12.05.2020

3.1.a $a \in \mathbb{C}$, $\quad f : \mathbb{C} \setminus \{a\} \longrightarrow \mathbb{C}$, $\quad z_0 \in \mathbb{C}$

$$z \longmapsto \frac{1}{z-a}$$

Many people computed $f^{(n)}(z_0)$ $\forall n \geq 0$. and said that for z in a nbd of z_0 $f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$.

But this is incomplete because you don't know that f is analytic.

$f : \mathbb{R} \longrightarrow \mathbb{R}$

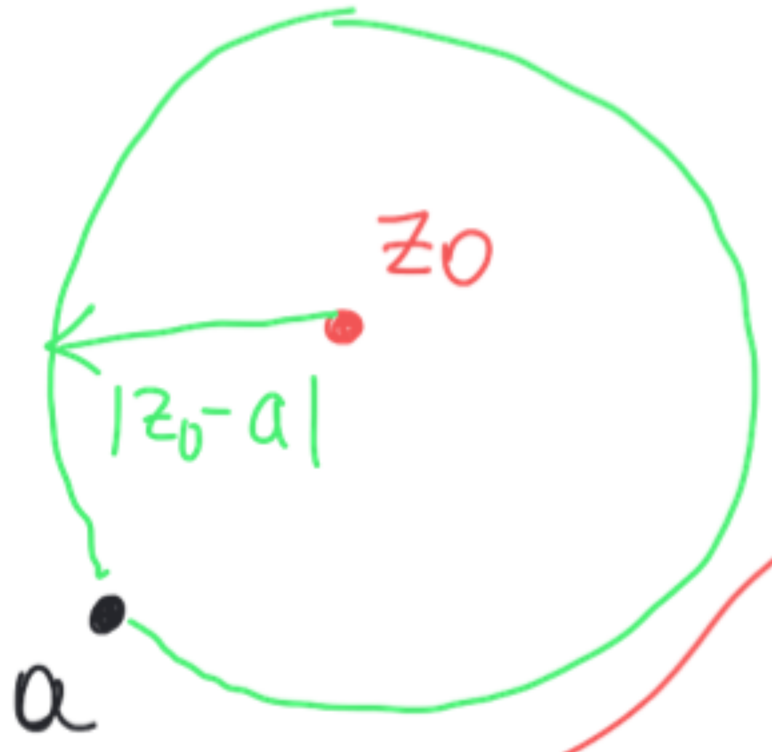
$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f C^∞, $\forall n \geq 0$ $f^{(n)}(0) = 0$

but one cannot find $\varepsilon > 0$ s.t. $\forall x \in (-\varepsilon, \varepsilon)$ $f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$

f is not real-analytic.

$$z \in B_{|z_0-a|}(z_0) \qquad |z-z_0| < |z_0-a|$$



$$\left|\frac{z-z_0}{z_0-a}\right| < 1$$

$$f(z) = \frac{1}{z-a} = \frac{1}{z_0-a+z-z_0} = \frac{1}{(z_0-a)}\cdot\frac{1}{\left(1-\frac{z-z_0}{a-z_0}\right)}$$

geometric series

$$= \frac{1}{z_0-a}\cdot\sum_{n=0}^{+\infty}\left(\frac{z-z_0}{a-z_0}\right)^n = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(z_0-a)^{n+1}}\cdot(z-z_0)^n$$

3.1 b $\varphi = \frac{1+\sqrt{5}}{2}$ golden ratio

(F_n) Fibonacci sequence

$$h : B_{1/\varphi}(0) \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{z}{1-z-z^2}$$

$$a = \sum_{n \geq 0} F_n T^n \in \mathbb{C}[\![T]\!]$$

$$\lim_{n \to +\infty} \frac{F_{n+1}}{F_n} = \varphi \Rightarrow \lim_{n \to +\infty} \sqrt[n]{F_n} = \varphi \Rightarrow \text{Konvergenzradius von } a = \frac{1}{\varphi}.$$

$$\Phi_a : B_{1/\varphi}(0) \longrightarrow \mathbb{C}$$
$$z \longmapsto a(z) = \sum_{n=0}^{+\infty} F_n z^n.$$

Previous problem Sheet: $(1-T-T^2) \cdot a = T$ in $\mathbb{C}[\![T]\!]$.

$$\Rightarrow \forall z \in B_{\frac{1}{\varphi}}(0) \quad (1-z-z^2) \cdot a(z) = z$$

$1 - z - z^2 = 0$?

$z^2 + z - 1 = 0$

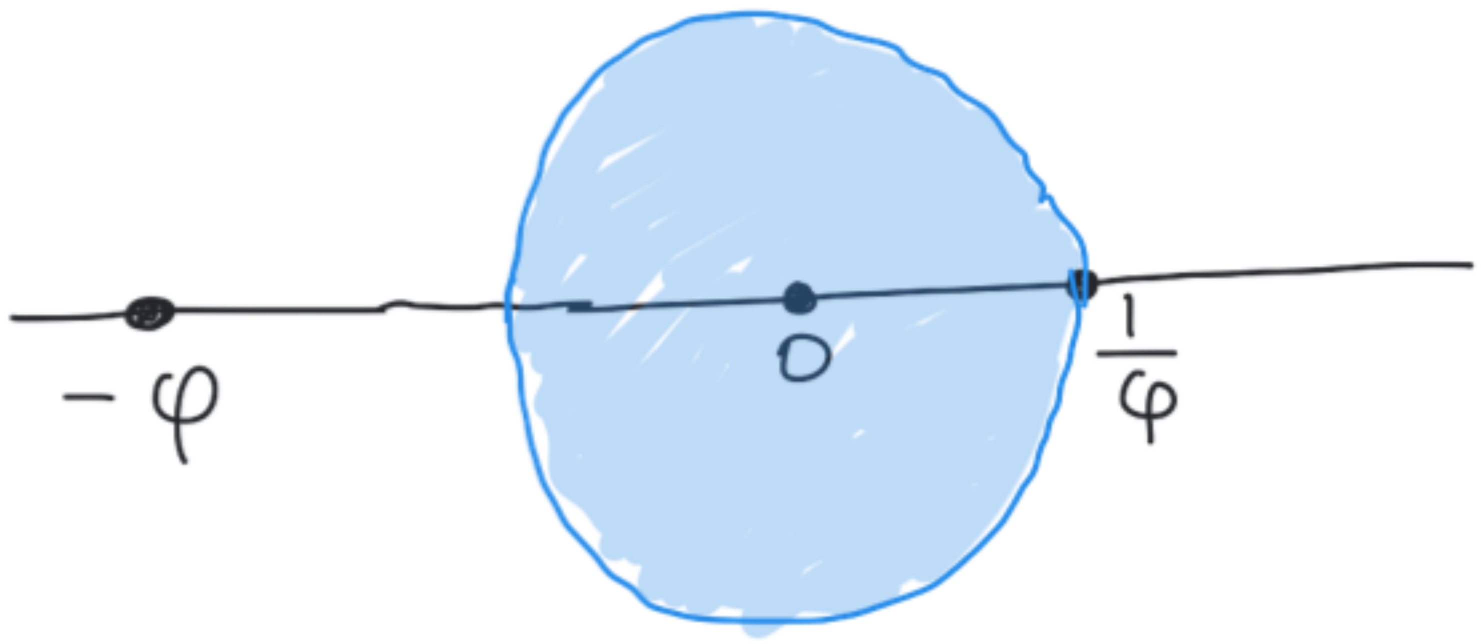
$$z = \frac{-1 \pm \sqrt{1+4}}{2} = \begin{cases} \dfrac{-1+\sqrt{5}}{2} = \dfrac{1}{\varphi} \simeq 0.6 \\[2mm] -\dfrac{1+\sqrt{5}}{2} = -\varphi \simeq -1.6 \end{cases}$$

$$\forall z \in B_{\frac{1}{\varphi}}(0), \quad 1 - z - z^2 \neq 0$$

$$\Rightarrow \forall z \in B_{\frac{1}{\varphi}}(0), \quad \underset{\sum_{n=0}^{+\infty} F_n z^n}{\underset{\|}{a(z)}} = \frac{z}{1-z-z^2} = h(z).$$

$$z \longmapsto \frac{z}{1-z-z^2}$$

is defined over $\mathbb{C} \setminus \left\{\frac{1}{\varphi}, -\varphi\right\}$



3.1.c) $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(i) = i$, $\forall n \in \mathbb{N}^+ \; f(\frac{1}{n}) = \frac{1}{n^3}$
analytic.

f stetig $\Rightarrow$ $f(0) = 0$
$\lim_{n \to +\infty} \frac{1}{n} = 0$

$g : \mathbb{C} \longrightarrow \mathbb{C}$ analytic
$z \longmapsto z^3$

$A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$ is not discrete

$f|_A = g|_A$ $\xRightarrow{\text{Principle of analytic continuation}}$ $f = g \Rightarrow f(i) = g(i)$ ↯
$\mathbb{C}$ connected (applied $f-g$)

$f(i) = i$, $g(i) = -i$

__Example__ $f : \mathbb{C}^* \longrightarrow \mathbb{C}$

$$z \longmapsto \sin\left(\frac{\pi}{z}\right)$$

$\sin : \mathbb{C} \longrightarrow \mathbb{C}$

$$z \longmapsto \frac{e^{iz} - e^{-iz}}{2i}$$

holomorphic

upcoming theorem $\nearrow$ $\Rightarrow$ analytic

$$\forall n \in \mathbb{N}^+ \quad f\left(\frac{1}{n}\right) = \sin\left(\frac{\pi}{\frac{1}{n}}\right) = \sin(\pi \cdot n) = 0.$$

But $f \neq 0$.

$$\left(\frac{1}{n}\right) \qquad \lim_{n \to +\infty} \frac{1}{n} = 0 \notin \mathbb{C}^*$$

$\left\{\frac{1}{n} \mid n \in \mathbb{N}^+\right\}$ discrete!

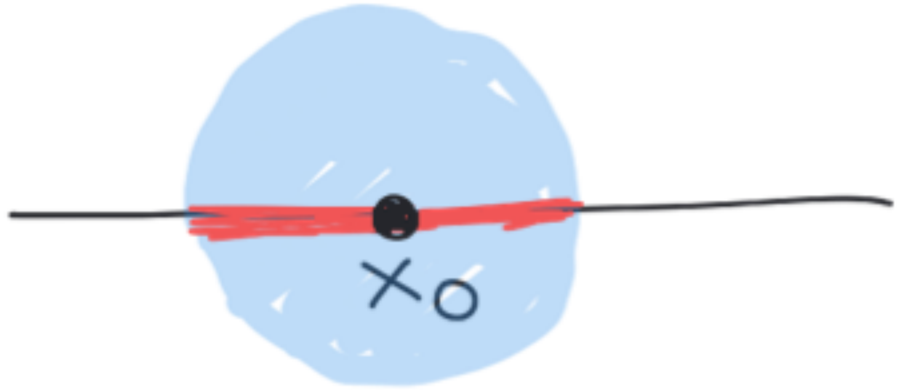
3.2.a $U \subseteq \mathbb{C}$ open, $f: U \to \mathbb{C}$ analytic, $f(U \cap \mathbb{R}) \subseteq \mathbb{R}$.
$x_0 \in U \cap \mathbb{R}$. Want to show that the power series expansion of f in x_0 has real coefficients:

$\sum_{n \geq 0} a_n^{x_0} T^n$ formal power series of the power series expansion of f in x_0

- radius of convergence $\geq \varepsilon_{x_0}$
- $\forall z \in B_{\varepsilon_{x_0}}(x_0)$ $\quad f(z) = \sum_{n=0}^{+\infty} a_n^{x_0} (z - x_0)^n$.

If $(a_n^{x_0})_{n \geq 0} \subseteq \mathbb{R}$, then $\forall x \in (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})$

$$\varphi(x) = f(x) = \sum_{n=0}^{+\infty} a_n^{x_0} (x - x_0)^n$$



1st way $a_n^{x_0} = \frac{1}{n!} f^{(n)}(x_0)$. Need to show

$\forall n \geq 0,\ f^{(n)}(x_0) \in \mathbb{R}$.

Lemma $f: U \to \mathbb{C}$ holomorphic, $f(U \cap \mathbb{R}) \subseteq \mathbb{R} \Longrightarrow$
$f'(U \cap \mathbb{R}) \subseteq \mathbb{R}$

Pf $x_0 \in U \cap \mathbb{R}$ $\quad f'(x_0) = \lim\limits_{\substack{z \to x_0 \\ z \in \mathbb{C}}} \frac{f(z) - f(x_0)}{z - x_0} = \lim\limits_{\substack{x \to x_0 \\ x \in \mathbb{R}}} \frac{\overbrace{f(x) - f(x_0)}^{\text{real}}}{\underbrace{x - x_0}_{\text{real}}} \in \mathbb{R}$

$\mathbb{R}$ is closed in $\mathbb{C}$.

Apply the lemma to $f, f', f'', f^{(3)}, \ldots$

2nd way $x_0 \in U \cap \mathbb{R}$, $f(z) = \sum_{n=0}^{+\infty} a_n^{x_0} (z-x_0)^n$ for $|z-x_0| < \varepsilon_{x_0}$.

$\sum_{n\geq 0} a_n^{x_0} T^n$, $\sum_{n\geq 0} \overline{a_n^{x_0}} T^n \in \mathbb{C}[[T]]$ have the same convergence radius

$|a_n^{x_0}| = |\overline{a_n^{x_0}}|$

$\overline{} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ $\mathbb{R}$-linear $\Rightarrow$ continuous $\Rightarrow$ ($z_n \to z_\infty$ in $\mathbb{C}$ $\Rightarrow \bar{z}_n \to \bar{z}_\infty$)

$$\Rightarrow \overline{f(z)} = \overline{\sum_{n=0}^{+\infty} a_n^{x_0} (z-x_0)^n} = \sum_{n=0}^{+\infty} \overline{a_n^{x_0} (z-x_0)^n} = \sum_{n=0}^{+\infty} \overline{a_n^{x_0}} (\bar{z}-x_0)^n$$

$$\Rightarrow \forall z \in (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0}) \quad \sum_{n=0}^{+\infty} a_n^{x_0} (z-x_0)^n = f(z) = \overline{f(z)} = \sum_{n=0}^{+\infty} \overline{a_n^{x_0}} (z-x_0)^n$$

$h = (z - x_0)$

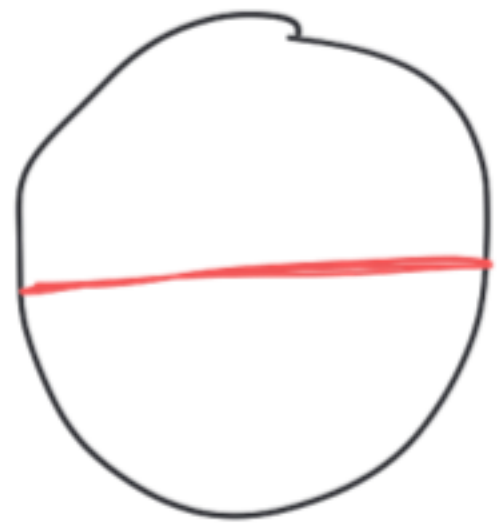
$f(z) \in \mathbb{R}$ $\quad$ $z \in \mathbb{R}$

$\forall h \in (-\varepsilon_{x_0}, \varepsilon_{x_0})$ $\quad \sum_{n=0}^{+\infty} a_n^{x_0} h^n = \sum_{n=0}^{+\infty} \overline{a_n^{x_0}} h^n$

$$\sum_{n=0}^{+\infty} \left(a_n^{x_0} - \overline{a_n^{x_0}}\right) h^n = 0$$

$\Rightarrow$ the formal power series $\sum_{n\geq 0} \left(a_n^{x_0} - \overline{a_n^{x_0}}\right) T^n$ induces a function on $B_{\varepsilon_{x_0}}(0)$ which is 0 on $(-\varepsilon_{x_0}, \varepsilon_{x_0})$

doesn't have isolated zeros



Lemma 1 of analytic continuation

$$\Rightarrow \sum_{n\geq 0} \left(a_n^{x_0} - \overline{a_n^{x_0}}\right) T^n = 0 \in \mathbb{C}[[T]]$$

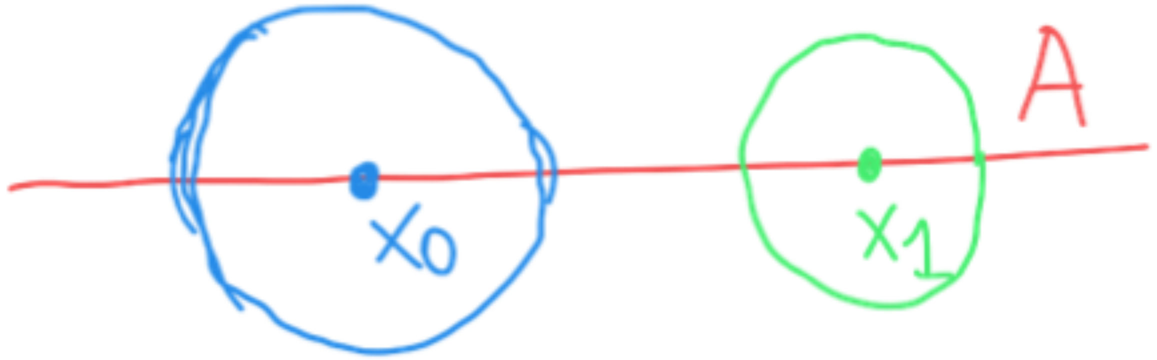
$$\Rightarrow a_n^{x_0} = \overline{a_n^{x_0}} \quad \forall n \geq 0 .$$

3.2.f

$\varphi : A \to \mathbb{R}$ real analytic.

$\forall x_0 \in A$, $\exists\, \varepsilon_{x_0} > 0$ and $\sum_{n\geq 0} a_n^{x_0} T^n \in \mathbb{R}[[T]]$ with convergence radius $\geq \varepsilon_{x_0}$:

$\forall x \in (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})$ $\varphi(x) = \sum_{n=0}^{+\infty} a_n^{x_0} (x - x_0)^n$.



Fix $x_0 \in A$.

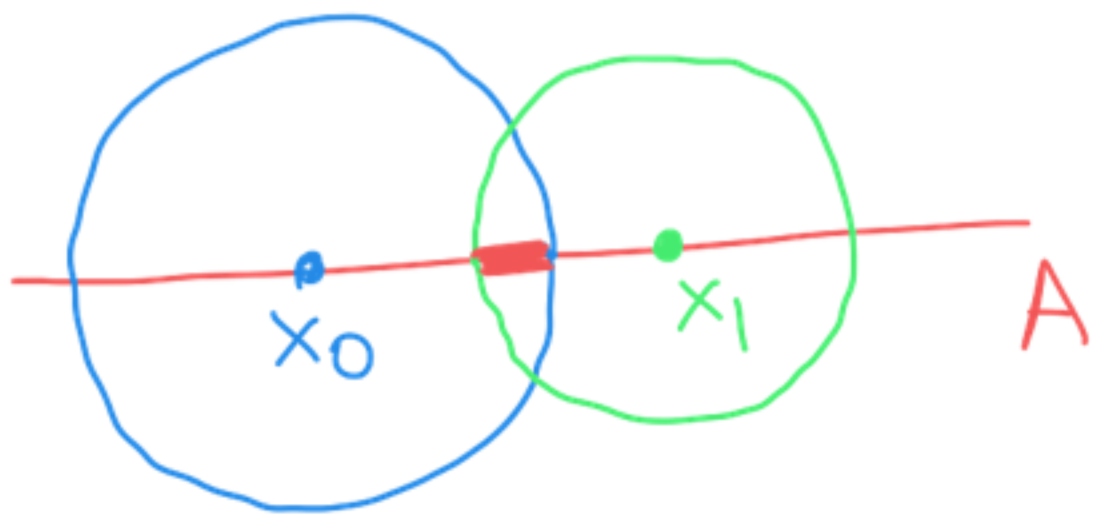
$$B_{\varepsilon_{x_0}}(x_0) = \{ z \in \mathbb{C} \mid |z - x_0| < \varepsilon_{x_0} \}$$

$B_{\varepsilon_{x_0}}(x_0) \cap \mathbb{R} = $
$= (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})$

we define
$$f^{x_0} : B_{\varepsilon_{x_0}}(x_0) \longrightarrow \mathbb{C}$$
$$z \longmapsto \sum_{n=0}^{+\infty} a_n^{x_0} (z - x_0)^n$$

f^{x_0} analytic

$$f^{x_0}\big|_{(x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})} = \varphi\big|_{(x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})},$$



$$x_0 \rightsquigarrow f^{x_0} : B_{\varepsilon_{x_0}}(x_0) \longrightarrow \mathbb{C}$$

$$x_1 \rightsquigarrow f^{x_1} : B_{\varepsilon_{x_1}}(x_1) \longrightarrow \mathbb{C} .$$

$$f^{x_0}\Big|_{B_{\varepsilon_{x_0}}(x_0) \cap B_{\varepsilon_{x_1}}(x_1)} = f^{x_1}\Big|_{B_{\varepsilon_{x_0}}(x_0) \cap B_{\varepsilon_{x_1}}(x_1)}$$

because: these two are analytic functions on the connected open set $B_{\varepsilon_{x_0}}(x_0) \cap B_{\varepsilon_{x_1}}(x_1)$ which coincide on the non-discrete set $B_{\varepsilon_{x_0}}(x_0) \cap B_{\varepsilon_{x_1}}(x_1) \cap \mathbb{R} =$
$= (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0}) \cap (x_1 - \varepsilon_{x_1}, x_1 + \varepsilon_{x_1})$, because on $(x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0}) \cap (x_1 - \varepsilon_{x_1}, x_1 + \varepsilon_{x_1})$ they coincide with φ.

$$U = \bigcup_{x_0 \in A} B_{\varepsilon_{x_0}}(x_0) \quad , \qquad U \cap \mathbb{R} = \bigcup_{x_0 \in A} (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0})$$

$$\bigcup_{x_0 \in A} (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0}) = A$$

$$f : U \longrightarrow \mathbb{C}$$
$$z \longmapsto f^{x_0}(z) \quad \text{if } z \in B_{\varepsilon_{x_0}}(x_0).$$

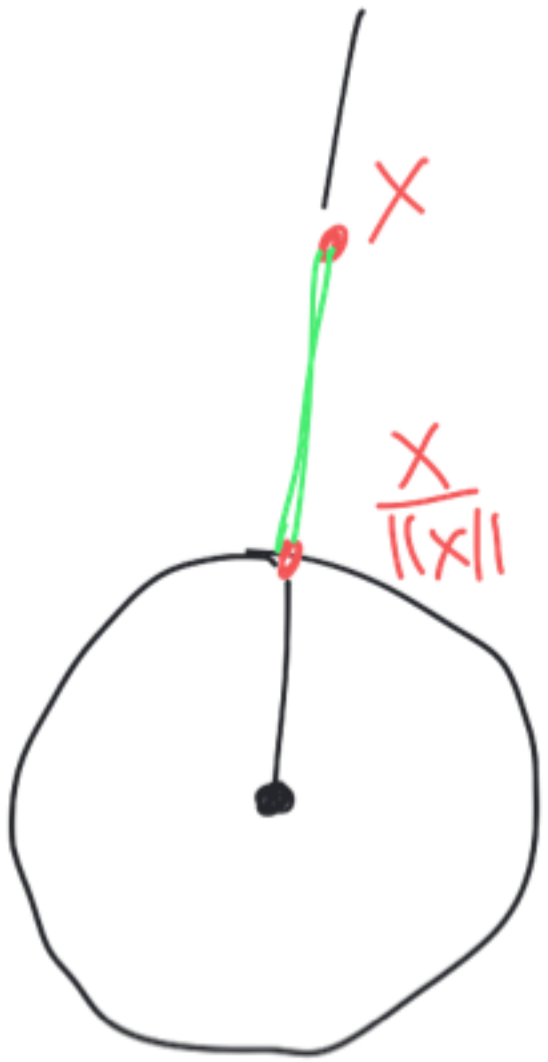
f wohldefiniert : previous slide

f analytic

3.5.6 Generalisation of the fact S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$:

$$R: (\mathbb{R}^n \setminus \{0\}) \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}$$

$$(x,s) \longmapsto (1-s)\frac{x}{\|x\|} + sx$$



$\frac{x}{\|x\|} \in S^{n-1}$ and lies on the ray/halfline containing x

$A \subseteq \mathbb{R}^n$ affine k-dimensional subspace.

Want to show that $\mathbb{R}^n \setminus A$ is homotopy eq. to S^{n-k-1}.

Translations are homeomorphisms $\Rightarrow$ wlog $0 \in A$

A linear subspace of $\mathbb{R}^n$.

linear transformations are homeomorphisms $\Rightarrow$ Wlog
$GL_n(\mathbb{R})$

$A = \mathrm{Span}(e_1, \dots, e_k)$.

$= \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$

(Homeomorphisms are homotopy equivalences)

$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ $\qquad X = \mathbb{R}^n \setminus A$

$A = \mathbb{R}^k \times \{0\}$

$x \in \mathbb{R}^k$

$y \in \mathbb{R}^{n-k}$

$(x,y) \in \mathbb{R}^n$

$$R: X \times [0,1] \longrightarrow X$$

$$(x,y,s) \longmapsto \left((1-s)x, \, (1-s)y + \frac{y}{\|y\|} \right)$$

never equal to the zero vector

deformation-retraction of X onto $\{0\} \times S^{n-k-1}$

$n=3, \; k=1$

$A = \mathrm{Span}(e_1)$

unitary sphere on $\mathrm{Span}(e_2, e_3)$

X contractible $\Rightarrow$ X path-connected:

↳ homotopy equivalent to a space with one element.

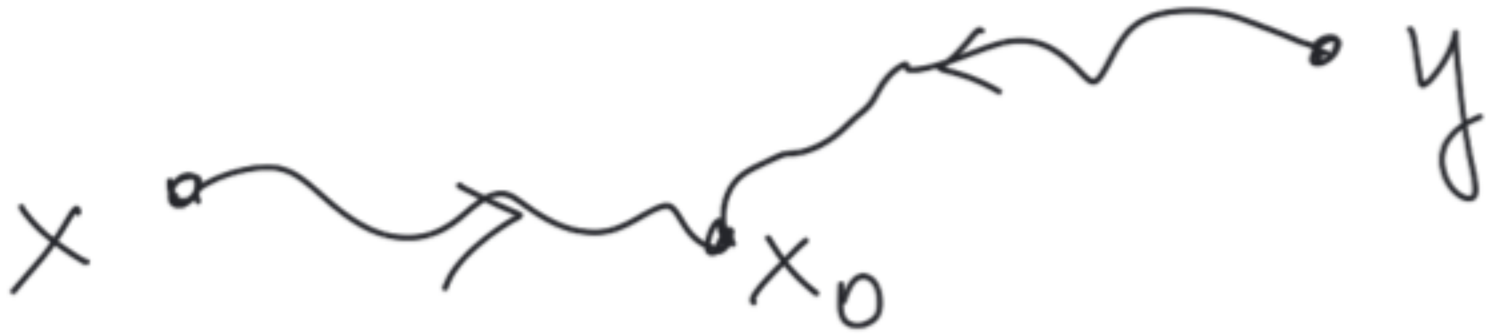
3.3 (i) $\Leftrightarrow$ (ii): $id_X : X \to X$ homotopic to a constant map $c_{x_0}: X \to X$, $x \mapsto x_0$

$F: X \times [0,1] \to X$

$F(\cdot, 0) = id_X$

$F(\cdot, 1) = c_{x_0}$

$\forall x \in X \Rightarrow F(x, \cdot): [0,1] \to X$ path from x to x_0.



$$X \xrightarrow{f} Y \qquad \rightsquigarrow \qquad \pi_1(f,x_0): \pi_1(X,x_0) \longrightarrow \pi_1(Y,f(x_0))$$
$$x_0 \in X,\ f(x_0) \in Y \qquad\qquad [\gamma] \longmapsto [f\circ\gamma]$$

If $f = id_X$:
$$\pi_1(id_X, x_0): \pi_1(X,x_0) \longrightarrow \pi_1(X,x_0)$$
$$[\gamma] \longmapsto [id_X \circ \gamma] = [\gamma]$$

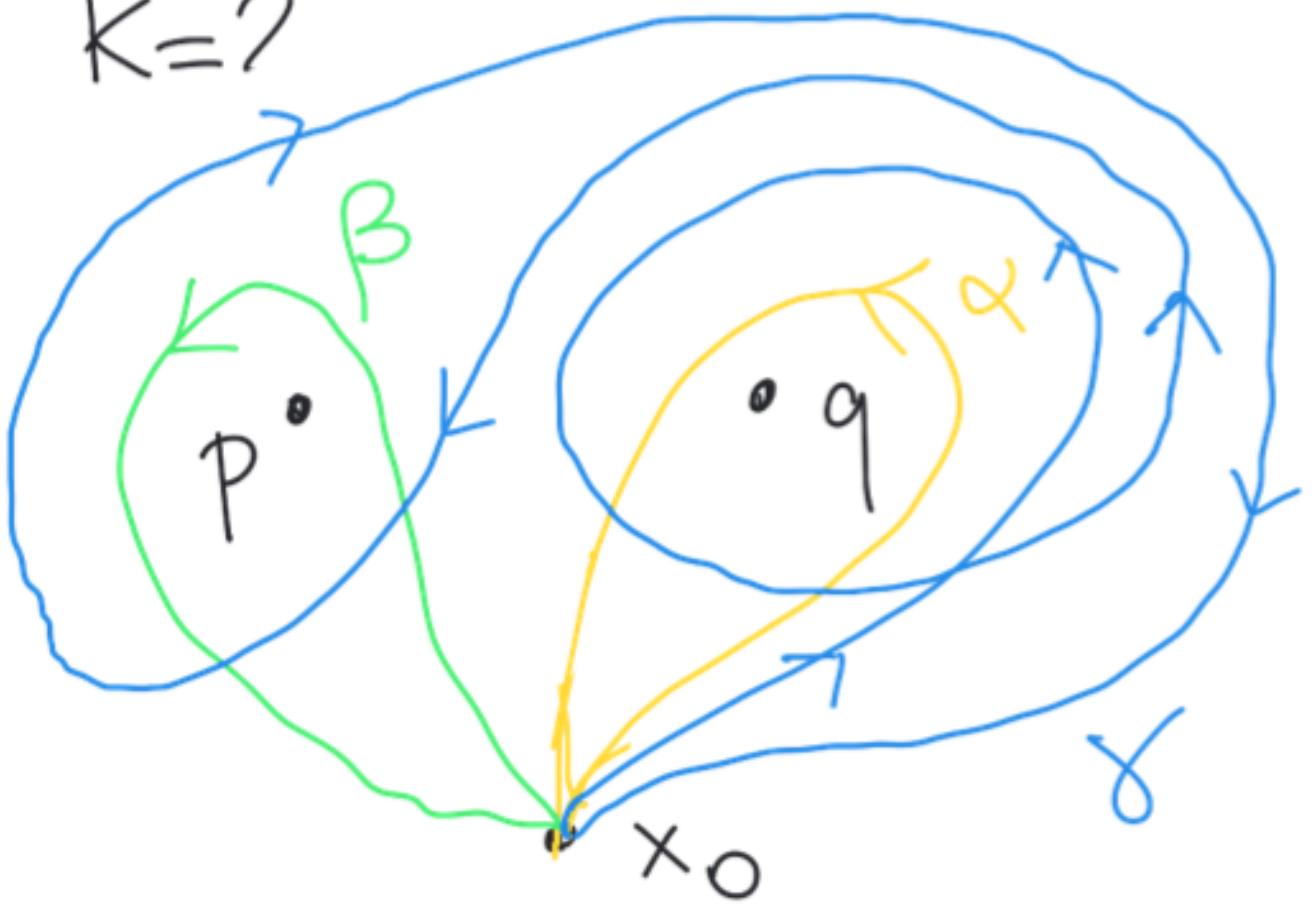
$$\Rightarrow \pi_1(id_X, x_0) = id_{\pi_1(X,x_0)}$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
x_0

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\pi_1(f{,}x_0)"] \arrow[dr, "\pi_1(g\circ f{,}x_0)"'] & \pi_1(Y, f(x_0)) \arrow[d, "\pi_1(g{,}f(x_0))"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \pi_1(Z, g(f(x_0)))
\end{tikzcd}$$

$$\pi_1(g\circ f, x_0) = \pi_1(g, f(x_0)) \circ \pi_1(f, x_0)$$

$k = ?$



$X = \mathbb{C} \setminus \{p, q\}$

$\pi_1(X, x_0)$

All the things below are not loops, but path-homotopy classes of loops, i.e. elements in $\pi_1(X, x_0)$

$\gamma = \alpha^2 \beta^{-1} \alpha^{-1}$
$= \alpha \alpha \beta^{-1} \alpha^{-1}$

homologous to $\alpha \beta^{-1}$

Every element in $\pi_1(X, x_0)$ is a "word" in the alphabet $\alpha, \beta, \alpha^{-1}, \beta^{-1}$.

- there is the empty word 1
- rule : $\alpha \alpha^{-1} = 1$, $\alpha^{-1} \alpha = 1$, $\beta \beta^{-1} = 1$, $\beta^{-1} \beta = 1$

$\Rightarrow \pi_1(X, x_0)$ is the free group with 2 generators $\mathbb{Z} * \mathbb{Z}$

G, H two groups

- $G \times H$ cartesian product

  $\forall g \in G, \forall h \in H \qquad (g,1)\cdot(1,h) = (1,h)\cdot(g,1)$

- $G * H$ free product:

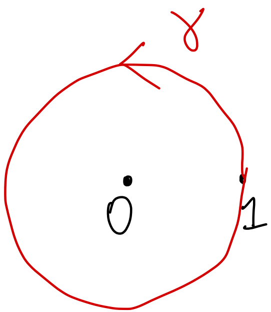
  elements are words in G or H

  $gh \neq hg$

$$\underset{\alpha}{\mathbb{Z}} * \underset{\beta}{\mathbb{Z}} = \text{free group with generators } \alpha, \beta$$

$$\begin{aligned} p_n : \mathbb{C}^* &\longrightarrow \mathbb{C}^* \\ z &\longmapsto z^n \end{aligned}$$ stetig, $p_n(1) = 1$. $n \in \mathbb{Z}$

$$\pi_1(p_n, 1) : \pi_1(\mathbb{C}^*, 1) \longrightarrow \pi_1(\mathbb{C}^*, 1)$$



$$\begin{aligned} \gamma : [0,1] &\longrightarrow \mathbb{C}^* \\ t &\longmapsto e^{2\pi i t} \end{aligned}$$

$\pi_1(\mathbb{C}^*, 1) \simeq \mathbb{Z}$ with generators $[\gamma]$

$$\begin{aligned} \Phi : \mathbb{Z} &\longrightarrow \pi_1(\mathbb{C}^*, 1) \\ k &\longmapsto [\gamma]^k \end{aligned}$$ Gruppenisom.

$$p_n \circ \gamma : [0,1] \longrightarrow \mathbb{C}^*$$
$$t \longmapsto p_n(\gamma(t)) = p_n(e^{2\pi i t}) = (e^{2\pi i t})^n = e^{2\pi i n t}$$

$n > 0$

$[0, \frac{1}{n}]$ $p_n \circ \gamma$ does one circle anticlockwise

$[\frac{1}{n}, \frac{2}{n}]$ " " "

$[\frac{2}{n}, \frac{3}{n}]$ " " "

$\vdots$

$[\frac{n-1}{n}, 1]$ " " "

$$(p_n \circ \gamma)\left(\tfrac{1}{n}\right) = e^{2\pi i n \cdot \frac{1}{n}} = e^{2\pi i} = 1$$

$$\begin{tikzcd}
{[0,\frac{1}{n}]} \arrow[rr, "(p_n \circ \gamma)|_{[0,\frac{1}{n}]}"] \arrow[dr, "\cdot n"', "\text{homeo}"] & & \mathbb{C}^* \\
& {[0,1]} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

$p_n \circ \gamma$ does the unitary circle $|n|$ times $\begin{cases} \text{anticlockwise if } n>0 \\ \text{clockwise if } n<0 \end{cases}$

$$\Rightarrow p_n \circ \gamma \text{ is path-homotopic to } \begin{cases} \underbrace{\gamma * \cdots * \gamma}_{n \text{ times}} & \text{if } n>0 \\ \underbrace{i(\gamma) * \cdots * i(\gamma)}_{|n| \text{ times}} & \text{if } n<0 \end{cases}$$

$$\Rightarrow [p_n \circ \gamma] = [\gamma]^n$$