7.2.d $f \in \mathcal{O}(\mathbb{C})$, $g \in \mathcal{O}(\mathbb{C})$ not constant

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$\Rightarrow$ $g \circ f : \mathbb{C} \to \mathbb{C}$ is not bounded.

Fact 1 $U \subseteq \mathbb{C}$ open connected, $f \in \mathcal{O}(U)$ not constant

$\Rightarrow$ $\forall c \in \mathbb{C}$ $\quad f^{-1}(c)$ is discrete

Pf You consider the function $g : U \to \mathbb{C}$, $z \mapsto f(z) - c$ $\quad$ g is not constant.

$g^{-1}(0) = f^{-1}(c)$

Principle of analytic continuation $g^{-1}(0)$ discrete. $\square$

Fact 2 X top space, discrete & connected $\Rightarrow$ X is a point.

Pf By contradiction $g \circ f$ is bounded. Liouville $\Rightarrow$ $g \circ f$ is constant. Therefore $\exists c \in \mathbb{C}$ s.t. $\forall z \in \mathbb{C}$ $(g \circ f)(z) = c$.

$\Rightarrow \forall z \in \mathbb{C} \quad g(f(z)) = c \quad \Rightarrow \quad \forall z \in \mathbb{C} \quad f(z) \in g^{-1}(c).$

$\Rightarrow \quad f(\mathbb{C}) \subseteq g^{-1}(c).$

g not constant $\overset{\text{Fact}}{\Longrightarrow}$ $g^{-1}(c)$ is discrete. $\Rightarrow$ $f(\mathbb{C})$ is discrete

because every subspace of a discrete top. space is discrete.

$\mathbb{C}$ connected, f continuous $\Rightarrow$ $f(\mathbb{C})$ is connected.

$f(\mathbb{C})$ connected & discrete $\overset{\text{Fact 2}}{\Longrightarrow}$ $f(\mathbb{C})$ is a point $\Rightarrow$

$\Rightarrow$ f constant. $\lightning$. $\square$

Another proof of 7.2 d

$f: \mathbb{C} \to \mathbb{C}$ not constant $\xRightarrow[\text{Gebietstreue}]{\text{open mapping theorem}}$ $f(\mathbb{C})$ is open in $\mathbb{C}$.

$\left.\begin{array}{l} f(\mathbb{C}) \subseteq \mathbb{C} \text{ open} \\ g: \mathbb{C} \to \mathbb{C} \text{ not constant} \end{array}\right\}$ $\xRightarrow[\text{Gebietstreue}]{\text{open mapping theorem}}$ $g(f(\mathbb{C}))$ is open in $\mathbb{C}$.

If $g \circ f$ were bounded, then by Liouville $g \circ f$ would be constant, so $g(f(\mathbb{C}))$ would be a point, so wouldn't be an open subset of $\mathbb{C}$

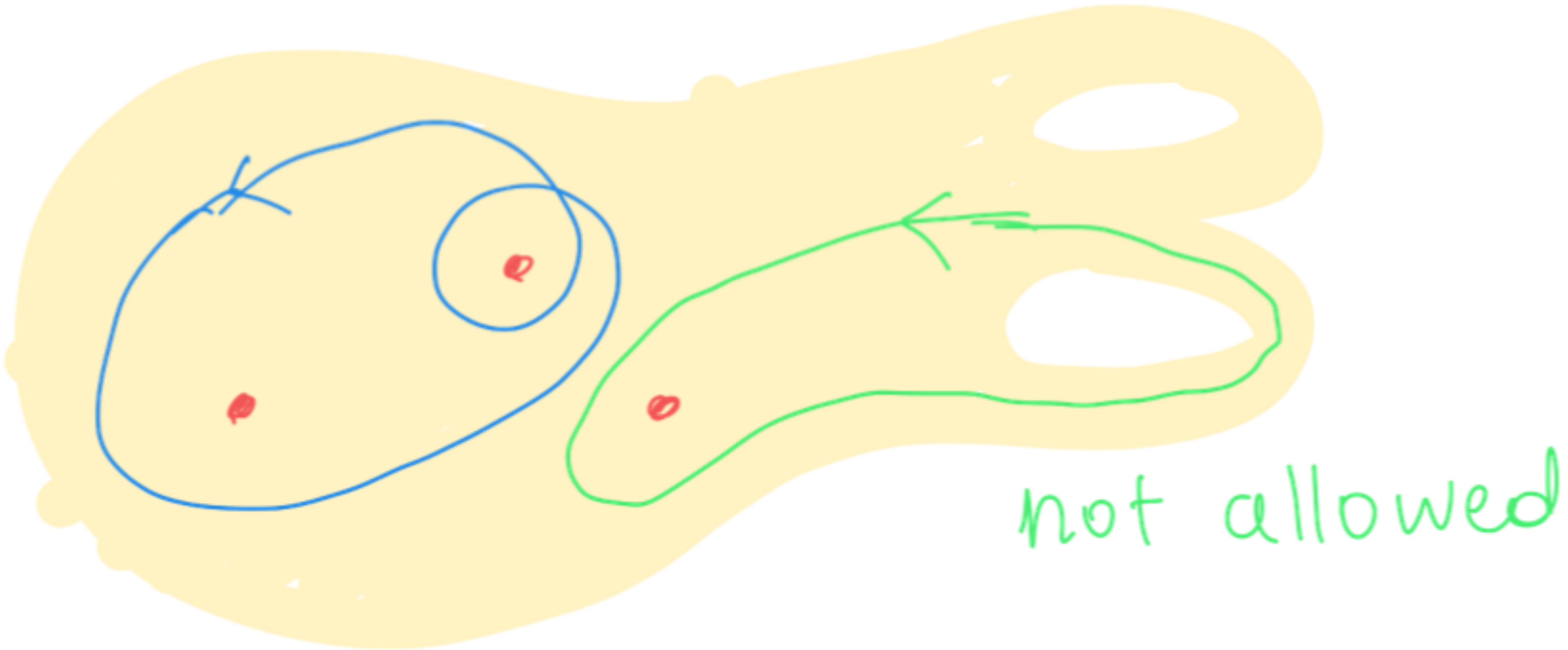
Therefore $g \circ f$ cannot be bounded. $\square$

Thm $U \subseteq \mathbb{C}$ connected open subset. $f \in \mathcal{O}(U)$.
The following statements are equivalent:

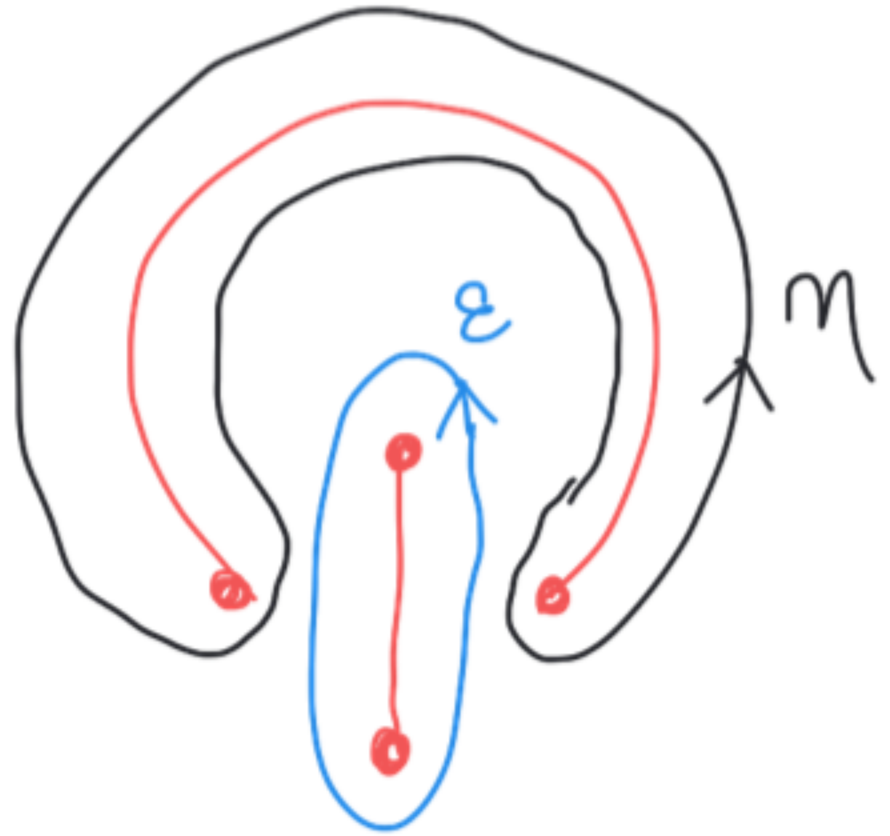
1) f is not constant
2) $\forall c \in \mathbb{C}$ $f^{-1}(c)$ is discrete
3) $f(U)$ is open in $\mathbb{C}$
4) $\forall V \subseteq U$ open, $f(V)$ is open in $\mathbb{C}$
5) $\forall z_0 \in U, \forall \varepsilon > 0$ with $B_\varepsilon(z_0) \subseteq U$, $f|_{B_\varepsilon(z_0)}$ is not constant
6) $\forall z_0 \in U$, $\exists n \geq 1$: $f^{(n)}(z_0) \neq 0$
7) $\bigcap_{n \geq 1} \{z \in U \mid f^{(n)}(z) = 0\} = \emptyset$

open mapping theorem
+
principle of analytic continuation

$X = U, Y = \mathbb{C}$
11.2, p.6.


not allowed

11.2 f

$C =$ 

η, ε are not nullhomologous in C

η, ε are nullhomologous in $\mathbb{C}$, because $\mathbb{C}$ is simply connected

$A = \mathbb{C} \setminus \{\pm 1, \pm i\}$

Residue theorem: $\mathbb{C}$

$1, i, -1, -i$

$\frac{h'}{h} : \mathbb{C} \setminus \{1, i, -1, -i\} \longrightarrow \mathbb{C}$ holomorph.

η loop in $\mathbb{C} \setminus \{1, i, -1, -i\}$, nullhomologous in $\mathbb{C}$

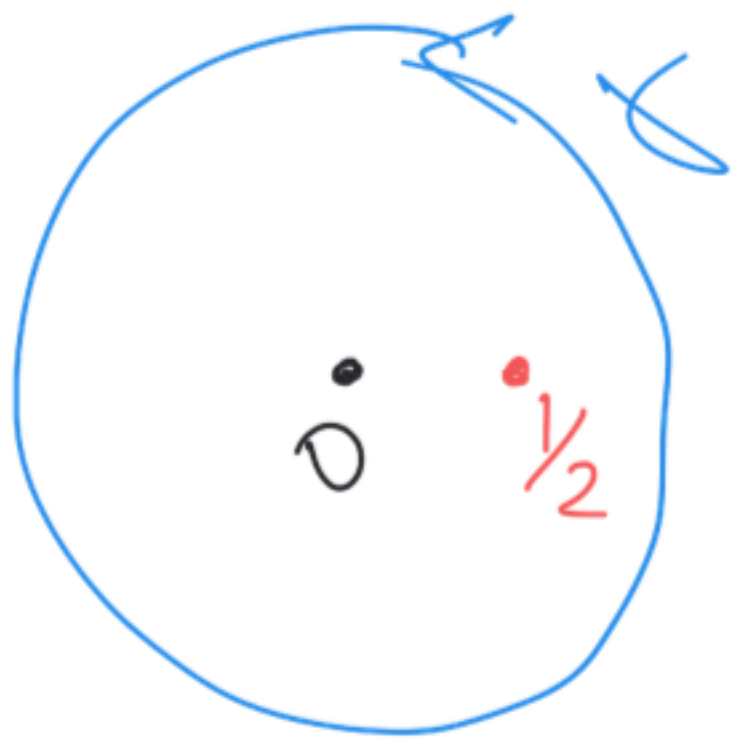
$$\Longrightarrow \int_\eta \frac{h'}{h} dz = 2\pi i \cdot \left[\operatorname{Res}\left(\frac{h'}{h}, 1\right) + \operatorname{Res}\left(\frac{h'}{h}, -1\right) \right]$$

Apply the Residue theorem:

$U = \mathbb{C}^*$, $\quad z_1 = \frac{1}{2}$

$f \in \mathcal{O}(U \setminus \{z_1\})$. $\quad f(z) = z^{-1}$

$\gamma = \partial B_1(0)$ loop in $\mathbb{C}^* \setminus \{z_1\}$



$$\int_\gamma f(z)\,dz = 2\pi i$$

$$2\pi i \cdot \underset{\substack{\| \\ 0}}{\mathrm{Res}(f, \tfrac{1}{2})} \cdot W(\gamma, \tfrac{1}{2}) = 0$$

but γ is not null homologous in $U = \mathbb{C}^*$

$U, V \subseteq \mathbb{C}$ open connected.

U and V are biholomorphic $\Rightarrow$ U and V are homeomorphic

$\Downarrow$

U and V are homotopy equivalent

$\Downarrow$

$\pi_1(U) \simeq \pi_1(V)$.
group isomorphism.

- $\mathbb{C}$ and $B_1(0)$ are homeomorphic but not biholomorphic
- $B_1(0) \setminus \{0\}$ and $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$ are homeomorphic but not biholomorphic.
- $B_1(0) \setminus \{0\}$ and $\mathbb{C}^*$ ...
- $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$ and $\mathbb{C}^*$ ...

Theorem U, V connected open subsets of $\mathbb{C}$.

U, V simply connected

$U \neq \mathbb{C}$ and $V \neq \mathbb{C}$

$\Rightarrow$ U and V are biholomorphic.

- Very important
- You are not allowed to use this theorem! But maybe it could be useful to get an idea.