5.5.2020

2.5.:

Satz $(a_n)_{n\geq 1} \subset \mathbb{R}_{>0}$ $\lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = L \in [0,+\infty]$

$\Rightarrow \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = L$

Beispiel $a_n = \begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$

$\frac{a_{n+1}}{a_n} = \begin{cases} 2 & n \text{ odd} \\ \frac{1}{2} & n \text{ even} \end{cases}$

$1 \leq \sqrt[n]{a_n} \leq \sqrt[n]{2}$

$\Rightarrow \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = 1$

$\lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n}$ existiert nicht

$\limsup\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = 2$

2.5d: in general $R \neq \frac{1}{\limsup \frac{a_{n+1}}{a_n}}$.

2.5e $n^{\log n} = e^{\log\left(n^{\log n}\right)} = e^{(\log n)^2}$

$\sqrt[n]{n^{\log n}} = \left(e^{(\log n)^2}\right)^{\frac{1}{n}} = e^{\left(\frac{(\log n)^2}{n}\right)} \longrightarrow e^0 = 1.$

2.2a) $U \xrightarrow{f} \mathbb{C}$, $V \xrightarrow{g} \mathbb{C}$ holomorph., $f(U) \subseteq V$, $z_0 \in U$

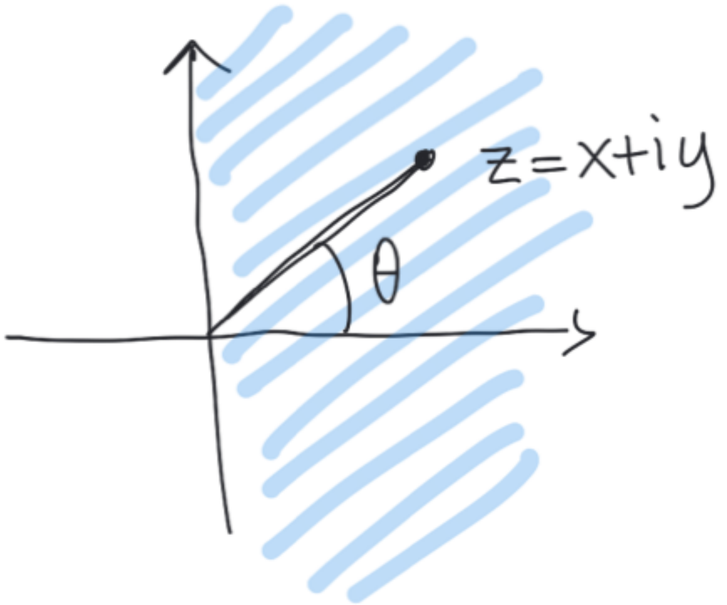
$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0}$$

ok if $\exists \delta > 0 : \forall z \in B_\delta(z_0) \cap U \setminus \{z_0\}$, $f(z) \neq f(z_0)$.

$$G(w) = \begin{cases} \dfrac{g(w) - g(f(z_0))}{w - f(z_0)} & \text{if } w \neq f(z_0) \\ g'(f(z_0)) & \text{if } w = f(z_0) \end{cases}$$

$G : U \longrightarrow \mathbb{C}$ $\quad \forall z \in U \setminus \{z_0\}$, $\dfrac{g(f(z)) - g(f(z_0))}{z - z_0} = G(f(z)) \cdot \dfrac{f(z) - f(z_0)}{z - z_0}$

2.3 a) $U = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$



$f: U \longrightarrow \mathbb{C}$ holomorph,

$\forall z \in U \quad e^{f(z)} = z$

"a logarithm on U"

$\equiv$

$f(z) = u(z) + i\,v(z)$

$u, v : U \longrightarrow \mathbb{R}$

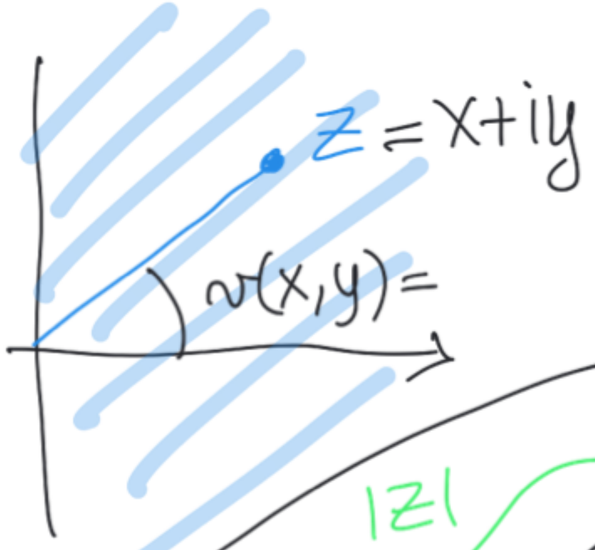
$e^{f(z)} = z \iff e^{u(x,y)+i\,v(x,y)} = x + iy$

$\|$

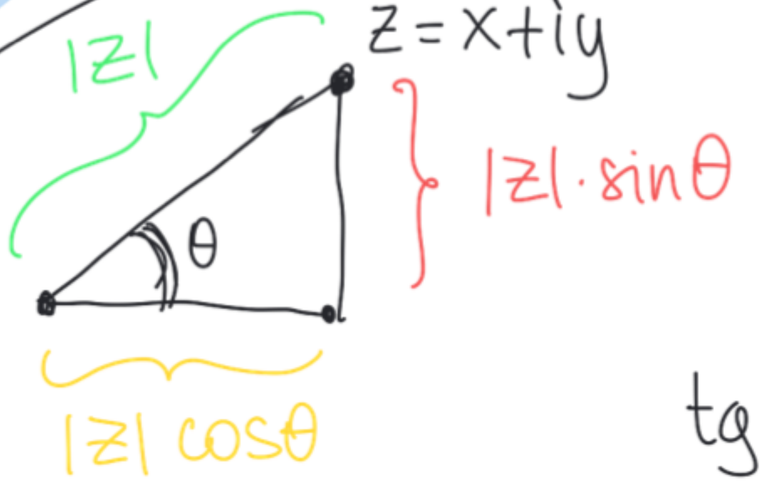
$e^{u(x,y)} \cdot (\cos v(x,y) + i \sin v(x,y))$

$$\sqrt{x^2+y^2} = |x+iy| = e^{u(x,y)} \Rightarrow u(x,y) = \log\sqrt{x^2+y^2}$$
$$= \frac{1}{2}\log(x^2+y^2)$$

$$v(x,y) = \arctan\left(\frac{y}{x}\right)$$



$$f(x,y) = \frac{1}{2}\log(x^2+y^2) + i\arctan\left(\frac{y}{x}\right)$$
$$= \frac{1}{2}\log(x^2+y^2) + i\arcsin\frac{y}{\sqrt{x^2+y^2}}$$



$$\sin\theta = \frac{y}{|z|} = \frac{y}{\sqrt{x^2+y^2}}, \quad \theta = \arcsin\frac{y}{\sqrt{x^2+y^2}}$$

$$\operatorname{tg}\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \Rightarrow \theta = \arctan\frac{y}{x}$$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ bijection

$\cos : \mathbb{R} \longrightarrow [-1,1]$ surjective

$\sin : \mathbb{R} \longrightarrow [-1,1]$ " "

$\cos|_{[0,\pi]} : [0,\pi] \longrightarrow [-1,1]$ bijective

$\sin|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1,1]$ bijective

$\arctan : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\arccos : [-1,1] \longrightarrow [0,\pi]$

$\operatorname{arcsin} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$U = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$$

$$f: U \longrightarrow \mathbb{C}$$

$$x+iy \longmapsto \frac{1}{2}\log(x^2+y^2) + i \arctan\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2) + i \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\neq \frac{1}{2}\log(x^2+y^2) + i \arccos\frac{x}{\sqrt{x^2+y^2}}$$

$$v(x,y) = \arctan\left(\frac{y}{x}\right) = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$w(x,y) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$v(1+i) = \frac{\pi}{4}$$

$$w(1+i) = \frac{\pi}{4}$$

$$v(1-i) = -\frac{\pi}{4}$$

$$w(1-i) = \frac{\pi}{4}$$

$$f : U \longrightarrow \mathbb{C}$$

$$x+iy \longmapsto \underbrace{\frac{1}{2}\log(x^2+y^2)}_{u(x,y)} + i\underbrace{\arctan\left(\frac{y}{x}\right)}_{v(x,y)}$$

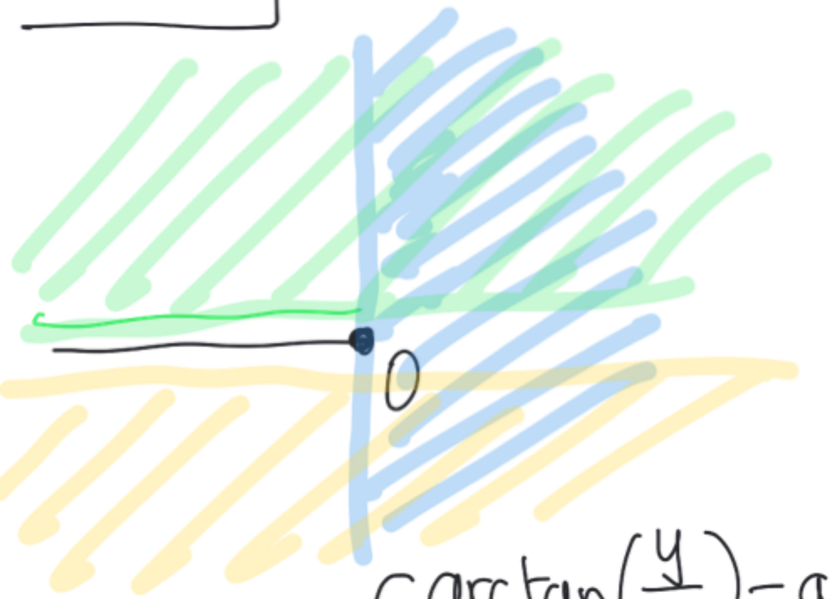
$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2}\cdot 2x \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

Cauchy-Riemann

2.3 b) $V = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$



$$f : V \longrightarrow \mathbb{C}$$
$$x+iy \longmapsto \frac{1}{2}\log(x^2+y^2) + i\, v(x,y)$$

$$v(x,y) = \begin{cases} \arctan\left(\frac{y}{x}\right) = \arcsin\frac{y}{\sqrt{x^2+y^2}} & \text{if } x>0 \\ \frac{\pi}{2} & \text{if } x=0, y>0 \\ \pi + \arctan\left(\frac{y}{x}\right) & \text{if } x<0, y>0 \\ -\frac{\pi}{2} & \text{if } x=0, y<0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & \text{if } x<0, y<0 \end{cases} = \begin{cases} \arctan\left(\frac{y}{x}\right) & x>0 \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & y>0 \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & y<0 \end{cases}$$

$$v(V) = (-\pi, \pi)$$

$$\forall t \in \mathbb{R}^* \text{, } \arctan\left(\frac{1}{t}\right) + \arctan t = \begin{cases} \pi/2 & t > 0 \\ -\pi/2 & t < 0 \end{cases}$$

$0 < \alpha < \frac{\pi}{2}$

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan \alpha}$$

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$$

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$$

$$\left(v(x,y) = \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}} & y > 0 \\ 0 & y = 0 \\ \arccos \dfrac{-x}{\sqrt{x^2+y^2}} - \pi & y < 0 \end{cases} \right.$$

hopefully

2.3c $n \in \mathbb{Z}$, $n \neq 0$. $g: V \longrightarrow \mathbb{C}$ holomorph.

$\forall z \in V \quad g(z)^n = z$

f from b)

$$g(z) = \exp\left(\frac{1}{n} f(z)\right)$$

$$g(z)^n = \left(\exp\left(\frac{1}{n} f(z)\right)\right)^n = \exp\left(n \cdot \frac{1}{n} f(z)\right) = \exp(f(z)) = z$$

$\exp\alpha \cdot \exp\beta = \exp(\alpha+\beta)$

$$\zeta_n = e^{2\pi i/n}$$

$$z \longmapsto \zeta_n^k \cdot g(z) \qquad 0 \leq k < n$$

$$W = \mathbb{C} \setminus \{iy \mid y \in \mathbb{R}, y \leq 0\}$$

$f_U : U \longrightarrow \mathbb{C}$ a)

$f_V : V \longrightarrow \mathbb{C}$ b) $f_V|_U = f_U$

$U \subseteq W$

Find a logaritm on W: $f_W : W \longrightarrow \mathbb{C}$

$f_W|_U = f_U$

$-1-i$
$\in$
$W \cap V$

$$\operatorname{Im} f_V(-1-i) = \operatorname{Im} f_W(-1-i)$$

$\|$ $\|$

$-\frac{3}{4}\pi$ $\frac{5}{4}\pi$

d) $f + 2\pi i k \qquad k \in \mathbb{Z}$

2.4.a]

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+1} = F_n + F_{n-1} \end{cases} \quad n \geq 1$$

$0, 1, 1, 2, 3, 5, 8, \ldots$

$f = T \cdot g$

$$f = \sum_{n \geq 0} F_n T^n = \cancel{F_0} + F_1 T + F_2 T^2 + \cdots$$

$$f - T = \sum_{n \geq 1} F_{n+1} T^{n+1} = \sum_{n \geq 1} (F_n + F_{n-1}) T^{n+1} =$$

$$= T \cdot \sum_{n \geq 1} F_n T^n + T^2 \cdot \sum_{n \geq 1} F_{n-1} T^{n-1} = T \cdot f + T^2 \cdot f$$

$$\Rightarrow (1 - T - T^2) \cdot f = T \qquad (1 - T - T^2) \cdot T \cdot g = T$$

$$\Rightarrow (1 - T - T^2) \cdot g = 1.$$

$$\begin{cases} a_0 = \\ \vdots \\ a_{k-1} = \\ a_{n+k} = \sum_{j=0}^{k-1} \lambda_j a_{n+j} \quad \forall n \geq 0 \end{cases}$$

$$f = \sum_{n \geq 0} a_n T^n \in \mathbb{C}[[T]]$$

$$\left(1 - \sum_{j=0}^{k-1} \lambda_j T^{k-j}\right) \cdot f \in \mathbb{C}[T]$$

z.B.: $k=1$

$$\begin{cases} a_0 \\ a_{n+1} = \lambda_0 a_n \end{cases}$$

$$a_n = (\lambda_0)^n a_0$$

$$(1 - \lambda_0 T) \cdot f = a_0$$

z.B.: $k=2$

$$\begin{cases} a_0 \\ a_1 \\ a_{n+2} = \lambda_0 a_n + \lambda_1 a_{n+1} \quad \forall n \geq 0 \end{cases}$$

$$(1 - \lambda_1 T - \lambda_0 T^2) \cdot f = a_0 + (a_1 + \lambda_1 a_0) T$$