

Y ist dicht in $X \iff \forall x \in X, \forall \varepsilon > 0, \ B_\varepsilon(x) \cap Y \neq \emptyset$.

Hinweis 7.2c:

by contradiction $\exists z_0 \in \mathbb{C}, \varepsilon > 0, \ B_\varepsilon(z_0) \cap f(\mathbb{C}) = \emptyset$.

$\Rightarrow \ldots$

$U \subseteq \mathbb{R}^2$ offen.

1-Form: $\quad P\,dx + Q\,dy \qquad P, Q : U \to \overset{\mathbb{C}}{\mathbb{R}}$ stetig

$\gamma : [a,b] \to U$ Weg, stückweise C^1

$$\int_\gamma P\,dx + Q\,dy := \underbrace{\int_a^b \big(P(\gamma(t))\,\dot{\gamma}_1(t) + Q(\gamma(t))\,\dot{\gamma}_2(t)\big)\,dt}_{\text{Aufgabe 5.1a}}$$

$$\gamma = (\gamma_1, \gamma_2) : [a,b] \to U \qquad \gamma(t) = (\gamma_1(t), \gamma_2(t))$$
$$= \gamma_1(t) + i\,\gamma_2(t)$$

$$dz = dx + i\,dy$$

$U \subseteq \mathbb{C}$ offen, $\quad f: U \to \mathbb{C}$ $\qquad f = u + iv \quad \begin{array}{l} u: U \to \mathbb{R} \\ v: U \to \mathbb{R} \end{array}$

$$f(z)dz = (u+iv)(dx+i\,dy)$$
$$= (u+iv)dx + (-v+iu)dy$$

$$\gamma = \gamma_1 + i\gamma_2$$
$$\dot\gamma = \dot\gamma_1 + i\dot\gamma_2$$

$\gamma: [a,b] \to U$ Weg, stückweise C^1

$$\boxed{\int_\gamma f(z)dz} = \int_\gamma (u+iv)dx + (-v+iu)dy =$$

$$= \int_a^b \Big[\big(u(\gamma(t)) + iv(\gamma(t))\big)\dot\gamma_1(t) + \big(-v(\gamma(t)) + iu(\gamma(t))\big)\dot\gamma_2(t)\Big]dt$$

$$= \int_a^b \big(u(\gamma(t)) + iv(\gamma(t))\big)\big(\dot\gamma_1(t) + i\dot\gamma_2(t)\big)dt = \boxed{\int_a^b f(\gamma(t))\,\dot\gamma(t)\,dt}$$

integral of a function with real variable

Aufgabe 5.1 b

$\boxed{5.2a}$ $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}^*$ $r > 0$ $m \in \mathbb{Z}$

$t \longmapsto re^{it}$

$$\int_\gamma z^m \, dz \underset{\substack{\uparrow \\ 5.1b}}{=} \int_0^{2\pi} (re^{it})^m \, re^{it} i \, dt = \int_0^{2\pi i} r^{m+1} i \, e^{i(m+1)t} \, dt$$

$\dot{\gamma}(t) = re^{it} i$

$\gamma(t) = re^{it} = r(\cos t + i \sin t)$

$\dot{\gamma}(t) = r(-\sin t + i \cos t) = ri(\cos t + i \sin t) = rie^{it}$

$\frac{dz}{z^2}$ auf $\mathbb{C}^*$

$-\frac{1}{z}$

$\forall n \geq 2 \quad \frac{dz}{z^n}$ ist exakt auf $\mathbb{C}^*$, mit Stammfunktion $\frac{(-1)^{n-1}}{z^{n-1}}$

$$\boxed{\begin{array}{l} f \in \mathcal{O}(U) \Rightarrow \\ f \text{ ist eine Stammfunktion von } \quad f'(z)\,dz \end{array}}$$

$$df = f'(z)\,dz$$

$U \subseteq \mathbb{C} = \mathbb{R}^2$ offen

$\mathbb{R}$-valued 1-Form on U: $\quad P\,dx + Q\,dy \quad P, Q : U \to \mathbb{R}$

$\mathbb{C}$-valued 1-Forms on U: $\quad P\,dx + Q\,dy \quad P, Q : U \to \mathbb{C}$

$P = A + iB \quad A, B, C, D : U \to \mathbb{R}$
$Q = C + iD$

$$A\,dx + iB\,dx + C\,dy + iD\,dy$$

$dz = dx + i\,dy \quad , \quad d\bar{z} = dx - i\,dy$

every $\mathbb{C}$-valued 1-Form on U can be written uniquely as $f \cdot dz + g \cdot d\bar{z}$ where $f, g : U \to \mathbb{C}$.

$f : U \to \mathbb{C}$ $\quad \mathbb{R}$-differenzierbar

$$\frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial \bar{z}}$$

$f = u + iv \quad u, v : U \to \mathbb{R}$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

- $$df := \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

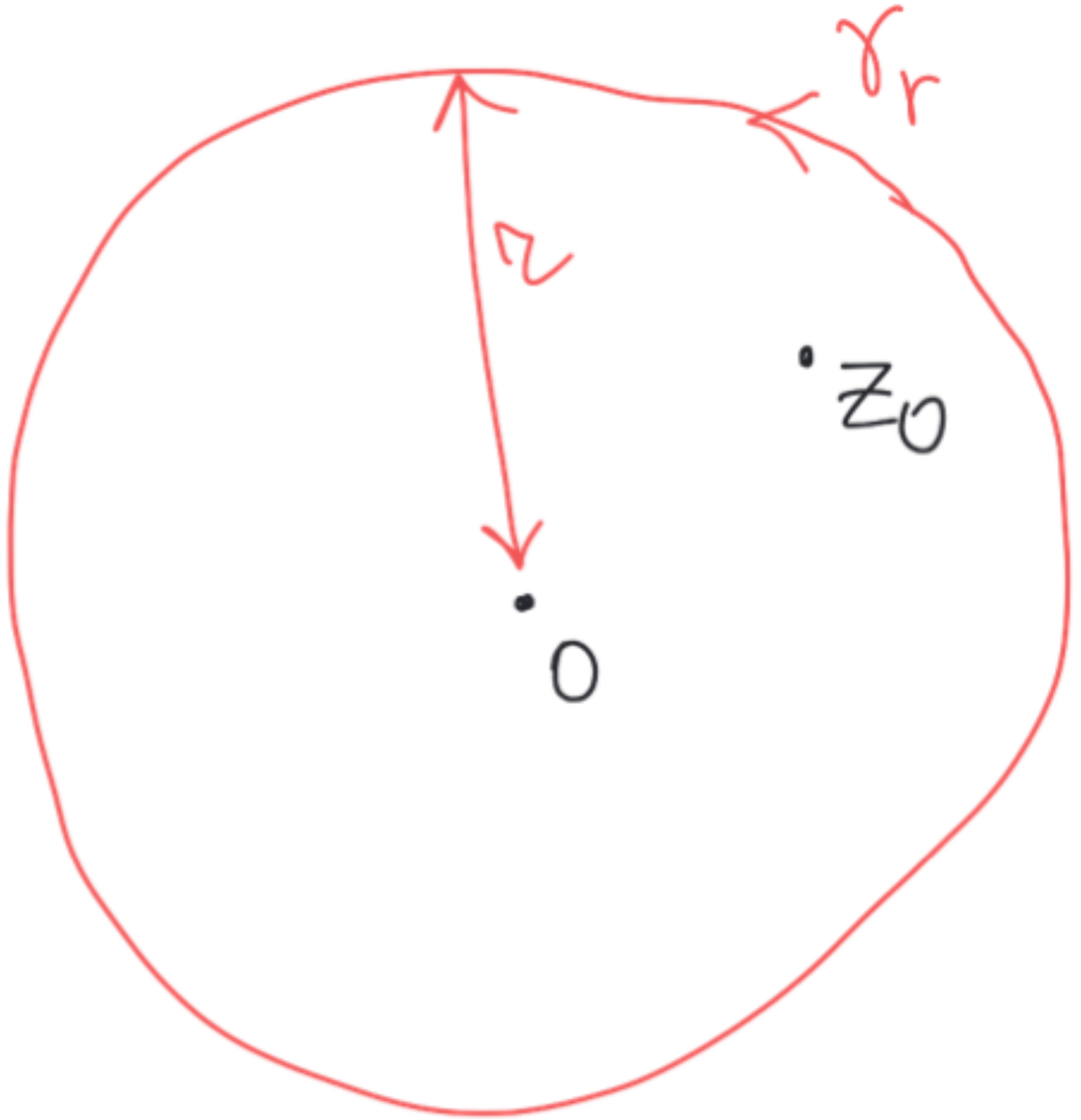
- f ist holomorph $\iff \dfrac{\partial f}{\partial \bar{z}} = 0 \iff df = \dfrac{\partial f}{\partial z} dz$

  If this is the case, $f' = \dfrac{\partial f}{\partial z}$.

Komplexe Ableitung

6.2.a $f \in \mathcal{O}(\mathbb{C})$, $R > 0$, $C > 0$, $\forall z \in \mathbb{C} \setminus B_R(0)$ $|f(z)| \le C|z|^p$.

I want to show that for every $z_0 \in \mathbb{C}$, $f^{(p+1)}(z_0) = 0$.



Choose $r > \max\{|z_0|, R\}$.

$$\gamma_r : [0, 2\pi] \to \mathbb{C}$$
$$t \mapsto re^{it}$$

$$f^{(p+1)}(z_0) = \frac{(p+1)!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z - z_0)^{p+2}} dz$$

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \forall z \in \mathbb{C}$$

$$\int_{\gamma_r} \frac{f(z)}{(z-z_0)^{p+2}} dz = \int_{\gamma_r} \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-p-2} dz$$

$$= \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} \int_{\gamma_r} \underbrace{(z-z_0)^{n-p-2} dz}_{} \quad \text{exact} \iff n-p-2 \neq -1$$

$$n \neq p+1$$

$$= \frac{f^{(p+1)}(z_0)}{(p+1)!} 2\pi i$$

$$f^{(p+1)}(z_0) = \frac{(p+1)!}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{p+2}}\,dz$$

$r > |z_0|$
$r > R$

$$|f^{(p+1)}(z_0)| \leq \frac{(p+1)!}{2\pi} \max_{z\in\partial B_r(0)} \frac{|f(z)|}{|z-z_0|^{p+2}} \cdot 2\pi r$$

$$\leq (p+1)! \cdot C r^{p+1} \cdot \max_{z\in\partial B_r(0)} \frac{1}{|z-z_0|^{p+2}} \leq$$

$|z| = r > |z_0|$

$z \in \partial B_r(0)$

$|z-z_0| \geq |z| - |z_0|$

$\frac{1}{|z-z_0|} \leq \frac{1}{|z|-|z_0|}$

$$\leq (p+1)! \frac{C r^{p+1}}{(r-|z_0|)^{p+2}}$$

$r \to +\infty$

$\forall z \in \mathbb{C} \setminus B_R(0) \quad |f(z)| \leq C|z|^p \Rightarrow \forall z \in \partial B_r(0),\ |f(z)| \leq C|z|^p$

$f: X \longrightarrow Y$ stetig, $x_0 \in X$

$$\Longrightarrow \quad \underset{\pi_1(f, x_0)}{\overset{}{f_*}} : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

$$[\gamma] \longmapsto [f \circ \gamma]$$

pathhomotopy class of γ as a loop in X

γ Schleife in X

$\gamma : [0,1] \longrightarrow X$

$\gamma(0) = \gamma(1) = x_0$

$f \circ \gamma : [0,1] \longrightarrow Y$

Schleife, $(f \circ \gamma)(0) = (f \circ \gamma)(1) = f(x_0)$

$i : \mathbb{C}^* \hookrightarrow \mathbb{C}$ Inklusionsabbildung $x_0 = 1$.

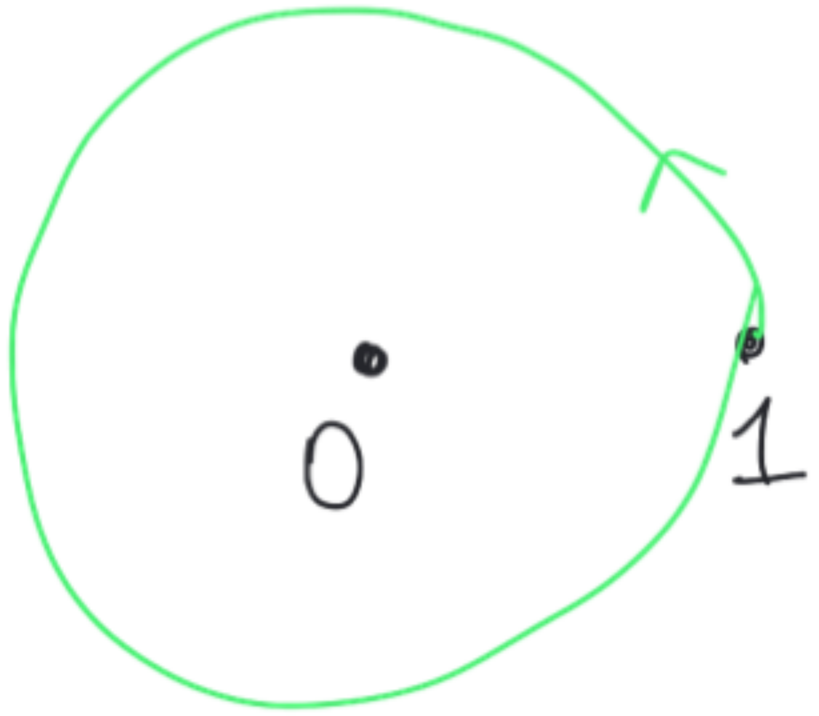
$\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$

$\mathbb{Z} \longrightarrow \pi_1(\mathbb{C}^*, 1)$

$k \longmapsto$ pathhomotopy class of the loop

$[0,1] \longrightarrow \mathbb{C}^*$

$t \longmapsto e^{2\pi i k t}$



$\pi_1(\mathbb{C}^*, 1) \longrightarrow \mathbb{Z}$

$[\gamma] \longmapsto W(\gamma, 0)$

$\pi_1(\mathbb{C}, 1) = \{e\}$
weil $\mathbb{C}$ konvex ist

konvex $\Rightarrow$ sternförmig $\Rightarrow$ einfach zusammenhängend

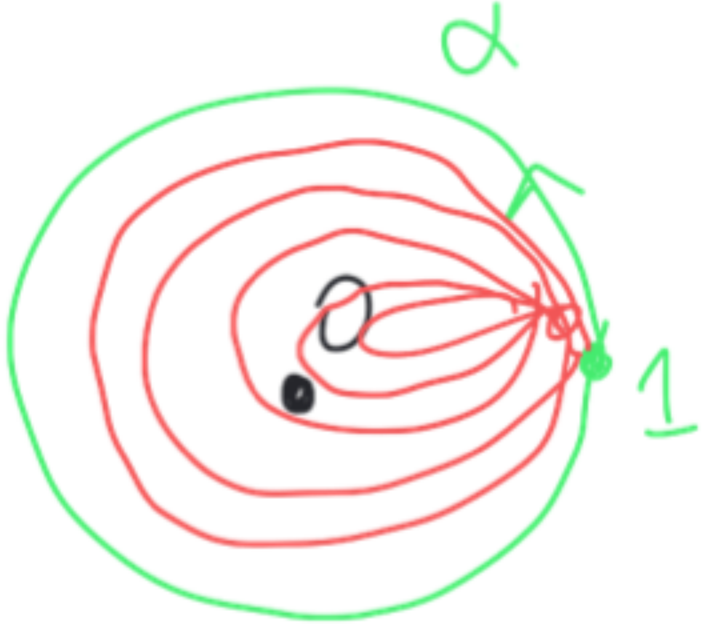
$i : \mathbb{C}^* \hookrightarrow \mathbb{C}$ Inklusionsabbildung

$i_* = \pi_1(i,1) : \quad \pi_1(\mathbb{C}^*, 1) \longrightarrow \pi_1(\mathbb{C}, 1)$

$[\gamma]_{\mathbb{C}^*} \longmapsto [\gamma]_{\mathbb{C}}$

pathhomotopy class of the loop γ as a loop in $\mathbb{C}^*$

pathhomotopy class of γ as a loop in $\mathbb{C}$

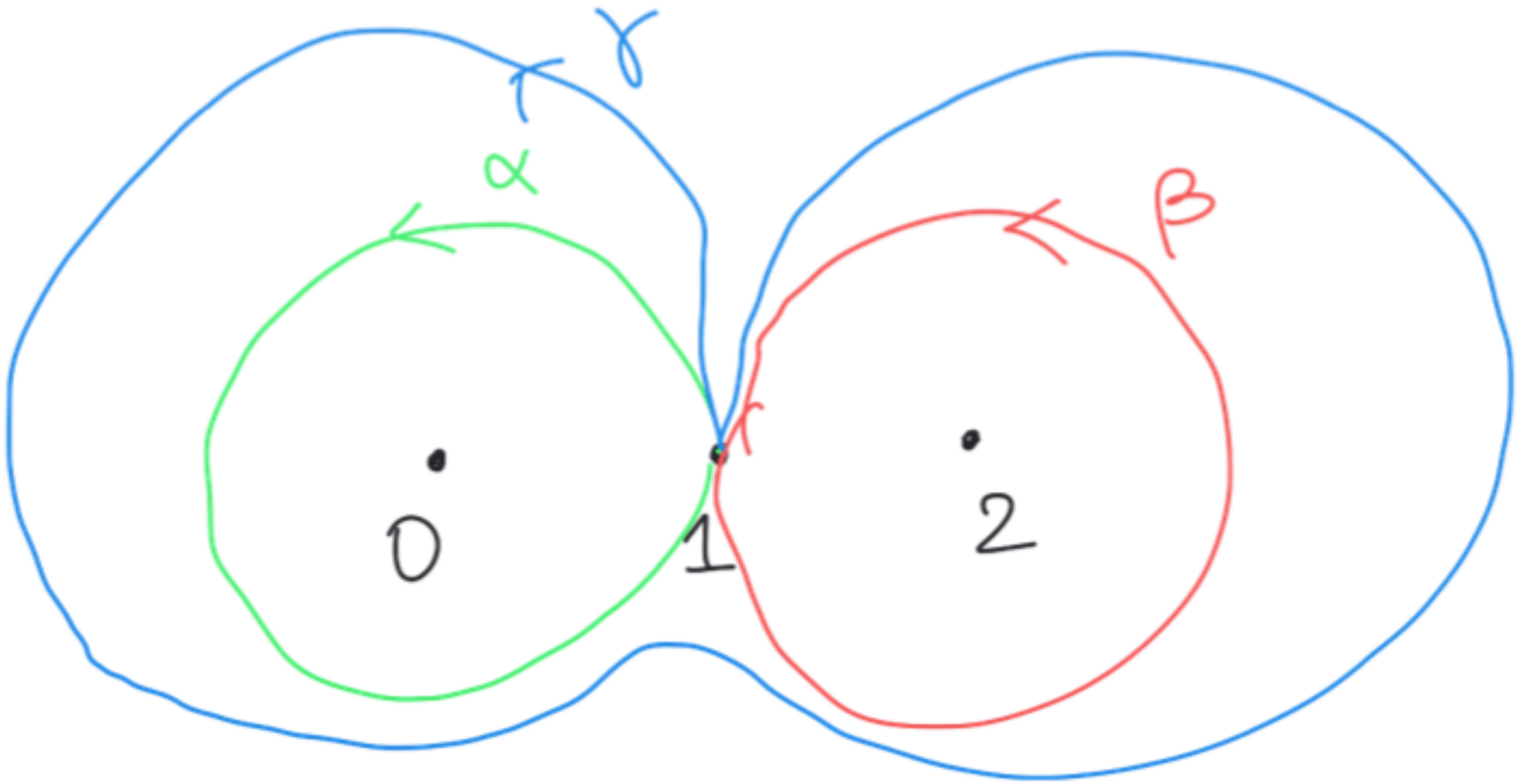


$\alpha : [0,1] \to \mathbb{C}^* \subseteq \mathbb{C}$
$t \mapsto e^{2\pi i t}$

$[\alpha]_{\mathbb{C}^*} \neq 0 \,, \quad [\alpha]_{\mathbb{C}} = 0 .$

$\pi_1(i,1)$ nicht injektiv.

$U = \mathbb{C} \setminus \{0, 2\}$ $\quad i : U \hookrightarrow \mathbb{C}^*$ $\quad z_0 = 1$



$\alpha : [0,1] \to U$
$t \mapsto e^{2\pi i t}$

$\beta : [0,1] \to U$
$t \mapsto 2 - e^{2\pi i t}$

$\pi_1(i, 1) : \pi_1(U, 1) \to \pi_1(\mathbb{C}^*, 1)$

$\pi_1(U,1)$ ist die frei Gruppe mit 2 Erzeuger $[\alpha]_U, [\beta]_U$

γ und $\alpha * \beta$ sind weghomotop in $U \Longrightarrow [\gamma]_U = [\alpha * \beta]_U = [\alpha]_U \cdot [\beta]_U$

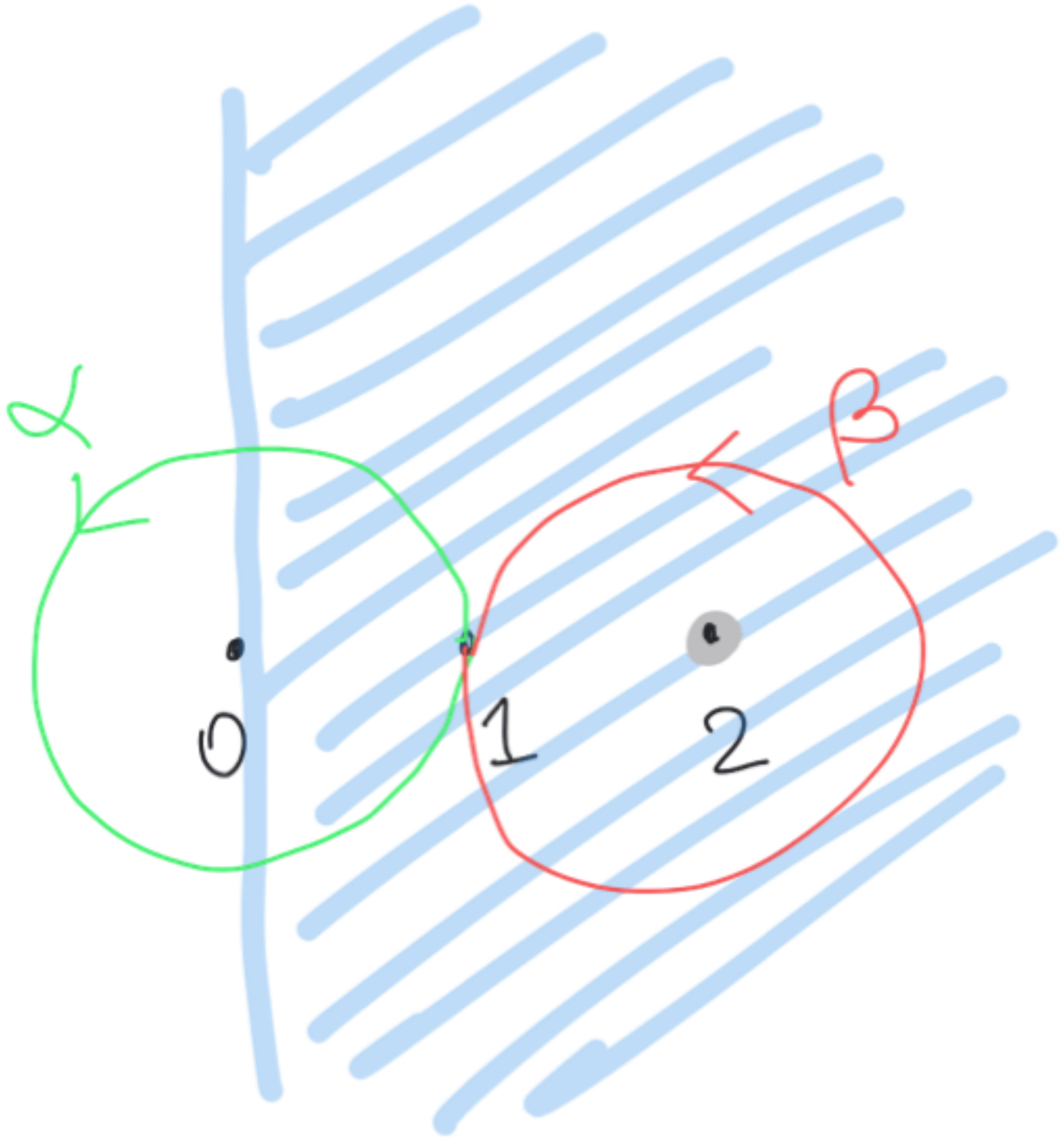
$\pi_1(\mathbb{C}^*, 1)$ ist die frei Gruppe mit 1 Erzeuger: $[\alpha]_{\mathbb{C}^*}$.

$$\pi_1(i,1) : \quad \begin{array}{l} [\alpha]_U \longmapsto [\alpha]_{\mathbb{C}^*} \\ [\beta]_U \longmapsto [\beta]_{\mathbb{C}^*} = e \end{array} \qquad [\gamma]_U \longmapsto [\gamma]_{\mathbb{C}^*} = [\alpha]_{\mathbb{C}^*}$$

$U = \{ z \in \mathbb{C} \mid \operatorname{Re} z > 0 \}$

$V = U \setminus \{2\}$

$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[rr, bend left, "k"] & U \arrow[r, hook, "j"] & \mathbb{C}^* \end{tikzcd}$$



$$\begin{array}{ccccc} \pi_1(V,1) & \longrightarrow & \pi_1(U,1) & \longrightarrow & \pi_1(\mathbb{C}^*,1) \\ \cong & & \| & & \| \\ \langle [\beta]_V \rangle & & \{e\} & & \langle [\alpha]_{\mathbb{C}^*} \rangle \\ \cong & & & & \cong \\ \mathbb{Z} & & & & \mathbb{Z} \end{array}$$

V ist nicht einfach zusammenhängend

$k: V \hookrightarrow \mathbb{C}^*$ Inklusionsabbildung

$\pi_1(k,1) : \pi_1(V,1) \to \pi_1(\mathbb{C}^*,1)$ Nullhom.