26.05.2020

$\omega = \frac{dz}{z}$ 1-Form auf $\mathbb{C}^*$

$\begin{aligned} \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$ holomorph $\overset{\text{Goursat}}{\Longrightarrow}$ $\omega$ lokal exakt

Aufgabe 2.2

$$\omega = \frac{dx + i\,dy}{x+iy} = \frac{(x-iy)(dx+i\,dy)}{x^2+y^2}$$

$$= \underbrace{\left(\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy\right)}_{\text{exakt nach Aufgabe 4.4a}} + i\underbrace{\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)}_{\text{Aufgabe 4.4}}$$

lokal exakt

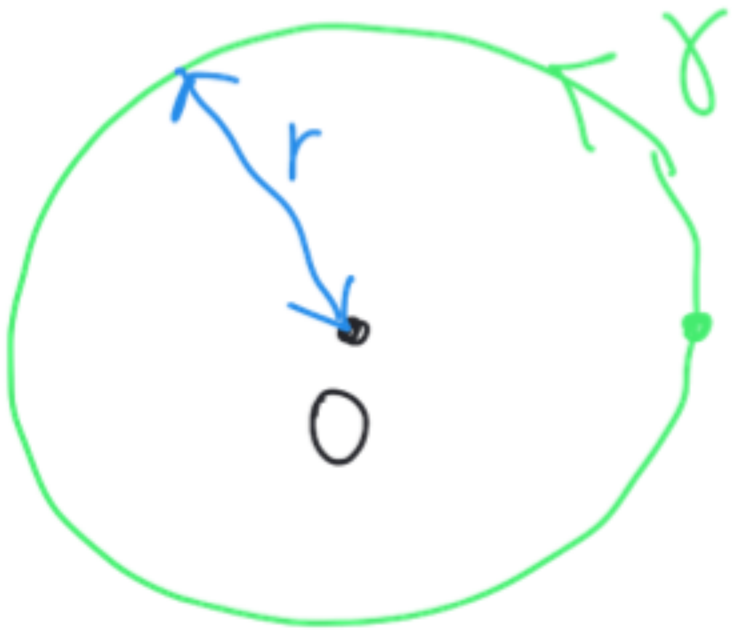
**5.2 a)** $m \in \mathbb{Z}$, $r \in \mathbb{R}_{>0}$

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}^*$$
$$t \longmapsto re^{it}$$

$\gamma$ $C^1$

$$\int_\gamma z^m \, dz = \int_0^{2\pi} (re^{it})^m \, re^{it} i \, dt$$

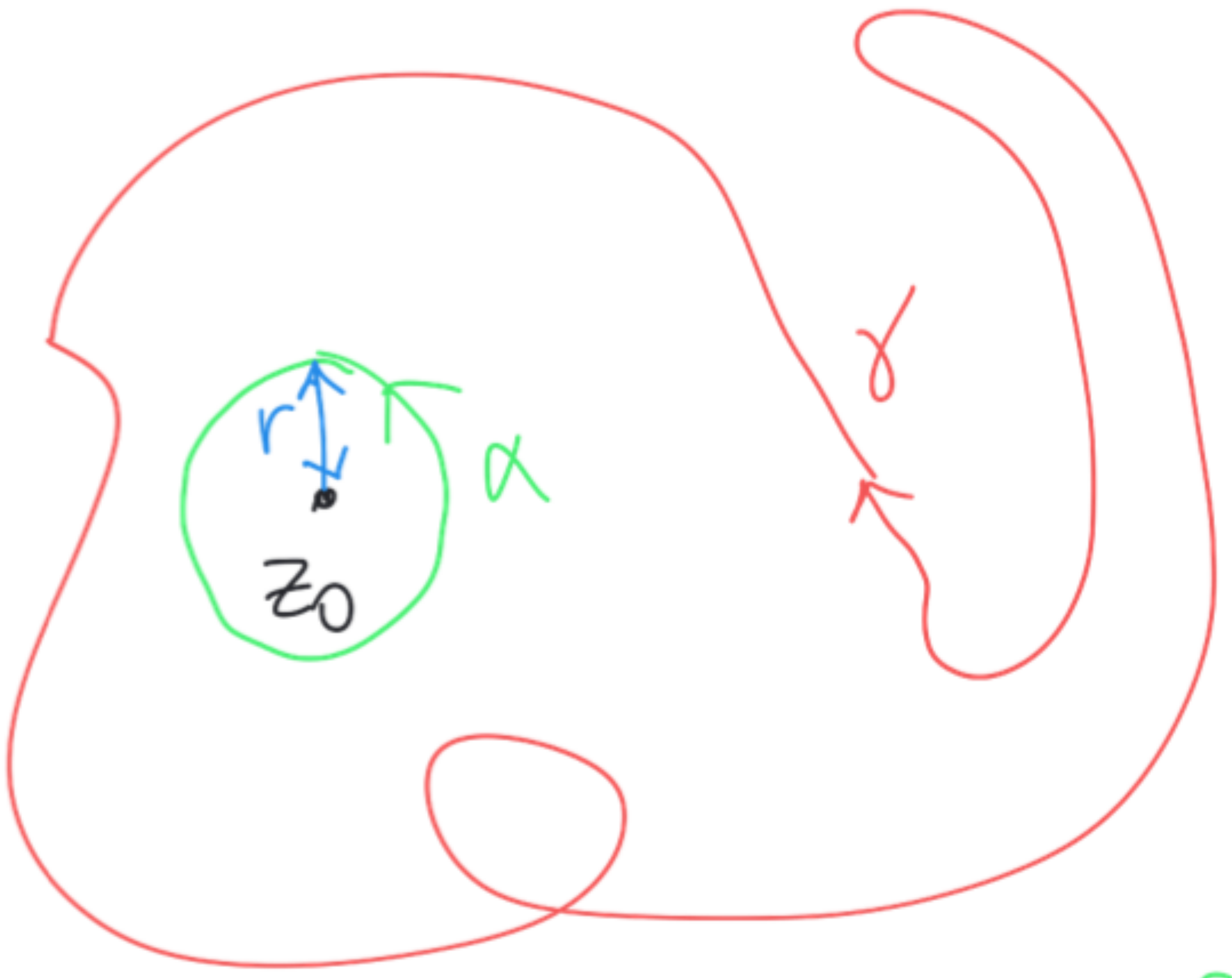
$$= \int_0^{2\pi} r^{m+1} e^{i(m+1)t} \, i \, dt$$




$$= \begin{cases} 0 & \text{wenn } m \neq -1 \\ \int_0^{2\pi} i \, dt = 2\pi i & \text{wenn } m = -1 \end{cases}$$

**5.2 b** $z_0 \in \mathbb{C}$, $\gamma$ Schleife in $\mathbb{C} \setminus \{z_0\}$

$W(\gamma, z_0) \in \mathbb{Z}$ Umlaufzahl (winding number) von $\gamma$ in Bezug auf $z_0$.




$W(\gamma, z_0) = k$, wobei $k$ is the unique integer s.t. $\gamma$ is homologous to

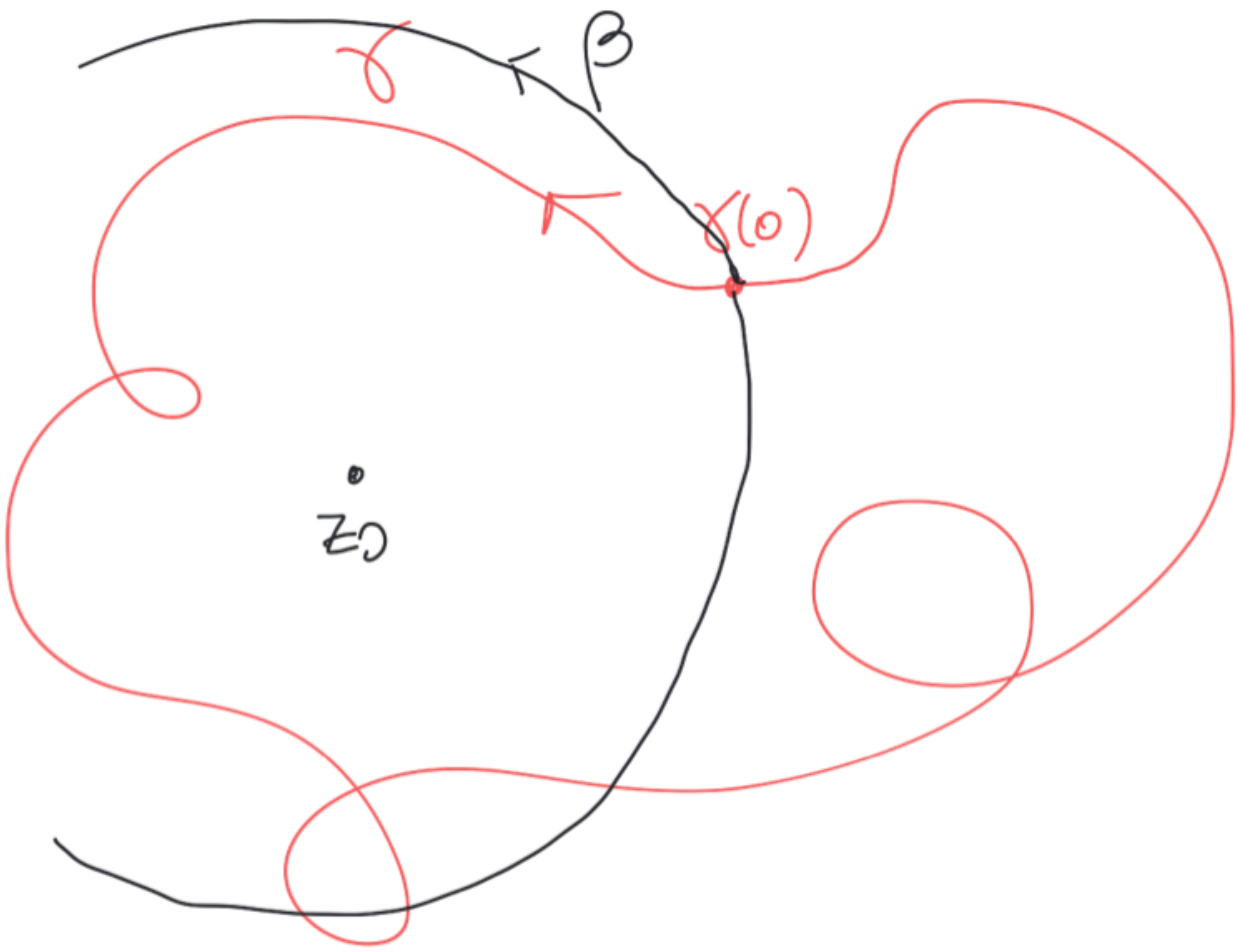
$$\underbrace{\alpha * \dots * \alpha}_{k \text{ mal}} \quad \text{in } \mathbb{C} \setminus \{z_0\}$$

$r > 0$ beliebig

$$\alpha : [0, 2\pi] \longrightarrow \mathbb{C} \setminus \{z_0\}$$
$$t \longmapsto z_0 + re^{it}$$

$$\beta : [0,1] \longrightarrow \mathbb{C} \setminus \{z_0\}$$
$$t \longmapsto z_0 + (\gamma(0) - z_0)e^{2\pi i t}$$

$\beta(0) = \gamma(0)$




$W(\gamma, z_0) = k$, where

$\gamma$ und $\underbrace{\beta * \cdots * \beta}_{k \text{ mal}}$ sind

weghomotop in
$\mathbb{C} \setminus \{z_0\}$

$\alpha : [0, 2\pi] \longrightarrow \mathbb{C} \setminus \{z_0\}$ $\quad r > 0$

$t \longmapsto z_0 + re^{it}$

$W(\gamma, z_0) = k \in \mathbb{Z} \Rightarrow \gamma$ und $\underbrace{\alpha * \cdots * \alpha}_{k \text{ mal}}$ sind homolog in $\mathbb{C} \setminus \{z_0\}$

$\mathbb{C} \setminus \{z_0\} \longrightarrow \mathbb{C}$

$z \longmapsto \frac{1}{z - z_0}$ holomorph $\overset{\text{Goursat}}{\Longrightarrow} \frac{dz}{z - z_0}$ lokal exakt auf $\mathbb{C} \setminus \{z_0\}$

$$\Rightarrow \int_\gamma \frac{dz}{z - z_0} = \int_{\underbrace{\alpha * \cdots * \alpha}_{k \text{ mal}}} \frac{dz}{z - z_0} = k \cdot \int_\alpha \frac{dz}{z - z_0} \overset{\alpha\ C^1}{=} k \cdot \int_0^{2\pi} \frac{re^{it} i\, dt}{re^{it}} =$$

$$= 2\pi i k = 2\pi i \cdot W(\gamma, z_0) \Rightarrow W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

## Aufgabe 5.3

$a, b \in \mathbb{C}$

$U = \mathbb{C} \setminus \{i, -i\}$

$V = \mathbb{C} \setminus \{iy \mid y \in \mathbb{R}, -1 \leq y \leq 1\}$

$A = \mathbb{C} \setminus \{iy \mid y \in \mathbb{R}, y \leq 1\}$

$\omega = \dfrac{az+b}{z^2+1} dz$ auf $U$. $\qquad U \supseteq V \supseteq A$.

$U \to \mathbb{C}$, $z \mapsto \dfrac{az+b}{z^2+1}$ holomorph $\overset{\text{Goursat}}{\Longrightarrow}$ $\omega$ lokal exakt auf $U$.

$\omega|_V$ " " " $V$

$\omega|_A$ " " " $A$

**Satz** $\omega$ 1-Form auf $U$, $U \subseteq \mathbb{R}^n$ offen, zusammenhängend

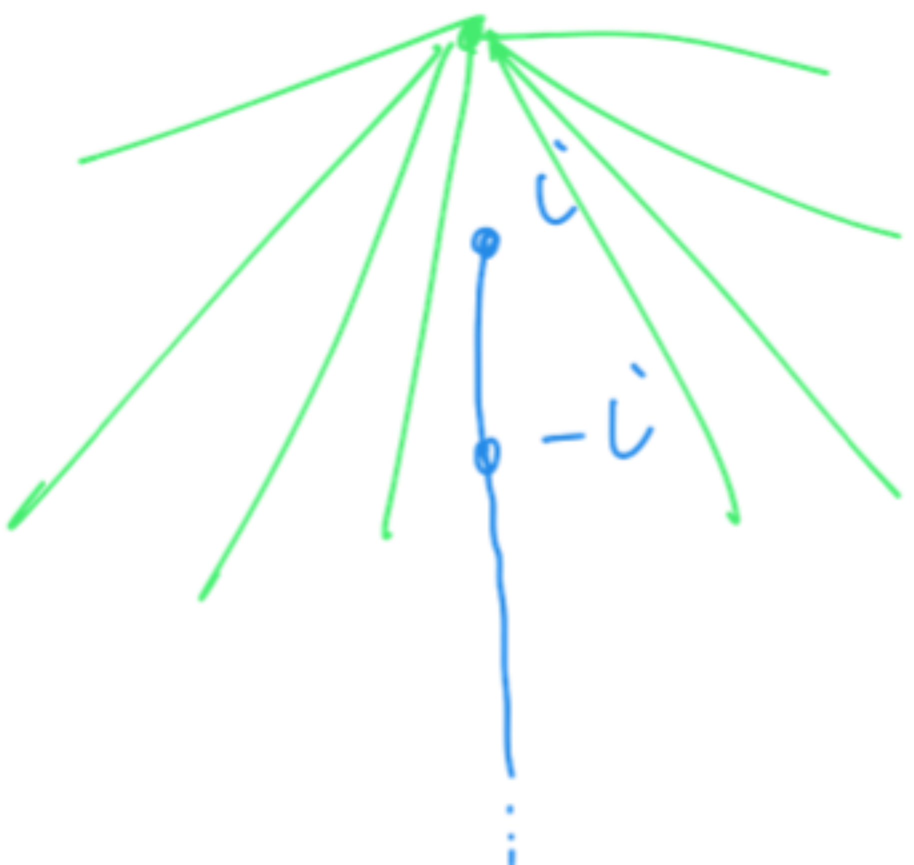
$\mathcal{F}$ collection of loops in $U$ s.t. every loop in $U$ is homologous to a linear combination of loops in $\mathcal{F}$.

$\omega$ exakt $\iff$ $\omega$ lokal exakt und $\forall \gamma \in \mathcal{F}$, $\int_\gamma \omega = 0$.

Kor U einfach zusammenhängend $\Rightarrow$
$\omega$ exakt $\iff$ $\omega$ lokal exakt.

Bws $\mathcal{F} = \emptyset$, jede Schleife in U ist nullhomolog. $\square$

i) A sternförmig $\Rightarrow$ A einfach zusammenhängend
$\Rightarrow$ $\omega$ exakt $\forall a, b \in \mathbb{C}$.



ii) $U$


$$\alpha : [0,2\pi] \longrightarrow U \qquad 0<r<2$$
$$t \longmapsto i + re^{it}$$

$$\beta : [0,2\pi] \longrightarrow U \qquad 0<r<2$$
$$t \longmapsto -i + re^{it}$$

Every loop in $U$ is homologous (in $U$) to a linear combination of $\alpha$ and $\beta$ with coefficients in $\mathbb{Z}$.

In the theorem one can take $\mathcal{F} = \{\alpha, \beta\}$.

$$\omega \text{ exakt auf } U \iff \int_\alpha \omega = 0 \text{ und } \int_\beta \omega = 0.$$

$$\omega = \frac{az+b}{z^2+1}\,dz \qquad \alpha(t) = i + re^{it} \quad 0 \le t \le 2\pi, \quad 0<r<2.$$

$$\int_\alpha \omega = \int_0^{2\pi} \frac{a\cdot(i+re^{it})+b}{(i+re^{it})^2+1}\, re^{it}\, i\, dt$$

difficult to compute

$$\omega = \frac{a+bi}{2}\,\frac{dz}{z+i} + \frac{a-bi}{2}\,\frac{dz}{z-i}$$

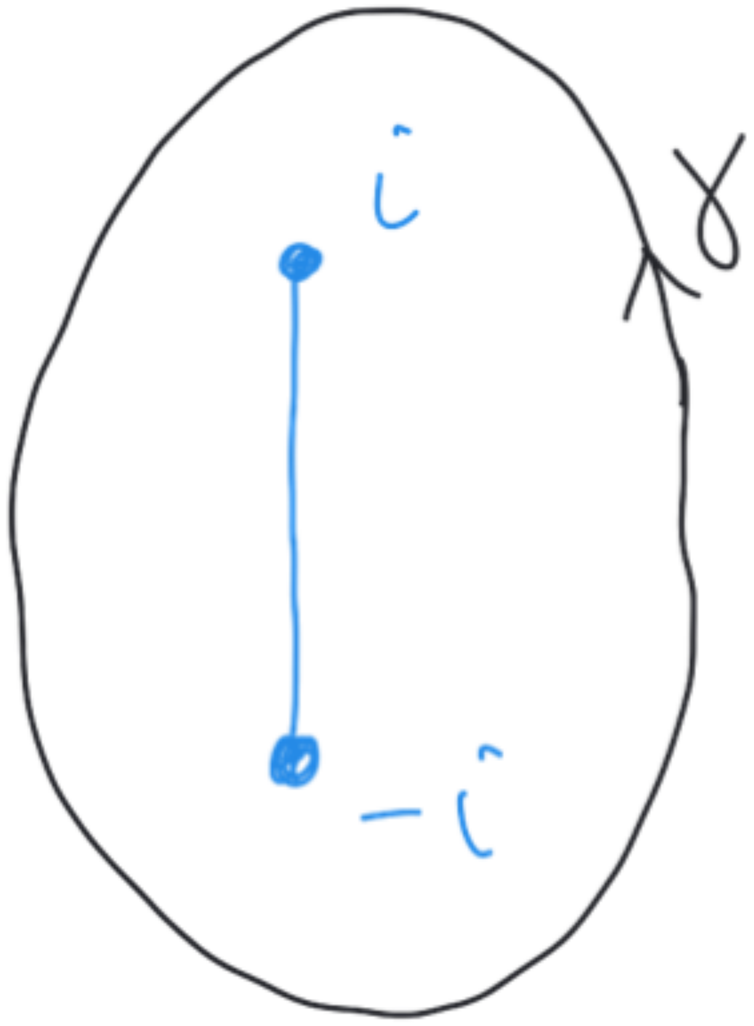
$$\int_\alpha \omega = \frac{a+bi}{2}\int_\alpha \frac{dz}{z+i} + \frac{a-bi}{2}\int_\alpha \frac{dz}{z-i}$$

$$= \frac{a+bi}{2}\cdot 2\pi i\, \underbrace{W(\alpha,-i)}_{=0} + \frac{a-bi}{2}\cdot 2\pi i\, \underbrace{W(\alpha,i)}_{=1} = (a-bi)\pi i$$

$$\int_\beta \omega = \pi i (a+bi)$$

$\omega$ exakt auf $U$ $\iff$ $\int_\alpha \omega = \int_\beta \omega = 0 \iff \begin{cases} a-bi=0 \\ a+bi=0 \end{cases} \iff a=b=0.$

iii) $V = \mathbb{C} \setminus \{iy \mid y \in \mathbb{R}, -1 \le y \le 1\}$




Every loop in $V$ is homologous in $V$ to $\underbrace{\gamma * \cdots * \gamma}_{k}$ for some $k \in \mathbb{Z}$.

$\mathcal{J} = \{\gamma\}$

$\omega$ exakt $\iff \int_\gamma \omega = 0.$

$\gamma$ and $\alpha+\beta$ are homologous in $U = \mathbb{C} \setminus \{i,-i\}$.

(notice: $\alpha, \beta$ are <u>not</u> loops in $V$).

$\omega$ lokal exakt auf $U \Longrightarrow \int_\gamma \omega = \int_\alpha \omega + \int_\beta \omega =$

$= (a-bi)\pi i + (a+bi)\pi i = 2a\pi i.$

$\omega|_V$ exakt auf $V \Longleftrightarrow \int_\gamma \omega = 0 \Longleftrightarrow a = 0.$

## Aufgabe 5.5

$U \subseteq \mathbb{C}^*$ offen, zusammenhängend. $z_0 \in U$.

$i : U \hookrightarrow \mathbb{C}^*$

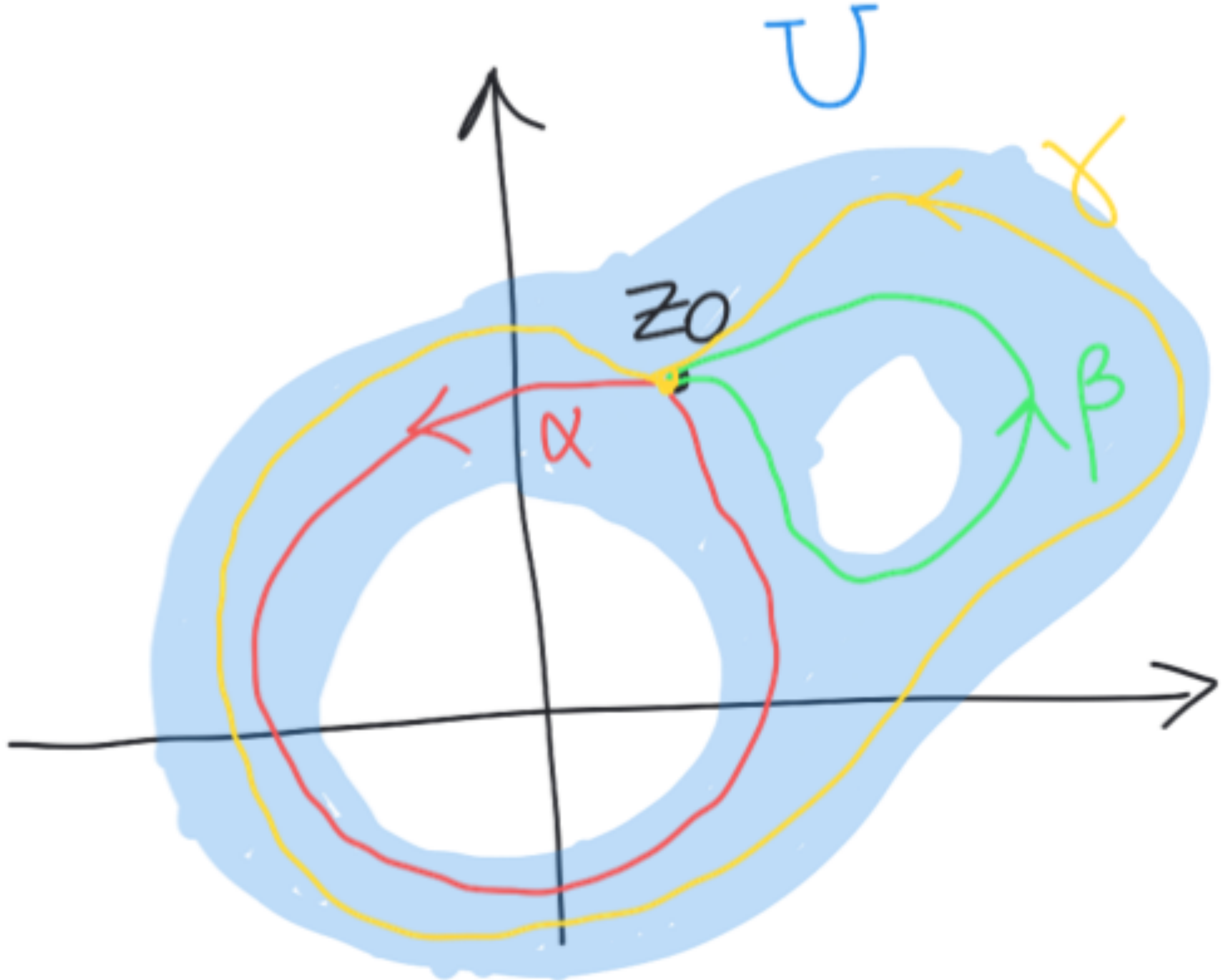
$$\pi_1(i, z_0) : \pi_1(U, z_0) \longrightarrow \pi_1(\mathbb{C}^*, z_0)$$

$$[\gamma]_U \longmapsto [i \circ \gamma] = [\gamma]_{\mathbb{C}^*}$$

$\gamma$ Schleife in $U$
mit $\gamma(0) = \gamma(1) = z_0$

path homotopy class of $\gamma$ in $U$

path homotopy class of $\gamma$ in $\mathbb{C}^*$




$$\begin{array}{cccc}
[\alpha]_U & [\beta]_U & [\gamma]_U = [\alpha]_U \cdot [\beta]_U & \pi_1(U, z_0) \\
\downarrow & \downarrow & \downarrow & \downarrow \pi_1(i, z_0) \\
[\alpha]_{\mathbb{C}^*} & 0 = [\beta]_{\mathbb{C}^*} & [\gamma]_{\mathbb{C}^*} & \pi_1(\mathbb{C}^*, z_0)
\end{array}$$

$2) \Rightarrow 3)$ $f : U \to \mathbb{C}$ stetig, $\forall z \in U \quad e^{f(z)} = z$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[ur, bend left, "f"] \arrow[r, "i"'] & \mathbb{C}^* \end{tikzcd}$$

Funktorialität $\Longrightarrow$ der Fundamentalgruppe

$$\begin{tikzcd} & \pi_1(\mathbb{C}, f(z_0)) = 0 \arrow[d] \\ \pi_1(U, z_0) \arrow[ur] \arrow[r, "\pi_1(i, z_0)"'] & \pi_1(\mathbb{C}^*, z_0) \end{tikzcd}$$

$i = \exp \circ f$

$\Rightarrow \pi_1(i, z_0)$ Nullhomomorphismus.

$3) \Rightarrow 4)$: $\forall \gamma \in \Omega(U, z_0, z_0)$, $\underbrace{\pi_1(i, z_0)([\gamma]_U)}_{= [\gamma]_{\mathbb{C}^*}} = [c_{z_0}]_{\mathbb{C}^*}$

$\forall \gamma \in \Omega(U, z_0, z_0)$, $\gamma$ ist nullweghomotop in $\mathbb{C}^*$

$\Rightarrow$ jede Schleife in $U$ ist nullhomolog in $\mathbb{C}^*$.

$\frac{dz}{z}$ lokal exakt auf $\mathbb{C}^* \Rightarrow \frac{dz}{z}$ lokal exakt auf $U$.

$\forall \gamma$ Schleife in $U$ $\quad \int_\gamma \frac{dz}{z} = 0$

$\gamma$ ist nullhomolog in $\mathbb{C}^*$

$\frac{dz}{z}$ lokal exakt auf $\mathbb{C}^*$.

$\frac{dz}{z}$ exakt auf $U$.

$4 \Rightarrow 1$: $\frac{dz}{z}$ exakt auf $U$.

Sei $\tilde{f}: U \to \mathbb{C}$ Stammfunktion von $\frac{dz}{z}$.

$\tilde{f}$ $C^1$, $d\tilde{f} = \frac{dz}{z} \Rightarrow \tilde{f}$ holomorphic, $\tilde{f}'(z) = \frac{1}{z}$ $\forall z \in U$.

$\exp: \mathbb{C} \to \mathbb{C}^*$ surjektiv $\Rightarrow \exists a \in \mathbb{C}$: $e^a = z_0 \cdot e^{-\tilde{f}(z_0)}$.

$f: U \to \mathbb{C}$
$z \mapsto \tilde{f}(z) + a$

- $f$ holomorphic
- $\forall z \in U$, $f'(z) = \frac{1}{z}$
- $e^{f(z_0)} = z_0$.

Man will zeigen, dass $\forall z \in U$ $e^{f(z)} = z$.

$h: U \to \mathbb{C}$, $z \mapsto \frac{e^{f(z)}}{z}$ wohldefiniert weil $0 \notin U$.

$r$ holomorph. $\quad r(z) = \dfrac{e^{f(z)}}{z}$

$f'(z) = \frac{1}{z}$

$$r'(z) = \frac{e^{f(z)} \cdot f'(z) \cdot z - e^{f(z)} \cdot 1}{z^2} = 0$$

Aufgabe 2.2.

$r$ holomorph, $r(z_0) = 1$, $\forall z \in U \;\; r'(z) = 0$.

$U$ zusammenhängend $\Rightarrow \forall z \in U \;\; r(z) = 1$

$$\frac{e^{f(z)}}{z} = 1$$

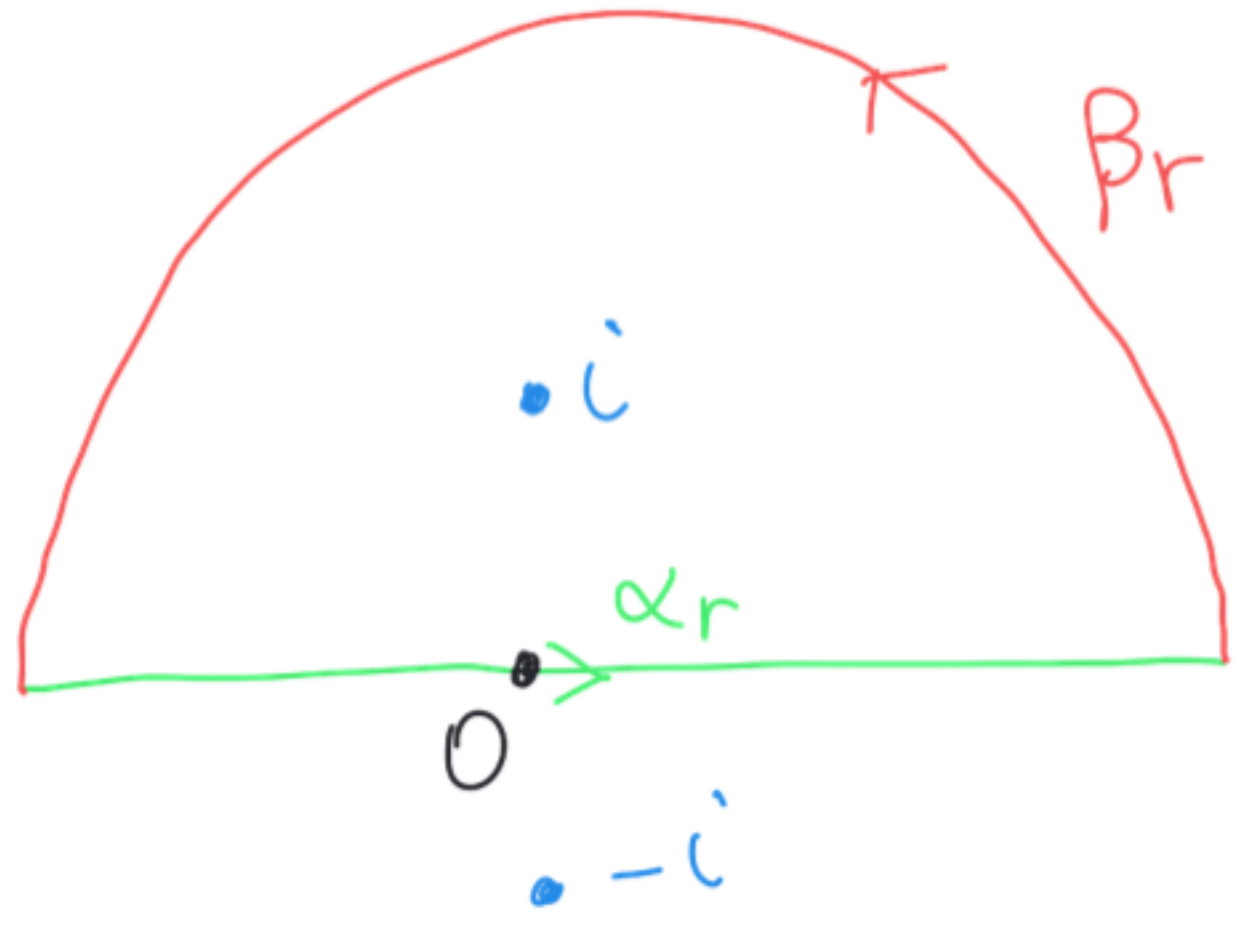
$$e^{f(z)} = z.$$

$$\begin{tikzcd}
1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[d] & 5 \arrow[r] & 6 \arrow[d] \\
4 \arrow[u] & 3 \arrow[l] & & 7 \arrow[ll]
\end{tikzcd}$$

Funktorialität von der Fundamentalgruppe

$1 \Rightarrow 5 \quad g(z) = \exp\left(\frac{1}{n} f(z)\right)$

# Aufgabe 5.4




$\alpha_r : [-r, r] \longrightarrow \mathbb{C}$
$t \longmapsto t$

$\beta_r : [0, \pi] \longrightarrow \mathbb{C}$
$t \longmapsto re^{it}$

$\alpha_r * \beta_r$ Schleife in $\mathbb{C} \setminus \{i\}$
$\forall r \in (0,1) \cup (1, +\infty)$.

$$\frac{dz}{z^2+1} = \frac{i}{2}\left(\frac{dz}{z+i} - \frac{dz}{z-i}\right)$$

a) $$\int_{\alpha_r * \beta_r} \frac{dz}{z^2+1} = \frac{i}{2}\left[\int_{\alpha_r * \beta_r} \frac{dz}{z+i} - \int_{\alpha_r * \beta_r} \frac{dz}{z-i}\right]$$

$$= \frac{i}{2}\left[2\pi i \cdot W(\alpha_r * \beta_r, -i) - 2\pi i \cdot W(\alpha_r * \beta_r, i)\right] =$$

$$= \begin{cases} 0 & \text{wenn } 0<r<1 \\ 2\pi i \cdot \frac{i}{2}(-1)\cdot 1 = \pi & \text{wenn } r>1 \end{cases}$$

b) $$\left|\int_{\beta_r} \frac{dz}{z^2+1}\right| \underset{5.1c}{\leq} L(\beta_r)\cdot \max_{z\in\beta_r([0,2\pi])} \left|\frac{1}{z^2+1}\right| \overset{r>1}{=}$$

$$= \pi r \cdot \max_{t\in[0,2\pi]} \frac{1}{|(re^{it})^2+1|} \leq \pi r \cdot \frac{1}{r^2-1}$$

$|z+w| \geq |z|-|w|$ $\quad$ $|(re^{it})^2+1| \geq |(re^{it})^2|-1 = r^2-1$

c) $\omega = \dfrac{dz}{z^2+1}$.

$$\forall r>1 \quad \int_{\alpha_r}\omega + \int_{\beta_r}\omega = \int_{\alpha_r * \beta_r}\omega = \pi$$

$$\forall r>1 \quad \left|\int_{\beta_r}\omega\right| \leq \frac{\pi r}{r^2-1}$$

$$\Rightarrow \lim_{r\to+\infty}\int_{\beta_r}\omega = 0$$

$$\underbrace{\int_{\alpha_r}\omega = \pi - \underbrace{\int_{\beta_r}\omega}_{\to 0}}_{\to \pi} \qquad r\to+\infty$$

$$\Rightarrow \pi = \lim_{r\to+\infty}\int_{\alpha_r}\omega = \lim_{r\to+\infty}\int_{-r}^{r}\frac{dt}{t^2+1} =: \int_{-\infty}^{+\infty}\frac{dt}{t^2+1}$$